

# Black Holes in Higher Derivative Gravity

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# Quantum Context

One-loop quantum corrections to General relativity in 4-dimensional spacetime produce ultraviolet divergences of curvature-squared structure.

G. 't Hooft and M. Veltman, *Ann. Inst. Henri Poincaré* **20**, 69 (1974)

Inclusion of  $\int d^4x \sqrt{-g}(-\alpha C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \beta R^2)$  terms ab initio in the gravitational action leads to a renormalizable  $D = 4$  theory, but at the price of a loss of *unitarity* owing to the ghost modes arising from the  $\alpha C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$  term, where  $C_{\mu\nu\rho\sigma}$  is the Weyl tensor. K.S.S., *Phys. Rev.* **D16**, 953 (1977).

[In  $D = 4$  spacetime dimensions, this  $(\text{Weyl})^2$  term is equivalent, up to a topological total derivative *via* the Gauss-Bonnet theorem, to the combination  $-\alpha(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2)$ ].

Despite the apparent nonphysical behavior, quadratic-curvature gravities continue to be explored in a number of contexts:

- *Cosmology*: Starobinsky's model for inflation was based on a  $\int d^4x \sqrt{-g}(R + \beta R^2)$  model. A.A. Starobinsky 1980; Mukhanov & Chibisov 1981

This early model has been quoted as a good fit to CMB fluctuation data from the Planck satellite.

J. Martin, C. Ringeval and V. Vennin, arXiv:1303.3787

- The *asymptotic safety scenario* considers a possibility that there may be a non-Gaussian renormalization-group fixed point and associated flow trajectories on which the ghost states arising from the  $(\text{Weyl})^2$  term could be absent.

S. Weinberg 1976, M. Reuter 1996, M. Niedermaier 2009

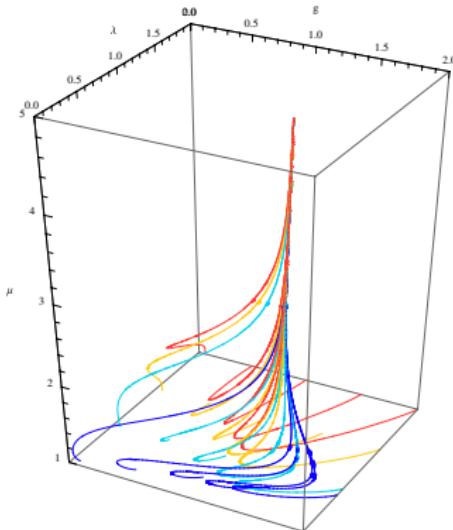
## Asymptotic Safety

In the gravitational context, a key question is whether there is a non-Gaussian fixed point with non-vanishing values for Newton's constant, and perhaps a cosmological constant as well. Another key question is whether the set of operators that are relevant as one approaches such a fixed point could avoid allowing coupling to the unwanted ghost states.

One point of view [Niedermaier 2010](#) focuses on the renormalizable set of operators in Einstein-plus-quadratic-curvature gravity and argues that the existence of a non-trivial fixed point can be determined just using perturbation theory.

A key feature of the Einstein-plus-quadratic-curvature system is that the  $(\text{Weyl})^2$  term is *asymptotically free* in the sense that for  $\alpha = 1/g_2^2$ , one finds  $g_2 \rightarrow 0$  at large momenta, as in Yang-Mills theory. [E.S. Fradkin & A.A. Tseytlin 1981, 1982; I.G. Avramidi & A.O. Barvinsky 1985](#) So the interactions permitting decay into the negative-energy states might turn off as one approaches the regime where this would be kinematically possible.

A claim is then that the Hessian matrix for the residues of the effective-theory propagator could be free of negative eigenvalues for appropriately chosen renormalization-group trajectories, so that instabilities arising from ghost states need not actually occur.



*Renormalization-group trajectories in coupling-constant space ending on a non-Gaussian fixed point with finite  $g_{\text{Newton}}$  and cosmological constant  $\Lambda$ . Niedermaier 2009*

## Classical gravity with higher derivatives

Instead of debating the various attitudes that can be taken towards the interpretation of quantum corrections in effective field theory, we shall simply adopt a point of view taking the higher-derivative terms and their consequences for gravitational solutions seriously in the classical effective action.

Consider the gravitational action

$$I = \int d^4x \sqrt{-g} (\gamma R - \alpha C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \beta R^2).$$

The field equations following from this higher-derivative action are

$$\begin{aligned} H_{\mu\nu} &= \gamma \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \frac{2}{3} (\alpha - 3\beta) \nabla_\mu \nabla_\nu R - 2\alpha \square R_{\mu\nu} \\ &\quad + \frac{1}{3} (\alpha + 6\beta) g_{\mu\nu} \square R - 4\alpha R^{\eta\lambda} R_{\mu\eta\nu\lambda} + 2 \left( \beta + \frac{2}{3}\alpha \right) R R_{\mu\nu} \\ &\quad + \frac{1}{2} g_{\mu\nu} \left( 2\alpha R^{\eta\lambda} R_{\eta\lambda} - \left( \beta + \frac{2}{3}\alpha \right) R^2 \right) = \frac{1}{2} T_{\mu\nu} \end{aligned}$$

## Separation of modes in the linearized theory

By expanding the action about flat space, one deduces the dynamical content of the linearized theory: *positive-energy* massless spin-two, *negative-energy* massive spin-two with mass  $m_2 = \gamma^{\frac{1}{2}}(2\alpha)^{-\frac{1}{2}}$  and *positive-energy* massive spin-zero with mass  $m_0 = \gamma^{\frac{1}{2}}[6\beta]^{-\frac{1}{2}}$ . K.S.S. 1978

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A simple model of what has happened can be made with a single scalar field and a higher-derivative action coupled to a source  $J$ :

$$I_{\text{hd}} = \int d^4x \left( -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\alpha\partial_\mu\phi\Box\partial^\mu\phi + J\phi \right)$$

Going over to momentum space  $k^\mu$ , one can solve for  $\phi$  and then separate the propagator into partial fractions:

$$\phi = \frac{J/\alpha}{k^2(k^2 + 1/\alpha)} = \frac{J}{k^2} - \frac{J}{k^2 + 1/\alpha}$$

similar to the structure found in quadratic gravity, but without the spin complications.

## Static and spherically symmetric solutions

Now consider what happens to spherically symmetric static solutions in the higher-curvature theory. One may choose to work in traditional Schwarzschild coordinates, for which the metric is given by

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

In the linearized theory, one then finds the general solution to the source-free field equations  $H_{\mu\nu}^L = 0$ , where  $C, C^{2,0}, C^{2,+}, C^{2,-}, C^{0,+}, C^{0,-}$  are six integration constants:

$$A(r) =$$

$$1 - \frac{C^{20}}{r} - C^{2+}\frac{e^{m_2 r}}{2r} - C^{2-}\frac{e^{-m_2 r}}{2r} + C^{0+}\frac{e^{m_0 r}}{r} + C^{0-}\frac{e^{-m_0 r}}{r} \\ + \frac{1}{2}C^{2+}m_2 e^{m_2 r} - \frac{1}{2}C^{2-}m_2 e^{-m_2 r} - C^{0+}m_0 e^{m_0 r} + C^{0-}m_0 e^{-m_0 r}$$

$$B(r) =$$

$$C + \frac{C^{20}}{r} + C^{2+}\frac{e^{m_2 r}}{r} + C^{2-}\frac{e^{-m_2 r}}{r} + C^{0+}\frac{e^{m_0 r}}{r} + C^{0-}\frac{e^{-m_0 r}}{r}$$

- As one might expect from the dynamics of the linearized theory, the general static, spherically symmetric solution is a combination of a massless Newtonian  $1/r$  potential plus rising and falling Yukawa potentials arising in both the spin-two and spin-zero sectors.
- When coupling to non-gravitational matter fields is made via standard  $h^{\mu\nu} T_{\mu\nu}$  minimal coupling, one gets values for the integration constants from the specific form of the source stress tensor. Requiring asymptotic flatness and coupling to a point-source positive-energy matter delta function

$T_{\mu\nu} = \delta_\mu^0 \delta_\nu^0 M \delta^3(\vec{x})$ , for example, one finds

$$A(r) = 1 + \frac{\kappa^2 M}{8\pi\gamma r} - \frac{\kappa^2 M(1+m_2 r)}{12\pi\gamma} \frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M(1+m_0 r)}{48\pi\gamma} \frac{e^{-m_0 r}}{r}$$

$$B(r) = 1 - \frac{\kappa^2 M}{8\pi\gamma r} + \frac{\kappa^2 M}{6\pi\gamma} \frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M}{24\pi\gamma} \frac{e^{-m_0 r}}{r}$$

with specific combinations of the Newtonian  $1/r$  and falling Yukawa potential corrections arising from the spin-two and spin-zero sectors.

Note that in the Einstein-plus-quadratic-curvature theory, there is *no Birkhoff theorem*. For example, in the linearized theory, coupling to the stress tensor for an extended source like a perfect fluid with pressure  $P$  constrained within a radius  $\ell$  by an elastic membrane,

$$T_{\mu\nu} = \text{diag}[P, [P - \frac{1}{2}\ell\delta(r-\ell)]r^2, [P - \frac{1}{2}\ell\delta(r-\ell)]r^2 \sin^2 \theta, 3M(4\pi\ell^3)^{-1}] ,$$

one finds for the external  $B(r)$  function

$$\begin{aligned} B(r) = & 1 - \frac{\kappa^2 M}{8\pi\gamma r} + \frac{\kappa^2 e^{-m_2 r}}{\gamma r} \left\{ \frac{M}{2\pi\ell^3} \left[ \frac{\ell \cosh(m_2\ell)}{m_2^2} - \frac{\sinh(m_2\ell)}{m_2^3} \right] \right. \\ & \left. - P \left[ \frac{\sinh(m_2\ell)}{m_2^3} - \frac{\ell \cosh(m_2\ell)}{m_2^2} + \frac{\ell^2 \sinh(m_2\ell)}{3m_2} \right] \right\} \\ & - \frac{\kappa^2 e^{-m_0 r}}{2\gamma r} \left\{ \frac{M}{4\pi\ell^3} \left[ \frac{\ell \cosh(m_0\ell)}{m_0^2} - \frac{\sinh(m_0\ell)}{m_0^3} \right] \right. \\ & \left. - P \left[ \frac{\sinh(m_0\ell)}{m_0^3} - \frac{\ell \cosh(m_0\ell)}{m_0^2} + \frac{\ell^2 \sinh(m_0\ell)}{3m_0} \right] \right\} \end{aligned}$$

which limits to the point-source result as  $\ell \rightarrow 0$ .

# Frobenius Asymptotic Analysis

Asymptotic analysis of the field equations near the origin leads to study of the *indicial equations* for behavior as  $r \rightarrow 0$ . K.S.S. 1978

Let

$$A(r) = a_s r^s + a_{s+1} r^{s+1} + a_{s+2} r^{s+2} + \dots$$

$$B(r) = b_t r^t + b_{t+1} r^{t+1} + b_{t+2} r^{t+2} + \dots$$

and analyze the conditions necessary for the lowest-order terms in  $r$  of the field equations  $H_{\mu\nu} = 0$  to be satisfied. This gives the following results, for the general  $\alpha, \beta$  higher derivative theory:

$$(s, t) = (1, -1) \quad \text{with 5 free parameters}$$

$$(s, t) = (0, 0) \quad \text{with 3 free parameters}$$

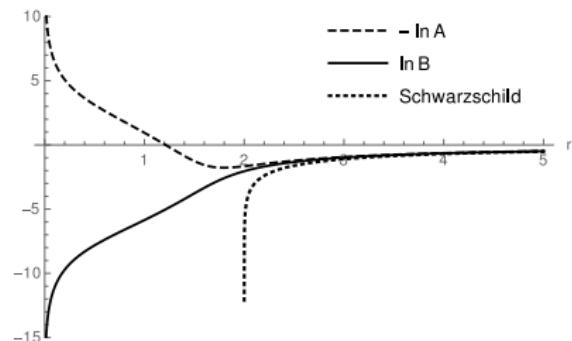
$$(s, t) = (2, 2) \quad \text{with 6 free parameters}$$

## (2,2) solutions without horizons

For asymptotically flat solutions with nonzero spin-two Yukawa coefficient  $C^{2-} \neq 0$ , one finds numerical solutions that can continue on in to mesh with the (2,2) family found by Frobenius asymptotic analysis around the origin. Such solutions have no horizon; numerical solutions have been found in the  $m_2 = m_0$  theory

B. Holdom, Phys. Rev. D66 (2002); hep-th 084010

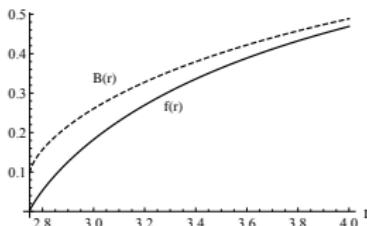
and in the  $R + C^2$  theory Lü, Perkins, Pope & K.S.S., 1508.00010



Horizonless solution in  $R - C^2$  theory, behaving as  $r^2$  in both  $A(r)$  and  $B(r)$  functions as  $r \rightarrow 0$ .

## Wormholes

Another solution type found numerically has the character of a “wormhole”. Such solutions can have either sign of  $M \sim -C^{20}$  and either sign of the falling Yukawa coefficient  $C^{2-}$ . As an example, one finds a solution with  $M < 0$  in the  $R - \alpha C^2$  theory



In this solution,  $f(r) = 1/A(r)$  reaches zero at a point where  $B(r) = a_0^2 > 0$ . Making a coordinate change  $r - r_0 = \frac{1}{4}\rho^2$ , one then has

$$ds^2 = -(a_0^2 + \frac{1}{4}B'(r_0)\rho^2)dt^2 + \frac{d\rho^2}{f'(r_0)} + (r_0^2 + \frac{1}{2}r_0\rho^2)d\Omega^2$$

which is  $\mathbb{Z}_2$  symmetric in  $\rho$  and can be interpreted as a “wormhole”, with the  $r < r_0$  region excluded from spacetime.

## Black hole solutions with horizons

If one assumes the existence of a horizon and assumes also asymptotic flatness at infinity, then a no-hair theorem for the trace of the field equations implies that the Ricci scalar must vanish:  $R = 0$ .

[W. Nelson, 1010.3986](#); [Lü, Perkins, Pope & K.S.S., 1508.00010](#) This significantly simplifies the analysis of the solutions. The field equations then become identical to those in the  $\beta = 0$  case, *i.e.* with just a (Weyl)<sup>2</sup> term and no  $R^2$  term in the action.

Counting parameters in an expansion around the horizon, subject to the  $R = 0$  condition, one finds just 3 free parameters. The Schwarzschild solution, satisfying  $R_{\mu\nu} = 0$ , is just such an asymptotically flat solution with a horizon, and it is characterized by two parameters: the mass  $M$  of the black hole, plus a trivial  $g_{00}$  normalization parameter at infinity. So in the higher-derivative theory, there is just one extra “non-Schwarzschild” parameter relevant to the family of asymptotically flat solutions with a horizon.

Now the question arises: what happens when one moves a finite distance away from a Schwarzschild solution in terms of the non-Schwarzschild parameter? In general, one expects the solution to violate asymptotic flatness at spatial infinity for small deviations from Schwarzschild. But what about increasing the non-Schwarzschild parameter further? Does the loss of asymptotic flatness persist, or does something else happen, with solutions arising that cannot be treated by a linearized analysis in deviation from Schwarzschild?

This can be answered numerically. In consequence of the trace no-hair theorem, the assumption of a horizon together with asymptotic flatness requires  $R = 0$  for the solution, so the calculations can effectively be done in the  $R - \alpha C^2$  theory with  $\beta = 0$ , in which the field equations, thankfully, can be reduced to a system of two second-order equations.

The study of non-Schwarzschild solutions is more easily carried out with a metric parametrization

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

i.e. by letting  $A(r) = 1/f(r)$ .

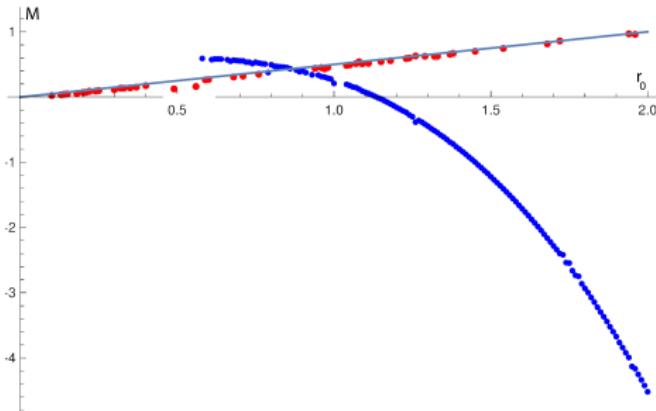
For  $B(r)$  vanishing linearly in  $r - r_0$  for some  $r_0$ , analysis of the field equations shows that one must then also have  $f(r)$  similarly linearly vanishing at  $r_0$ , and accordingly one has a horizon. One can thus make near-horizon expansions

$$\begin{aligned} B(r) &= c \left[ (r - r_0) + h_2 (r - r_0)^2 + h_3 (r - r_0)^3 + \dots \right] \\ f(r) &= f_1 (r - r_0) + f_2 (r - r_0)^2 + f_3 (r - r_0)^3 + \dots \end{aligned}$$

and the parameters  $h_i$  and  $f_i$  for  $i \geq 2$  can then be solved-for in terms of  $r_0$  and  $f_1$ . For the Schwarzschild solution, one has  $f_1 = 1/r_0$ , so it is convenient to parametrize the deviation from Schwarzschild using a non-Schwarzschild parameter  $\delta$  with

$$f_1 = \frac{1 + \delta}{r_0}.$$

The task then becomes one of finding values of  $\delta \neq 0$  for which the generic rising exponential behavior as  $r \rightarrow \infty$  is suppressed. What one finds is that there does indeed exist an asymptotically flat family of non-Schwarzschild black holes which crosses the Schwarzschild family at a special horizon radius  $r_0^{\text{Lich}}$ . For  $\alpha = \frac{1}{2}, \gamma = 1$ , one finds the following families of black holes:



*Black-hole masses as a function of horizon radius  $r_0$ , with a crossing point at  $r_0^{\text{Lich}} \simeq 0.876$ . The red family denotes Schwarzschild black holes and the blue family denotes non-Schwarzschild black holes.*

# The Lichnerowicz Operator

Now let us study in some more detail the point where the new black hole family crosses the classic Schwarzschild solution family. We can study solutions in the vicinity of the Schwarzschild family by looking at infinitesimal variations of the higher-derivative equations of motion around a Ricci-flat background. For the  $\delta R_{\mu\nu}$  variation of the Ricci tensor away from a background with  $R_{\mu\nu} = 0$ , one obtains

$$\begin{aligned}\gamma(\delta R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\delta R) + 2(\beta - \frac{1}{3}\alpha)(g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)\delta R \\ - 2\alpha\square(\delta R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\delta R) - 4\alpha R_{\mu\rho\nu\sigma}\delta R^{\rho\sigma} = 0.\end{aligned}$$

Restricting attention to asymptotically flat solutions with horizons, however, we know from the trace no-hair theorem that  $R = 0$  so  $\delta R = 0$  and the  $\delta R_{\mu\nu}$  equation simplifies, upon recalling that  $m_2^2 = \frac{\gamma}{2\alpha}$ , to

$$(\Delta_L + m_2^2) \delta R_{\mu\nu} = 0 ,$$

where the Lichnerowicz operator is given by

$$\Delta_L \delta R_{\mu\nu} \equiv -\square \delta R_{\mu\nu} - 2R_{\mu\rho\nu\sigma} \delta R^{\rho\sigma} .$$

Restricting attention to the  $m_2^2 > 0$  nontachyonic case, one sees that black hole solutions deviating from Schwarzschild must have a  $\lambda = -m_2^2$  negative Lichnerowicz eigenvalue for  $\delta R_{\mu\nu}$ .

## The Gross-Perry-Yaffe eigenvalue

In a study of the thermodynamic instability of the Euclideanised Schwarzschild solution in Einstein theory, Gross, Perry and Yaffe [Phys. Rev. D25 \(1982\), 330](#) found that there is just one normalisable negative-eigenvalue mode of the Lichnerowicz operator for deviations from the Schwarzschild solution. For a Schwarzschild solution of mass  $M$ , it is

$$\begin{aligned}\lambda &\simeq -0.192M^{-2} \\ \text{i.e. } m_2 M &\simeq 0.438 \simeq \sqrt{.192}\end{aligned}$$

- ▶ Comparing with the numerical results for the new black hole solutions of the higher-derivative gravity theory, this corresponds nicely with the point where the new black hole family crosses the Schwarzschild family.

## Gregory-Laflamme Instability

Analysis of the possibility of growing ( $e^{\nu t}$ ) perturbations can be approached using WKB methods [B.F. Schutz and C.M. Will, Ap.J. 291:L33 \(1985\)](#) or numerically. But in fact, the answer has been known for some time from the 5D string [R. Gregory and R. Laflamme, PRL 70 \(1993\) 2837](#). Considering perturbations about a 5D black string metric  $ds_{(5)}^2 = ds_{(4)}^2 + dz^2$

$$\begin{pmatrix} h_{\mu\nu}^{(4)} & h_{\mu z} \\ h_{z\nu} & h_{zz} \end{pmatrix} \quad (1)$$

where the  $z$  dependence is assumed to be of the form  $e^{ikz}$ , one finds that  $h_{\mu\nu}^{(4)}$  satisfies an equation of the same Lichnerowicz form  $(\Delta_L + k^2) h_{\mu\nu}^{(4)} = 0$  as for  $\delta R_{\mu\nu}$  [Y.S. Myung, Phys.Rev. D88 \(2013\)](#). This form is also found for perturbations about the Schwarzschild solution in dRGT nonlinear massive gravity.

The Gregory-Laflamme instability is an S-wave ( $\ell = 0$ ) spherically symmetric instability from the 4D perspective. In the higher-derivative theory, it exists for low-mass Schwarzschild black holes, but disappears for black hole masses  $M \geq M_{\max}$  where

$$\frac{m_2 M_{\max}}{M_{\text{Pl}}^2} = .438$$

This is precisely the crossing point between the family of new black holes and the Schwarzschild family.

Note that this monopole instability depends on the presence in the theory of the  $m_2$  massive spin-two mode.

# Thermodynamic Implications for Instability

The  $D = 4$  Wald entropy formula

$$S = -\frac{1}{8} \int_+ \sqrt{h} d\Sigma \epsilon^{ab} \epsilon^{cd} \frac{\partial L}{\partial R^{abcd}}$$

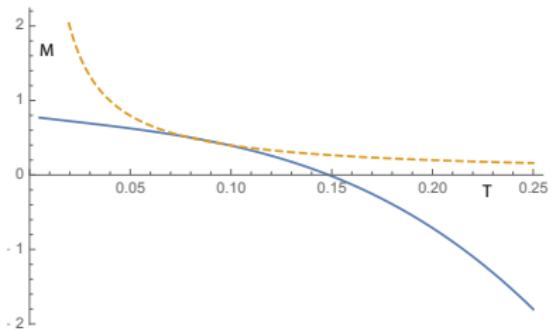
gives results that respect the first law of black-hole thermodynamics,  $dM = TdS$ .

For non-Schwarzschild black holes in  $D = 4$ , one obtains the following numerical relations between mass, temperature and entropy:

$$M_{\text{NSch}} \approx 0.168 + 0.131 S - 0.00749 S^2 - 0.000139 S^3 + \dots$$

$$T_{\text{NSch}} \approx 0.131 - 0.0151 S - 0.000428 S^2 + \dots$$

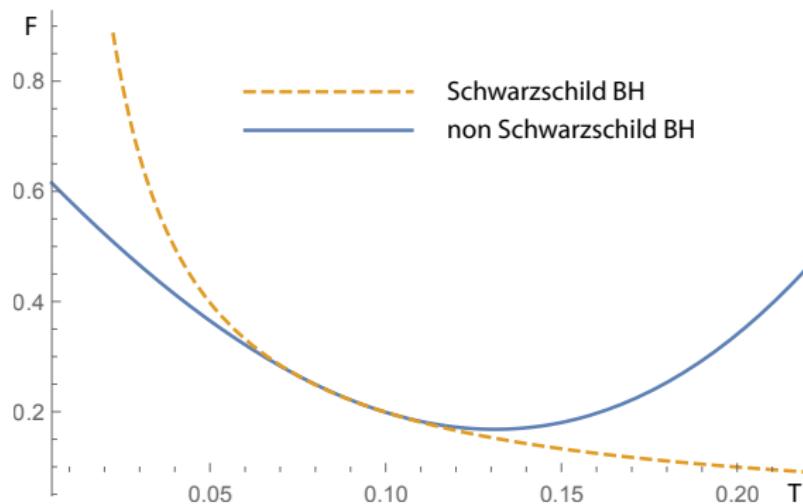
Recall that for Schwarzschild black holes, one has the classic mass-temperature relation  $M_{\text{Sch}} = \frac{1}{8\pi T}$ . Eliminating the entropy for the non-Schwarzschild black holes, one obtains the corresponding relations between black-hole mass and temperature for Schwarzschild and non-Schwarzschild black holes:



*Mass versus temperature relations for Schwarzschild (dashed red) and non-Schwarzschild (solid blue) black holes.*

# Free Energy

Consequently, for the free energy  $F = M - TS$ , one has the following situation, showing a switchover between the Schwarzschild and non-Schwarzschild solutions:



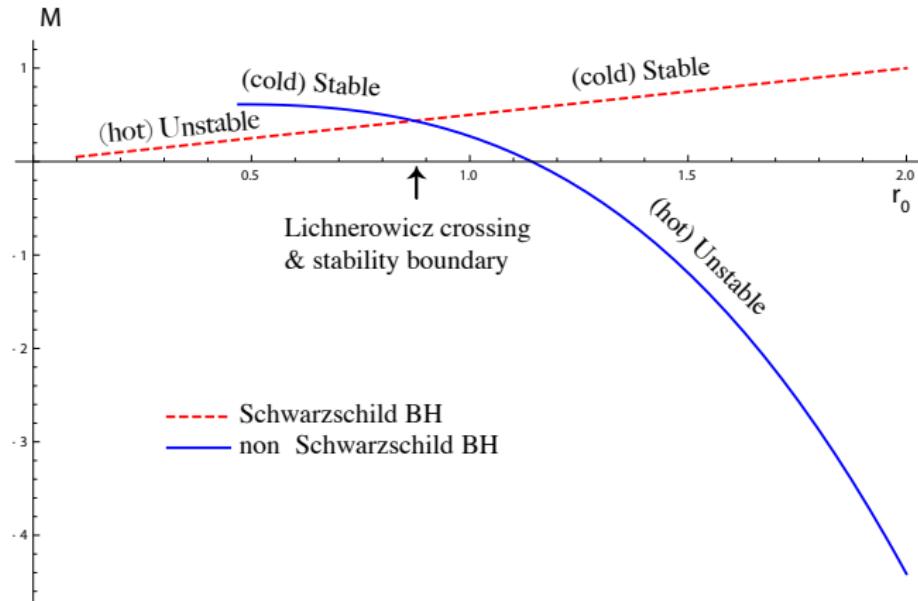
Free energy for Schwarzschild (dashed red) and non-Schwarzschild (blue) black holes.  
Lower free energy corresponds to greater stability.

## Thermodynamic versus dynamical instabilities

Gubser and Mitra proposed a relationship between thermodynamic and dynamical instabilities: time-dependent dynamical instability cannot occur without a corresponding thermodynamic instability in the related finite-temperature Euclidean theory. [JHEP 0108 \(2001\) 018](#)

This has been proved in the context of axisymmetric black holes in Einstein theory by Hollands and Wald [Commun.Math.Phys. 321 \(2013\) 629](#).

Assuming the same relation holds between dynamical and thermodynamic instabilities in the higher-derivative gravity theory, and taking into account the known Gregory-Laflamme instability for Schwarzschild black holes below the Lichnerowicz crossing point, one obtains a clear suggestion for the respective domains of stability and instability of the Schwarzschild and non-Schwarzschild black holes.



*Classical stability regimes. The dashed red line denotes Schwarzschild black holes and the solid blue line denotes non-Schwarzschild black holes.*

## Overview

- The  $I = \int d^4x \sqrt{-g}(\gamma R - \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2)$  theory has a richer static classical solution set than Einstein theory: in addition to the standard Einstein Ricci-flat static vacuum solutions, there are solutions without a horizon, wormhole solutions, and also a family of non-Schwarzschild black hole solutions.
- The Schwarzschild and non-Schwarzschild black-hole solution families cross at a mass  $M_{\text{Lich}}$  which is related to the Gross-Perry-Yaffe negative-eigenvalue mode  $\lambda$  of the Lichnerowicz operator by  $\lambda = -m_2^2 \simeq -0.192M_{\text{Lich}}^{-2}$ .
- The Schwarzschild solution family develops a Gregory-Laflamme S-wave instability for solutions with radii below a minimum radius  $r_0^{\text{Lich}} = 2M_{\text{Lich}}$  while thermodynamic analysis implies that the non-Schwarzschild black holes are stable for solutions with radii below  $r_0^{\text{Lich}}$ .
- For Starobinsky inflation,  $m_0^{\text{St}} \sim 10^{-6}M_{\text{Pl}}$ . Taking  $m_2 \sim m_0^{\text{St}}$  then gives  $M_{\text{Lich}} \simeq M_{\text{Pl}}^2/2m_2 \sim 10^{24}\text{GeV}/c^2 \sim 1\text{gr}$ .

## Some Related papers

K.S.S., Phys.Rev. D16 (1977) 953

K.S.S, Gen.Rel.Grav. 9 (1978) 353

H. Lü, A. Perkins, C.N. Pope & K.S.S.,  
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H. Lü, A. Perkins, C.N. Pope & K.S.S.,  
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H. Lü, A. Perkins, C.N. Pope & K.S.S., Phys. Rev. D96 (2017) 4,  
046006; 1704.05493

K. Kokkotas, R.A. Konoplya & A. Zhidenko, Phys. Rev. D96  
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A. Salvio, 1804.09944.