Soft-gluon and non-relativistic resummation in hadronic top and sparticle pair production

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Outline

- Pair production and resummation
- Top quarks near threshold in e^+e^- collisions: towards NNNLO
- Hadronic pair production (factorization of soft and Coulomb gluons, all-order colour structure, 2-loop anomalous dimension)
- Threshold expansion of the hadronic top-quark pair production cross section at NNLO
- Squark pair production: threshold resummation

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Massgebende Geschwindigkeit	52	km/h
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Pair production and resummation

$$e^+e^-,\ ij\,(q\bar{q}, qg, gg) \to HH' + X$$

near threshold: particle masses, spin, couplings

"Non-perturbative" despite small couplings:

- Strong Coulomb force: $g^2/v \sim 1$
- Sizeable decay widths of H, H': $\Gamma_H/m_H \sim g^2$
  Physical final states, "Dyson resummation", non-resonant backgrounds
- Soft gluon (photon) resummation, Sudakov logarithms: $g^2 \ln^2 v \sim 1$.

(Perturbative) Resummations in the frameworks of (P)NRQCD, SCET, unstable particle EFT.

Pair production and resummation

- $e^+e^- \longrightarrow t\bar{t}X$ – NNNLO in non-relativistic PT

  (MB, Y. Kiyo, K. Schuller, 0705.4518 [hep-ph], 0801.3464 [hep-ph] and in preparation; MB, Y. Kiyo and A.A. Penin, 0706.2733 [hep-ph]; MB, Y. Kiyo, 0804.4004 [hep-ph])

  Determined by strong interactions + Coulomb force

- $e^+e^- \longrightarrow W^+[\rightarrow u\bar{d}]W^-[\rightarrow \mu^-\bar{\nu}_\mu]X$ – NLO and dominant NNLO

  (MB, P. Falgari, C. Schwinn, A. Signer, G. Zanderighi, 0707.0773 [hep-ph]; S. Actis, MB, P. Falgari, C. Schwinn, 0807.0102 [hep-ph])

  Determined by weak interaction (enhanced Coulomb) + finite width ("four-fermion production") and non-resonant background

- $pp \rightarrow t\bar{t}X, \tilde{g}\tilde{g}X, \tilde{q}\tilde{q}X$ – NLO+NNLL (top) and NLL (SUSY)

  (MB, P. Falgari, C. Schwinn, 0907.1443 [hep-ph] and in preparation; MB, M. Czakon, P. Falgari, A. Mitov, C. Schwinn, 0911.5166 [hep-ph])

  Soft gluon resummation + Coulomb force, maybe finite width

Top quarks near threshold in e^+e^- collisions

In the absence of electroweak corrections:

$$(q_\mu q_\nu - q^2 g_{\mu\nu})\,\Pi(q^2) = i\int d^4x\, e^{iq\cdot x}\,\langle 0|T(j_\mu(x)j_\nu(0))|0\rangle, \qquad j^\mu(x) = [\bar{t}\gamma^\mu t](x)$$

$$R \equiv \frac{\sigma_{t\bar{t}X}}{\sigma_0} = 12\pi e_t^2\,\mathrm{Im}\,\Pi(s)$$

Relevant scales near threshold: $m_t \approx 175\,\mathrm{GeV}$, $m_t\alpha_s \approx 30\,\mathrm{GeV}$ and the ultrasoft scale $m_t\alpha_s^2 \approx 2\,\mathrm{GeV}$.

$$\mathcal{L}_{\mathrm{QCD}}\,[Q(h,s,p),\ g(h,s,p,us)] \qquad \mu > m_t$$

$$\downarrow$$

$$\mathcal{L}_{\mathrm{PNRQCD}}\,[Q(p),\ g(us)] \qquad \mu < m_t v$$

At NNLO: large uncertainty [up $\pm 25\%$] in the cross section in the resonance peak region (MB, Signer, Smrinov; Hoang, Teubner; Melnikov,Yelkovsky; Yakovlev; Nagano et al.; Penin, Pivovarov; 1998/99). Maybe less after $\log(v)$ resummation [$\pm 3\%$] (Hoang et al., 2002)

The ultrasoft scale appears explicitly only at NNNLO. A complete calculation of the (non-logarithmic) NNNLO correction is therefore needed.

Matching/effective Lagrangian at NNNLO

Matching of currents and interactions (potentials and ultrasoft)

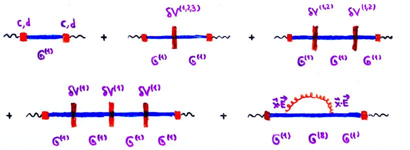
$$j^i = c_v\,\psi^\dagger\sigma^i\chi + \frac{d_v}{6m_t^2}\psi^\dagger\sigma^i\,\mathbf{D}^2\chi + \ldots$$

$$\mathcal{L}_{\rm QCD} \to \mathcal{L}_{\rm PNRQCD}$$

$$\begin{aligned}\mathcal{L}_{\rm PNRQCD} \quad = \quad & \psi^\dagger\Big(iD_0 + i\frac{\Gamma_t}{2} + \frac{\boldsymbol{\partial}^2}{2m} + \frac{\boldsymbol{\partial}^4}{8m^3}\Big)\psi + \chi^\dagger\Big(iD_0 - i\frac{\Gamma_t}{2} - \frac{\boldsymbol{\partial}^2}{2m} - \frac{\boldsymbol{\partial}^4}{8m^3}\Big)\chi \\ & + \int d^{d-1}\mathbf{r}\,\Big[\psi^\dagger\psi\Big](x+\mathbf{r})\,\Big(-\frac{\alpha_s C_F}{r} + \delta V(r,\boldsymbol{\partial})\Big)\,\Big[\chi^\dagger\chi\Big](x) \\ & - g_s\psi^\dagger(x)\boldsymbol{x}\boldsymbol{E}(t,\mathbf{0})\psi(x) - g_s\chi^\dagger(x)\boldsymbol{x}\boldsymbol{E}(t,\mathbf{0})\chi(x)\end{aligned}$$

- Almost everything needed at NNNLO known ... (Manohar, 1997; Wüster, 2003, Luke, Savage, 1997; Marquard et al., 2006, 2009; MB, Signer, Smirnov, 1999; Kniehl et al., 2001, 2002; Wüster, 2003).
- ... including the three-loop correction to the Coulomb potential since very recently (Anzai, Kiyo, Sumino; Sminrov, Smirnov, Steinhauser, 2009).
- Only missing piece: non-fermionic and singlet pieces of three-loop $c_v^{(3)}$.

Third-order PNRQCD correlation function with resummed Coulomb propagators:



$$G_c^{(1,8)}(\boldsymbol{r},\boldsymbol{r}',E) = \frac{my}{2\pi} e^{-y(r+r')} \sum_{l=0}^{\infty} (2l+1)(2yr)^l (2yr')^l P_l\left(\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{rr'}\right) \sum_{s=0}^{\infty} \frac{s! L_s^{(2l+1)}(2yr) L_s^{(2l+1)}(2yr')}{(s+2l+1)!(s+l+1-\lambda)}$$

$$y = \sqrt{-m(E+i\epsilon)},\ \lambda = \frac{m\alpha_s}{2y} \times \{C_F\ (\text{singlet});\ C_F - C_A/2\ (\text{octet})\}$$

The ultrasoft contribution is $(D = d-1)$

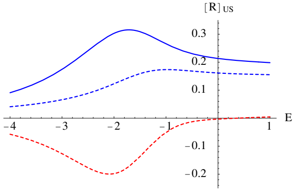
$$\delta G_{\rm us} = (-i)(ig_s)^2 C_F \int \frac{d^d k}{(2\pi)^d} \frac{-i}{k^2}\left(\frac{k^i k^j}{k_0^2} - \delta^{ij}\right) \int \prod_{n=1}^{6} \frac{d^D p_n}{(2\pi)^D}\, i\tilde{G}^{(1)}(p_1,p_2;E)\, i\tilde{G}^{(8)}(p_3,p_4;E+k^0)\, i\tilde{G}^{(1)}(p_5,p_6;E)$$

$$\times\, i\left[\frac{2p_3^i}{m_t}(2\pi)^D \delta^{(D)}(p_3-p_2) + (ig_s)^2 \frac{C_A}{2}\frac{2(p_2-p_3)^i}{(p_2-p_3)^4}\right] i\left[-\frac{2p_4^j}{m_t}(2\pi)^D \delta^{(D)}(p_4-p_5) + (ig_s)^2 \frac{C_A}{2}\frac{2(p_4-p_5)^j}{(p_4-p_5)^4}\right]$$

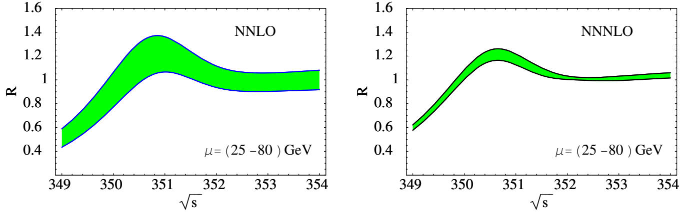
Difficulty is the extraction of the divergence in dim. reg. to cancel the poles.

Results

Non-logarithmic ultrasoft correction (solid blue) is very large [up to 30%]



Scale dependence at NNNLO versus NNLO:



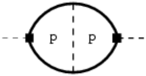
(Missing non-fermionic c_3 terms shift the NNNLO result, but do not change the scale uncertainty. Preliminary results (Steinhauser et al., RADCOR09) indicate that the shift may be large.)

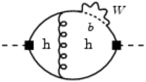
“Electroweak effects”

The QCD-only result usually discussed is far from reality

- Significant “non-resonant” background from off-shell top decay (unless tight invariant mass cuts are applied), not described by NRQCD
- Initial state radiation
- Finite-width divergences (overall divergence, already at NNLO):

$$[\delta^{us} G(E)]_{\text{overall}} \propto \frac{\alpha_s}{\epsilon} \cdot E$$





With $E = \sqrt{s} - 2m_t + i\Gamma$ the divergence survives in the imaginary part, and is

$$\text{Im}\,[\delta^{us} G(E)]_{\text{overall}} \propto m_t \times \frac{\alpha_s \alpha_{ew}}{\epsilon}$$

The systematic study of a realistic cross section prediction has only just started (Hoang, Reisser, 2004; Actis et al., 2008; Hoang, Reisser, Ruiz-Femenia (in preparation); MB, Jantzen, Ruiz-Femenia (work in progress))

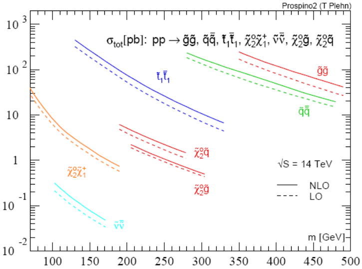
Soft gluon resummation in hadronic production (MB, Falgari, Schwinn)

Partonic cross sections $i + j \to HH' + X$ contain

$$\left[\alpha_s \ln^2 \beta\right]^n, \qquad \beta^2 = 1 - z = 1 - (m_H + m_H')^2/\hat{s}$$

which should be resummed, if the total hadronic cross section is dominated by the partonic threshold. ($t\bar{t}$: Catani et al., 1996; Bonciani et al, 1998; Kidonakis et al, 2001; ...; Moch, Uwer, 2008; ...; Hagiwara et.al, 2008; Kiyo et al., 2008; Sparticle pairs: Kulesza, Motyka, 2008; Langenfeld, Moch, 2009; Beenakker et al; 2009)

- Not clear why the partonic threshold should be relevant at LHC energies for $t\bar{t}$ despite PDF fall-off. Even for sparticles really only for $m_H \geq 3$ TeV.
- Main reason is probably empirical observation of improved scale dependence.
- Anyway an interesting problem due to colour exchange and interplay with Coulomb singularities $(\alpha_s/\beta)^n$.



Soft gluon resummation for the total cross section

Formalism for $2 \rightarrow 2$ scattering processes with massless coloured particles (s, t-dependent anomalous dimensions) was set up by (Kidonakis, Sterman; 1997).

But threshold (Sudakov) resummation for total $t\bar{t}$ cross section (or any heavy coloured particle pairs) was in fact never performed accurately.

In the following aim at

- Treatment of colour exchange to all orders
  Separate short-distance coefficients in each independent colour channel
- Proof of factorization of soft gluons from Coulomb exchange and combined resummation of $\alpha_s \ln^2 \beta$ and α_s/β
- Threshold resummation for heavy particles with sizeable decay width
  (not discussed in this talk)

Solution of RGE equation done in momentum space using the formalism developed by (Becher, Neubert, 2007) for Drell-Yan production.

Systematics and the factorization formula

Expansion of the partonic cross section

$$\hat{\sigma}(\beta) \quad = \quad \hat{\sigma}^{(0)} \sum_{k=0} \left(\frac{\alpha_s}{\beta}\right)^k \exp\Big[\underbrace{\ln\beta\, g_0(\alpha_s \ln\beta)}_{\text{(LL)}} + \underbrace{g_1(\alpha_s \ln\beta)}_{\text{(NLL)}} + \underbrace{\alpha_s g_2(\alpha_s \ln\beta)}_{\text{(NNLL)}} + \ldots\Big]$$

$$\times \left\{1 \text{ (LL,NLL)};\ \alpha_s,\ \beta \text{ (NNLL)};\ \alpha_s^2,\ \alpha_s\beta,\ \beta^2 \text{ (NNNLL)};\ \ldots\right\},$$

In fixed order [Note: NNLL can be α_s/β[Coulomb] $\times$ β[sub-leading soft] $\times$ $\alpha_s \ln^2\beta$ – beyond the standard soft gluon approximation!]

LL	$\alpha_s \left\{\frac{1}{\beta}, \ln^2\beta\right\};\ \alpha_s^2 \left\{\frac{1}{\beta^2}, \frac{\ln^2\beta}{\beta}, \ln^4\beta\right\};\ldots,$
NLL	$\alpha_s \ln\beta;\ \alpha_s^2 \left\{\frac{\ln\beta}{\beta}, \ln^3\beta\right\};\ldots,$
NNLL	$\alpha_s \left\{1, \beta\times\ln^{2,1}\beta\right\};\ \alpha_s^2 \left\{\frac{1}{\beta}, \ln^{2,1}\beta, \beta\times\ln^{4,3}\beta\right\};\ldots,$

Factorization formula [sum over a includes sub-leading powers in β.]

$$\hat{\sigma}(\beta,\mu) = \sum_a \sum_{i,i'} H^a_{ii'}(M,\mu) \int d\omega \sum_{R_\alpha} J^a_{R_\alpha}(E - \frac{\omega}{2})\, W^{a,R_\alpha}_{ii'}(\omega,\mu).$$

Hard amplitude

Hard sub-process always $2 \to 2$. Operators with more fields are power-suppressed in $(1-z)$.

$$\mathcal{A}(pp' \to HH'X) = \sum_{\ell} C^{(\ell)}_{\{a;\alpha\}}(\mu)\, \langle HH'X|\mathcal{O}^{(\ell)}_{\{a;\alpha\}}(\mu)|pp'\rangle_{\text{EFT}}$$

$$\mathcal{O}^{(\ell)}_{\{a;\alpha\}}(\mu) = \left[\phi_{c;a_1\alpha_1}\phi_{\bar{c};a_2\alpha_2}\psi^{\dagger}_{a_3\alpha_3}\psi'^{\dagger}_{a_4\alpha_4}\right](\mu).$$

- Collinear fields (initial state)

$$\Phi_c \in \{W_c^{\dagger}\xi_c,\ \bar{\xi}_c W_c,\ \mathcal{A}_c^{\mu\perp} = g_s^{-1}(W_c^{\dagger}[iD_c^{\mu\perp}W_c])\}$$

  and non-relativistic scalar, spinor, vector, ... fields (final state)
- Not necessary to perform a spin decomposition of the operators, since leading anomalous dimensions are spin independent.
- Require a suitable colour basis: $\mathcal{O}^{(\ell,i)}_{\{a;\alpha\}} = c^{(i)}_{\{a\}}\mathcal{O}^{(\ell)}_{\{a;\alpha\}}$
- Soft gluons interact with everything, and "in between" Coulomb exchange. Factorization?

Soft-gluon decoupling

From the initial state:

$$\mathcal{L}_c = \bar{\xi}_c \left(in \cdot D + i \not{D}_{\perp c} \frac{1}{i\bar{n} \cdot D_c} i\not{D}_{\perp c} \right) \frac{\not{\bar{n}}}{2} \xi_c - \frac{1}{2} \,\mathrm{tr} \left(F_c^{\mu\nu} F^c_{\mu\nu} \right)$$

by the SCET field redefinitions (Bauer, Pirjol, Stewart, 2001) $\xi_c(x) = S_n^{(3)}(x_-)\, \xi_c^{(0)}(x)$, $A^A_{c\mu}(x) = S_n^{(8)}(x_-)\, A^{A(0)}_{c\mu}(x)$, such that $n \cdot D \to n \cdot D_c$.

From the final state:

$$\mathcal{L}_{\mathrm{PNRQCD}} = \psi^\dagger \left(iD_s^0 + \frac{\vec{\partial}^2}{2m_H} + \frac{i\Gamma_H}{2} \right) \psi + \psi'^\dagger \left(iD_s^0 + \frac{\vec{\partial}^2}{2m_{H'}} + \frac{i\Gamma_{H'}}{2} \right) \psi' + \int d^3\vec{r} \left[\psi^\dagger \mathbf{T}^{(R)a} \psi \right](\vec{r}) \left(\frac{\alpha_s}{r} \right) \left[\psi'^\dagger \mathbf{T}^{(R')a} \psi' \right](0)$$

by the PNRQCD field redefinition $\psi_a(x) = S_v^{(R)}(x^0)_{ab}\, \psi_b^{(0)}(x)$, such that $D_s^0 \to \partial^0$.

[S_v drops out from the Coulomb interaction, since $S_v^{(R)\dagger} \mathbf{T}^{(R)a} S_v^{(R)\dagger} = [S_{\mathrm{ad}}^T]^{ab}\, \mathbf{T}^{(R)b}$ in any rep R; S_{ad} is real and independent of $\vec{r}$.]

Proves decoupling of soft gluon and Coulomb resummation, since soft gluons disappear from the leading-order Lagrangians for the other fields. Sub-leading interactions can be treated as perturbations in β.

Leading term

$$\hat{\sigma}(\beta,\mu) = \sum_{i,i'} H_{ii'}(M,\mu) \int d\omega \sum_{R_\alpha} J_{R_\alpha}(E-\frac{\omega}{2})\, W_{ii'}^{R_\alpha}(\omega,\mu).$$

- $H_{ii'} = C_i C_{i'}^*$ – colour-unaveraged partonic (hard) cross section directly at threshold.
  C_i related to matching coefficient of $\mathcal{O}^{(\ell)}_{\{a;\alpha\}}(\mu) = \left[\phi_{c;a_1}\phi_{\bar{c};a_2}\psi^\dagger_{a_3}\psi'^\dagger_{a_4}\right](\mu)$
  [No spin-separation needed at NNLL.]
- J_{R_α} – sums Coulomb-exchange to all orders for HH' in irreducible product rep R_α. Related to a correlation function of non-relativistic fields in PNRQCD.

$$J_{R_\alpha}(E) \propto \text{Im}\left[-\frac{(2m_{\text{red}})^2}{4\pi}\left\{\sqrt{-\frac{E}{2m_{\text{red}}}} + \alpha_s(-D_{R_\alpha})\left[\frac{1}{2}\ln\left(-\frac{8\,m_{\text{red}}E}{\mu^2}\right) + \psi\left(1-\frac{\alpha_s(-D_{R_\alpha})}{2\sqrt{-E/(2m_{\text{red}})}}\right)\right]\right\}\right]$$

- $W_{ii'}^{R_\alpha}$ – generalized soft function (one for each rep R_α). Fourier transform of
  $\hat{W}_{ii'}^{R_\alpha}(z,\mu) = P^{R_\alpha}_{\{k\}} c^{(i)}_{\{a\}}\,\hat{W}^{\{k\}}_{\{ab\}}(z,\mu) c^{(i')*}_{\{b\}}$. $c^{(i)}_{\{a\}}$ - colour basis element. $\{a\} = a_1a_2a_3a_4$.

$$\hat{W}^{\{k\}}_{\{ab\}}(z,\mu) = \langle 0|\overline{\text{T}}[S_{v,b_4k_2}S_{v,b_3k_1}S^\dagger_{\bar{n},jb_2}S^\dagger_{n,ib_1}](z)\text{T}[S_{n,a_1i}S_{\bar{n},a_2j}S^\dagger_{v,k_3a_3}S^\dagger_{v,k_4a_4}](0)|0\rangle\,,$$

All-order diagonal colour basis for $W_{ii'}^{R_\alpha}$

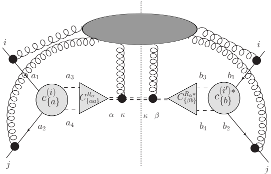
- Decompose initial and final state rep product into irreducible reps: $r \otimes r' = \sum_\alpha r_\alpha$, and $R \otimes R' = \sum_\beta R_\beta$.

  Number of basis elements ($i = 1 \ldots n$) = number of pairs $P_i = (r_\alpha, R_\beta)$ of equivalent representations r_α and R_β, e.g. for $8 \otimes 8 \to 8 \otimes 8$:
  $P_i \in \{(1,1),\ (8_S, 8_S),\ (8_A, 8_S),\ (8_A, 8_A),\ (8_S, 8_A),\ \ (10,10),\ (\overline{10}, \overline{10}),\ (27,27)\}.$

- $W_{ii'}^{R_\alpha}$ is diagonal to all orders in the orthonormal basis:

$$c_{\{a\}}^{(i)} = \frac{1}{\sqrt{\dim(r_\alpha)}}\, C_{\alpha a_1 a_2}^{r_\alpha} C_{\alpha a_3 a_4}^{R_\beta *}$$

  [$C_{\alpha a_1 a_2}^{r_\alpha}$ Clebsch-Gordan coefficient]

$W_{ii'}^{R_\alpha} =$ 

- Properties of CGCs imply that element $W_{ii'}^{R_\alpha}$ is non-zero only, if in $P_i = (r_\alpha, R_\beta)$ and $P_{i'} = (r_{\alpha'}, R_{\beta'})$ the final state reps R_β, $R_{\beta'}$ are *identical* to R_α, and r_α, r'_α are *equivalent* to R_α. This leaves only $gg[8_S]$ coupling to $HH'[8_A]$ (and interchanged), but this vanishes by Bose symmetry for the Wilson line operators.

[Explicit construction of the colour bases for all interesting cases, see MB, P. Falgari, C. Schwinn, 0907.1443 [hep-ph] appendix.]

Two-loop soft anomalous dimension at threshold

- NNLL soft-gluon resummation needs the 3-loop cusp anomalous dimension and 2-loop soft anomalous dimension in

$$\frac{d}{d\ln\mu}\hat{W}_i^{R_\alpha}(L) = \left((\Gamma^r_{\rm cusp}+\Gamma^{r'}_{\rm cusp})L - 2\gamma^{R_\alpha}_{W,i}\right)\hat{W}_i^{R_\alpha}(L),$$

[We use a generalization of the momentum space formalism developed for Drell-Yan production by (Becher, Neubert, Xu, 2007).]

- At threshold the Wilson lines of the two heavy particle combine to a sum of single particle soft functions $C^{R_\alpha}_{\alpha a_1 a_2} S^{(R)}_{v,a_1 b_1} S^{(R')}_{v,a_2 b_2} = S^{(R_\alpha)}_{v,\alpha\beta} C^{R_\alpha}_{\beta,b_1 b_2}$. The soft function is the "square" of the soft function for the amplitude discussed recently by (Becher, Neubert, 2009; Mitov et al, 2009).

  For a $2 \to 1$ process the three-particle correlations $f^{abc}T^aT^bT^c$ vanish by colour conservation. The anomalous dimension is a sum over single particle terms: $\gamma^{R_\alpha}_{W,i} = \gamma^{R_\alpha}_{H,s} + \gamma^r_s + \gamma^{r'}_s$. The 2-loop anomalous dimension satisfies Casimir scaling and can be extracted from (Becher, Neubert, 2009; Korchesmky, Radyushkin, 1992; Kidonakis, 2009).

- In Mellin space resummation formalism:

$$D^{(1)R_\alpha}_{HH'} = -C_{R_\alpha}C_A\left(\frac{460}{9} - \frac{4\pi^2}{3} + 8\zeta_3\right) + \frac{176}{9}C_{R_\alpha}T_F n_f.$$

[Confirmed independently by Czakon et al., 2009]

Resummation

Resummation of soft gluon threshold logs via renormalization group equations.
Since the soft functions are diagonal (to all orders) in the chosen colour basis, can use the formalism developed for Drell-Yan (and Higgs) production by (Becher, Neubert, Xu. 2007)

$$\hat{\sigma}^{\rm res}_{pp'}(\hat{s},\mu_f) = \frac{\hat{\sigma}^{(0)}_{pp'}(\hat{s},\mu_f)}{\beta} \sum_{i,R_\alpha} h_i(M,\mu_h)\;\; U_i^{R_\alpha}(M,\mu_h,\mu_s,\mu_f) \left(\frac{2M}{\mu_s}\right)^{-2\eta}$$
$$\times\, \tilde{s}_i^{R_\alpha}(\partial_\eta,\mu_s) \frac{e^{-2\gamma_E\eta}}{\Gamma(2\eta)} \int_0^\infty d\omega\; \frac{J_{R_\alpha}(E-\frac{\omega}{2})}{\omega} \left(\frac{\omega}{\mu_s}\right)^{2\eta} \qquad (1)$$

$$U_i^{R_\alpha} = \exp[4S(\mu_h,\mu_s) - 2a_i^V(\mu_h,\mu_s) + 2a^{\phi,r}(\mu_s,\mu_f) + 2a^{\phi,r'}(\mu_s,\mu_f)] \left(\frac{4M^2}{\mu_h^2}\right)^{-2a_\Gamma(\mu_h,\mu_s)}$$

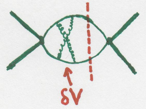
$$\tilde{s}_i^{R_\alpha}(\rho,\mu) = \int_{0_-}^\infty d\omega e^{-s\omega}\; \overline{W}_i^{R_\alpha}(\omega,\mu)$$

[$a(\mu_1,\mu_2)$ denote integrated anomalous dimensions]

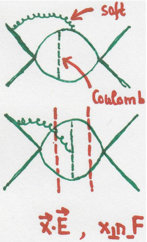
Expansion of the resummation formula allows one to compute the Coulomb and logarithmically enhanced terms at $\mathcal{O}(\alpha_s^2)$ from soft gluon resummation. Add non-relativistic logs to obtain all.

NNLL at $\mathcal{O}(\alpha_s^2)$ for $t\bar{t}$ (MB, Czakon, Falgari, Mitov, Schwinn; 2009)

2-loop anomalous dimension not enough for NNLL resummation.
Consider $\alpha_s^2 \ln \beta$ terms for $t\bar{t}$:



- $\hat{\sigma}^{(0)} \times \alpha_s^2 \ln \beta$ from singular heavy-quark potentials ($1/r^2$ etc.)
  Can be obtained from $e^+e^- \longrightarrow t\bar{t}X$ calculation [MB, Signer, Smirnov, 1999] (+ colour, spin adjustment)
  Diagonal in the singlet-octet basis.



- $\hat{\sigma}^{(0)} \times \alpha_s^2 \ln^{2,1} \beta$ from sub-leading (non-eikonal) β-suppressed soft interactions in SCET and NRQCD. Implies new soft functions with operator insertions between Wilson lines.
  Off-diagonal in the singlet-octet basis.
  Corresponds to three-particle correlations.
  Vanish for the total cross section to all orders in α_s (Lorentz-invariance + scaling), but not for the amplitude.
- Above + diagonal leading soft function consistent with (Ferroglia et al., 2009). No extra terms for $t\bar{t}$ total cross section, though.

[NNLL resummation for the total cross section requires non-relativistic factorization. 2-loop anomalous dimension from (Ferroglia et al., 2009) applies to relativistic production.]

Expansion of the partonic top-pair production section

$$\hat{\sigma}(\beta)^{(2)} = \hat{\sigma}(\beta)^{(0)} \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{c_{20}}{\beta^2} + \frac{1}{\beta}\left\{c_{12}\ln^2\beta + c_{11}\ln\beta + c_{10}\right\} + c_{04}\ln^4\beta + c_{03}\ln^3\beta + c_{02}\ln^2\beta + c_{01}\ln^1\beta\right]$$

- c_{20} – double Coulomb
  c_{12} – Coulomb$\times$ 1-loop soft
  c_{11} – Coulomb$\times$ soft, running Coulomb potential
  c_{10} – ..., 1-loop Coulomb potential, 1-loop hard matching coefficient [extracted from Czakon, Mitov, 2008]
  c_{04}, c_{03}, c_{02} – 2-loop soft c_{01} – ..., NRQCD logs from singular potentials

Example: $gg \rightarrow [t\bar{t}]_8 X$

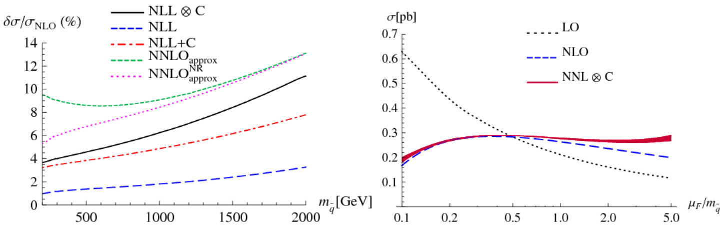
$$\hat{\sigma}(\beta)^{(2)}_{gg\rightarrow[t\bar{t}]_8X} = \hat{\sigma}(\beta)^{(0)} \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{\pi^4}{27\beta^2} + \frac{\pi^2}{\beta}\left\{-32\ln^2\beta + \left(\frac{118}{9} - 32\ln 2\right)\ln\beta + \frac{46}{9}\ln 2 - \frac{5\pi^2}{18} - \frac{127}{27}\right\}\right.$$
$$+ 4608\log^4\beta + \left(27648\ln 2 - \frac{65152}{3}\right)\log^3\beta + \left(59904\ln 2^2 - 96576\ln 2 - 3088\pi^2 + \frac{204944}{3}\right)\log^2\beta$$
$$\left. + \left(55296\ln 2^3 - 137952\ln 2^2 - 9264\pi^2\ln 2 + 202352\ln 2 + 33120\zeta(3) + \frac{65908\pi^2}{9} - \frac{1244776}{9}\right)\log\beta\right]$$

Formula available for arbitrary heavy particles (e.g. sparticles).

Results at NLL for $pp \to$ squark+antisquark $+ X$ at $\sqrt{s} = 14$ TeV

NLL = Tree C and W, 1-loop anomalous dim. + LO Coulomb Green function + matching to NLO fixed order ((Beenakker et al, 1997; Prospino code), fitting functions from (Langenfeld, Moch, 2009))

Size of corrections and scale dependence at LO, NLO, NLL



($\sqrt{s} = 14$ TeV; $m_{\tilde{g}}/m_{\tilde{q}} = 1.25$; (right) $m_{\tilde{q}} = 1$ TeV (right); Red band: variation of the soft scale.)

Resummation is a few to 10% effect at the natural scale, but shows less scale variation.

Conclusion

1) The NNNLO QCD prediction for the $e^+e^- \to t\bar{t}X$ cross section near threshold is (nearly) complete.

$$\frac{\delta\sigma}{\sigma} \approx 5\%$$

Surprise due to $c_v^{(3)}$ possible.

2) Factorization of soft and Coulomb gluon summation proved (SCET $\times$ NRQCD)
Leading soft function diagonal to all orders in a simple colour basis
2-loop anomalous dimension at threshold determined.

3) Soft gluon resummation at NNLL for total cross section possible.
For complete NNLL combine with non-relativstic log resummation.

4) Top pair cross section at threshold known at $\mathcal{O}(\alpha_s^2)$ at threshold up to the constant term.
For squark-antisquark production find (once again) improvement of scale dependence at NLL. Few percent corrections beyond NLO.