Spontaneous breaking of restricted Weyl symmetry in pure R^2 gravity

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Pure R^2 gravity

The action of pure R^2 gravity is given by

$$S_0 = \int d^4x \sqrt{-g}\, \alpha\, R^2 \,.$$

Recently discovered that it is

- Invariant under the transformation $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$, with $\Box\Omega(x) \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\Omega(x) = 0$. This was dubbed *restricted Weyl invariance*.
- Equivalent to Einstein gravity with non-zero cosmological constant and massless scalar field

Restricted Weyl symmetry

Under a Weyl transformation, $g_{\mu\nu} \rightarrow \Omega^2(x) g_{\mu\nu}$, we have the following transformations (in four spacetime dimensions)

$$\begin{aligned} \sqrt{|g|} &\rightarrow \Omega^4 \sqrt{|g|} \\ R &\rightarrow \Omega^{-2} R - 6\,\Omega^{-3} \,\Box\, \Omega \,. \end{aligned}$$

It follows that under a Weyl transformation we have

$$\sqrt{|g|}\, R^2 \rightarrow \sqrt{|g|}\, R^2 - \sqrt{|g|}\, 12\, R\, \Omega^{-1} \,\Box \Omega + \sqrt{|g|}\, 36\, \Omega^{-2} \,(\Box\, \Omega)^2 \,.$$

Invariant if $\Box \Omega(x) = 0$; this is called restricted Weyl invariance.

R^2 gravity: equivalent form of the action

We introduce the auxiliary field φ and rewrite the action into the equivalent form

$$S_1 = \int d^4x\sqrt{-g}\left[-\alpha\,(c_1\varphi + R)^2 + \alpha\,R^2\right]$$

where c_1 is a non-zero arbitrary constant.

The first term (which is squared) does nothing after performing the Gaussian integral over φ in the path integral i.e.

$$\int \mathcal{D}\varphi e^{-i\alpha\,c_1^2\int d^4x\sqrt{-g}(\varphi - f(x))^2} = \mathrm{const}$$

Expanding the action

Expanding the squared term in the action yields

$$S_2 = \int d^4x\sqrt{-g}\left(-c_1^2\,\alpha\,\varphi^2 - 2\alpha\,c_1\,\varphi\,R\right)$$

The above action maintains restricted Weyl invariance if the auxiliary field φ transforms as $\varphi \to \varphi/\Omega^2$

i.e. action is invariant under the transformations $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ and $\varphi \to \varphi/\Omega^2$ with $\Box\Omega = 0$.

Conformal transformation

We now perform the conformal transformation

$$g_{\mu\nu} \rightarrow \varphi^{-1} g_{\mu\nu}$$

$$\sqrt{-g} \rightarrow \varphi^{-2} \sqrt{-g}$$

$$R \rightarrow \varphi R - 6\varphi^{3/2} \Box \varphi^{-1/2}$$

Einstein gravity with cosmological constant and massless scalar field

In this new frame, we obtain Einstein gravity with cosmological constant and massless scalar field

$$
\begin{aligned}
S_3 &= \int d^4x\sqrt{-g}\Big(-\alpha c_1^2 - 2\,\alpha c_1 R + 3\,\alpha c_1 \frac{1}{\varphi^2}\partial_\mu\varphi\,\partial^\mu\varphi\Big)\\
&= \int d^4x\sqrt{-g}\Big(-\alpha c_1^2 - 2\,\alpha c_1 R - \partial_\mu\psi\,\partial^\mu\psi\Big)
\end{aligned}
$$

where $\psi \equiv \sqrt{-3\,\alpha\,c_1}\,\ln\varphi$.

- $-2\,\alpha c_1 R$ is the Einstein-Hilbert term with $-2\,\alpha\,c_1$ determining Newton's constant (with αc_1 negative in our convention).
- the cosmological constant is $\Lambda = -\dfrac{c_1}{4}$ and can be positive (de Sitter) or negative (anti-de Sitter). It cannot be zero.
- ψ is a minimally coupled massless scalar field.

Spontaneous breaking of restricted Weyl symmetry

The final Einstein gravity action is not scale or restricted Weyl invariant while the original R^2 action is. What happened?

- The conformal transformation is not valid for $\varphi = 0$ and therefore excludes solutions with $R = 0$.
- The vacuum (background) spacetime must have $R \neq 0$ and this vacuum state does not obey the restricted Weyl symmetry of the original action.
- The original scale or restricted Weyl symmetry is *spontaneously broken* and the massless scalar field ψ is identified as the *Nambu Goldstone boson* of the broken symmetry.

Shift symmetry of Goldstone boson

Note that the final Einstein action is invariant under

$$g_{\mu\nu} \to g_{\mu\nu} \quad ; \quad \varphi \to \varphi/\Omega^2 \text{ with } \tilde{\Box}\Omega = 0 \,.$$

Here $\tilde{g}_{\mu\nu}$ is the original metric. In terms of the final metric $g_{\mu\nu}$ the condition becomes $\Box\Omega - \partial_\mu(\ln\varphi)\,\partial^\mu\Omega = 0$.

In terms of the Goldstone boson ψ the Einstein action is invariant under

$$g_{\mu\nu} \to g_{\mu\nu} \quad ; \quad \psi \to \psi - 2\sqrt{-3\,\alpha\,c_1}\,\ln\Omega \,.$$

This is a shift symmetry of the Goldstone boson ψ.

The restricted Weyl symmetry of the original action is therefore realized in the shift symmetry of the Goldstone boson in the final Einstein action. This is a well known feature of spontaneously broken theories.

Part II

Generating a massive Higgs field via the spontaneous breaking of restricted Weyl symmetry

Adding a massless Higgs field to the original action

We now add a non-minimally coupled Higgs boson field Φ (with no explicit mass) to the R^2 gravity action

$$S_0 = \int d^4x\sqrt{-g}\left(\alpha R^2 - \xi R|\Phi|^2 - (\partial_\mu \bar{\Phi}\partial^\mu \Phi) - \lambda|\Phi|^4\right) .$$

The above action is restricted Weyl invariant: it is invariant under the transformations

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu},\ \Phi \to \Phi/\Omega(x) \ \text{ with } \ \Box\Omega(x) = 0.$$

Equivalent form of action: second constant c_2

Like before we introduce the auxiliary field φ and rewrite the action into the equivalent form

$$S_1 = \int d^4x\sqrt{-g}\Big[-\alpha\left(c_1\varphi + R + \frac{c_2}{\alpha}|\Phi|^2\right)^2 + \alpha R^2 - \xi R|\Phi|^2 - (\partial_\mu\bar{\Phi}\partial^\mu\Phi) - \lambda|\Phi|^4\Big]$$

where besides c_1 we now have the constant c_2.

Expanding the squared term in the action yields

$$S_2 = \int d^4x\sqrt{-g}\,\big(-c_1^2\alpha\varphi^2 - 2\alpha c_1\varphi R$$
$$-(\xi + 2c_2)R|\Phi|^2 - (\partial_\mu\bar{\Phi}\partial^\mu\Phi) - 2c_1c_2\varphi|\Phi|^2 - (\alpha^{-1}c_2^2 + \lambda)|\Phi|^4\big) \;.$$

The above action maintains restricted Weyl invariance if the auxiliary field φ transforms as $\varphi \to \varphi/\Omega^2$

i.e. the action is invariant under the transformations $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, $\Phi \to \Phi/\Omega$ and $\varphi \to \varphi/\Omega^2$ with $\Box\Omega = 0$.

Conformal transformation including transformation of Higgs field

We now perform the conformal transformation

$$g_{\mu\nu} \rightarrow \varphi^{-1} g_{\mu\nu}$$

$$\sqrt{-g} \rightarrow \varphi^{-2}\sqrt{-g}$$

$$R \rightarrow \varphi R - 6\varphi^{3/2}\Box\varphi^{-1/2}$$

$$\Phi \rightarrow \varphi^{1/2}\Phi \ .$$

Emergence of Higgs mass

In this new frame, we obtain like before Einstein gravity with cosmological constant

$$S_3 = \int d^4x\sqrt{-g}\Big(-\alpha c_1^2 - 2\,\alpha c_1 R - \partial_\mu\bar{\Phi}\partial^\mu\Phi - 2\,c_1 c_2|\Phi|^2 - (\alpha^{-1}c_2^2+\lambda)|\Phi|^4 - (\xi+2c_2)R\,|\Phi|^2 + 3\,\alpha c_1\frac{1}{\varphi^2}\partial_\mu\varphi\,\partial^\mu\varphi + (6(\xi+2c_2)-1)\varphi^{1/2}\Box\varphi^{-1/2}|\Phi|^2\Big)\,.$$

The new thing here is the emergence of a Higgs mass $m^2 = 2\,c_1\,c_2$.

Coupling of scalar field φ with Higgs field Φ

The auxiliary field φ couples to the Higgs field and gravity. If its coupling to the Higgs field is experimentally small, we are free to eliminate it by choosing $\xi + 2c_2$ to be $1/6$.

The action then simplifies to

$$S_4 = \int d^4x\sqrt{-g}\Big(-\alpha c_1^2 - 2\,\alpha c_1 R - \partial_\mu \bar{\Phi}\partial^\mu \Phi - 2\,c_1 c_2 |\Phi|^2 \\ - (\alpha^{-1} c_2^2 + \lambda)|\Phi|^4 - \frac{1}{6} R\,|\Phi|^2 - \partial_\mu \psi\,\partial^\mu \psi\Big)\,.$$

This final model contains Einstein gravity, cosmological constant, a massive Higgs and the Goldstone boson ψ.

One can then include all of the other standard model fields in the original restricted Weyl invariant action.

Thank You

Inclusion of other Standard Model fields

We now briefly discuss the inclusion of the other standard model fields into the original action. They are collectively given by gauge fields A_μ with gauge group $G = U(1) \times SU(2) \times SU(3)$ and (Weyl) fermions Ψ under various representations of G. We may consider the (restricted) Weyl invariant action schematically given by

$$-\int d^4x\sqrt{-g}\Big(\frac{1}{2g^2}\,\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \theta\,\mathrm{Tr}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + \bar{\Psi}D_\mu\gamma^\mu\Psi + \lambda(\Psi\Psi\Phi) + \bar{\lambda}(\bar{\Psi}\bar{\Psi}\bar{\Phi}) + \text{Higgs}\Big).$$

The Higgs sector is essentially given in the previous action by replacing the derivative ∂_μ with the gauge covariant one. The gauge coupling constant g, the theta angle θ, and Yukawa coupling λ are all dimensionless and it is easy to see that the above action does not change under a restricted Weyl transformation supplemented by $\Psi \to \varphi^{3/4}\Psi$. Note that there is no direct coupling to the extra massless scalar field φ (except for the Higgs sector we have already discussed).

Symmetry of final action

The restricted Weyl symmetry of our original action manifests itself in the final action through a symmetry under a transformation of the auxiliary field φ only. The action is invariant under

$\varphi \to \varphi/\Omega^2$, $g_{\mu\nu} \to g_{\mu\nu}$, $\Phi \to \Phi$ with condition $\Box\Omega - \partial_\mu(\ln\varphi)\partial^\mu\Omega = 0$.

This can be understood as follows. We replaced $g_{\mu\nu}$ with $\hat{g}_{\mu\nu} = \varphi^{-1} g_{\mu\nu}$. Originally, the restricted Weyl symmetry acts as $\hat{g}_{\mu\nu} \to \hat{g}_{\mu\nu}\Omega^2$, $\varphi \to \varphi/\Omega^2$ and $\Phi \to \Phi/\Omega$ with $\hat{\Box}\Omega = 0$. After the replacement, it acts as $g_{\mu\nu} \to g_{\mu\nu}$, $\Phi \to \Phi$, $\varphi \to \varphi/\Omega^2$ with the condition $\hat{\Box}\Omega = 0$. In terms of $g_{\mu\nu}$, the condition $\hat{\Box}\Omega = 0$ is $\Box\Omega - \partial_\mu(\ln\varphi)\partial^\mu\Omega = 0$.

Composition law of restricted Weyl transformations in four dimensions

Consider the two consecutive restricted Weyl transformations in general d space-time dimensions

$$\begin{aligned} \tilde{g}_{\mu\nu} &= \Omega^2 g_{\mu\nu} \\ \tilde{\tilde{g}}_{\mu\nu} &= \tilde{\Omega}^2 \tilde{g}_{\mu\nu} = \tilde{\Omega}^2 \Omega^2 g_{\mu\nu} \end{aligned} \tag{1}$$

where $\Box \Omega = 0$ and $\tilde{\Box} \tilde{\Omega} = 0$.

In d=4 dimensions (and only d=4) one can show that $\Box(\tilde{\Omega}\Omega) = 0$ so that consecutive restricted Weyl transformations obey a composition law.

Restricted Weyl flat metrics

- If a metric $g_{\mu\nu}$ is Weyl flat (conformal to flat) so that $g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$, it is well known that the Weyl tensor is zero.
- However, if Ω obeys the condition $\Box\Omega = 0$, where $\Box$ here is the flat space d'Alembertian, then the Ricci scalar is also zero.
- We will call a Weyl flat metric which obeys $\Box\Omega = \eta^{\mu\nu}\partial_\mu\partial_\nu\Omega = 0$, a restricted Weyl flat metric.

Both the Weyl tensor and the Ricci scalar are zero in a restricted Weyl flat metric.

In GR, restricted Weyl flat spacetimes have traceless matter

Though $R = 0$ in a restricted Weyl flat spacetime, the Ricci tensor $R_{\mu\nu}$ is not zero (except for the trivial case when Ω is a constant).

In the context of General Relativity,

restricted Weyl flat metrics correspond to non-vacuum spacetimes ($T_{\mu\nu} \neq 0$) with traceless matter ($T = 0$).

Examples of restricted Weyl flat metrics

Examples of spacetimes that are restricted Weyl flat include

- $AdS_2 \times S^2$ (near-horizon limit of an extremal RN black hole)
- Radiation-dominated era of FLRW cosmology

The metric of $AdS_2 \times S^2$ is given by

$$ds^2 = -\frac{dt^2}{r^2} + \frac{dr^2}{r^2} + (d\theta^2 + \sin^2\theta d\phi^2) = \frac{1}{r^2}\left(-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right).$$

It is Weyl flat with Weyl (conformal) factor $\Omega = 1/r$. Since $\Box\Omega = 0$, this is a restricted Weyl flat metric.

The flat space FLRW metric can be expressed in the following equivalent forms

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) = \Omega^2(\tau)(-d\tau^2 + dx^2 + dy^2 + dz^2)$$

where $a(t)$ is the scale factor and $\Omega(\tau)$ is the conformal factor. In the radiation dominated era, $a(t) \propto t^{1/2}$ and $\Omega(\tau) \propto \tau$. Since $\Box\Omega = 0$, this is a restricted Weyl flat metric.

One can readily check that the above two cases have a Weyl tensor and Ricci scalar of zero with a non-zero Ricci tensor. This agrees with the fact that both spacetimes have traceless matter.

Transformation of kinetic term under Weyl transformation

We now evaluate how the kinetic term for the scalar field transforms under a Weyl transformation.

$$
\begin{aligned}
&\sqrt{|g|}\, g^{\mu\nu}\, \nabla_\mu\phi\, \nabla_\nu\phi \\
&\to \sqrt{|g|}\, g^{\mu\nu}\, \Omega^2\, \nabla_\mu(\phi\,\Omega^{-1})\, \nabla_\nu(\phi\,\Omega^{-1}) \\
&= \sqrt{|g|}\, \Big(\nabla_\mu\phi\, \nabla^\mu\phi - 2\,\phi\,\Omega^{-1}\, \nabla_\mu\phi\, \nabla^\mu\Omega + \phi^2\,\Omega^{-2}\, \nabla_\mu\Omega\, \nabla^\mu\Omega\Big) \\
&= \sqrt{|g|}\, \Big(\nabla_\mu\phi\, \nabla^\mu\phi - \nabla_\mu(\phi^2)\, \nabla^\mu(\ln\Omega) + \phi^2\,\Omega^{-2}\, \nabla_\mu\Omega\, \nabla^\mu\Omega\Big) \\
&= \sqrt{|g|}\, \Big(\nabla_\mu\phi\, \nabla^\mu\phi - \nabla_\mu(\phi^2\, \nabla^\mu(\ln\Omega)) + \phi^2\nabla_\mu\nabla^\mu(\ln\Omega) + \phi^2\,\Omega^{-2}\, \nabla_\mu\Omega\, \nabla^\mu\Omega \\
&= \sqrt{|g|}\, \Big(\nabla_\mu\phi\, \nabla^\mu\phi + \phi^2\,\Omega^{-1}\,\Box\,\Omega - \nabla_\mu(\phi^2\, \nabla^\mu(\ln\Omega))\Big)\,. \qquad (2)
\end{aligned}
$$