Scanning the Earth with neutrino oscillations

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Zurich, January 2010

1. Neutrinos are unique particles that can propagate inside the Earth
2. Matter changes oscillation properties of neutrinos: amplitudes and oscillation lengths (Wolfenstein, Wyler)
3. The density profile of the Earth's core is unknown
4. Semi-analytic expressions for the Neutrino propagation in matter with non constant density
5. Results and Conclusions

$$S_{\nu_i \to \nu_j} \equiv S_{ji}$$
$$S_{\nu_i \to \nu_\alpha} = U_{\alpha i} S_{ji}$$
$$S_{\nu_\alpha \to \nu_\beta} = U_{\beta j} S_{ji} U^\dagger_{i\alpha}$$

$$i \frac{d\ S(x, x_0)}{d\ x} = H(x) S(x, x_0)$$

$$H(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Delta m^2_\odot}{2E} & 0 \\ 0 & 0 & \frac{\Delta m^2_{atm}}{2E} \end{pmatrix} + U^\dagger \begin{pmatrix} \mathrm{V(x)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U$$

$$S \;=\; Te^{-i \int_{x_0}^{x_f} H(x)\ dx} \equiv \lim_{n \to \infty} e^{-iH(x_n)\Delta x} \cdot e^{-iH(x_{n-1})\Delta x} \cdots e^{-iH(x_1)\Delta x}$$

$$(\Delta x = \frac{x_f - x_0}{n}), \quad [H(x_i), H(x_j)] \neq 0, \quad e^a \cdot e^b = e^{a+b+\frac{1}{2}[a,b]+\dots}$$

$$S = e^{-iC[H]} = e^{-i(C_1+C_2+C_3+...)}$$

$$\begin{aligned} S = Te^{-i\int_{x_0}^{x_f} H(x)\ dx} &= 1 - i\int_{x_0}^{x_f} H(x)\ dx - \int_{x_0}^{x_f} dx \int_{x_0}^{x} H(x)\ H(y)\ dy \\ &+ i\int_{x_0}^{x_f} H(x)\ dx - \int_{x_0}^{x_f} dx \int_{x_0}^{x} dy \int_{x_0}^{y} H(x)\ H(y)\ H(z) \end{aligned}$$

$$-i(C_1 + C_2 + C_3 + ...) = \log\ [(S-1)+1] = S - 1 - \frac{(S-1)^2}{2} + \frac{(S-1)^3}{3} + \ldots$$

$$\begin{aligned} C_1 &= \int_{x_0}^{x_f} dx\ H(x)\ , \\ C_2 &= \frac{-i}{2}\int_{x_0}^{x_f} dx \int_{x_0}^{x} dy\ [H(x),\ H(y)]\ , \end{aligned}$$

$$C_3 = \frac{(-i)^2}{6}\int_{x_0}^{x_f} dx \int_{x_0}^{x} dy \int_{x_0}^{y} dz \left([H(x),[H(y),H(z)]] + [[H(x),H(y)],H(z)]\right).$$

The Magnus expansion is the integral version of the Baker-Campbell-Hausdorff equality.

Properties of Magnus expansion.

1) If $H(x) =$ constant, then $C_i = 0$ for $i > 1$.in the uniform medium

$$S = e^{-i\int_{x_0}^{x_f} dx \;\; H(x)} = e^{-iH \;\; (x_f - x_0)}.$$

This reproduces immediately the standard oscillation MSW result.

2) The Magnus expansion coefficients contain factorials in denominator.

3) Commutators themselves may contain an additional smallness.

4) If $H(x)$ is symmetric function with respect to the middle point of neutrino trajectory, one can show that $C_{2n} = 0$ $(n = 1, 2...)$ so that

only the odd terms in the expansion are non-zero.

Magnus expansion in the "interaction" representation

Lets split the Hamiltonin into two parts

$$H(x) = H_0(x) + \Upsilon(x)$$

$$[H_0(x_i),\ H_0(x_j)] = 0, \qquad H_0(x) = f(x)\ M$$

$$\psi = U_I(x)\ \psi_I = e^{-i\int_{x_0}^{x} dt\ \ H_0(t)}\psi_I$$

$$\Upsilon_I(x, x_0) = U_I^{\dagger}\Upsilon(x)U_I(x) = e^{i\int_{x_0}^{x} H_0(t)\ dt}\Upsilon(x)e^{-i\int_{x_0}^{x} H_0(t)dt}.$$

Transformation to the new basis is equivalent to transition to the "interaction representation" if H_0 is interpreted as the Hamiltonian of free propagation. Υ_I can be considered as the operator in the interaction representation.

$$S_I(x_f, x_0) = e^{-iC[\Upsilon_I(x,x_0)]},$$

The $S-$ matrix in the original basis equals

$$S(x_f, x_0) = U_I(x_f) S_I(x_f, x_0) U_I(x_0)^\dagger,$$

or explicitly

$$S(x_f, x_0) = e^{-i\int_{x_0}^{x_f} dt H_0(t)} e^{-iC[\Upsilon_I(x,x_0)]}.$$

$$e^{A+B} = e^A T e^{\int_0^1 e^{-At} B \; e^{At} dt}$$

$$S(x_f, x_0) = e^{-i\int_{x_0}^{x_f} dt H_0(t)} T e^{-i\int_{x_0}^{x_f} \Upsilon_I(x,x_0) \; dx}$$

$$C[\Upsilon_I(x, x_0)] = C_1^I + C_2^I + \dots$$

$$C_1^I = \int_{x_0}^{x_f} dx \; \Upsilon_I(x, x_0) \; ,$$

$$C_2^I = \frac{-i}{2} \int_{x_0}^{x_f} dx \int_{x_0}^{x} dy \; [\Upsilon_I(x, x_0), \Upsilon_I(y, x_0)] \; ,$$

Evolution in symmetric potential

the middle point of neutrino trajectory, $\bar{x}$,

$$\bar{U}_I(x) = e^{-i \int_{\bar{x}}^{x} dt H_0(t)}.$$

$$S_I(x_f, x_0) = e^{-iC[\Upsilon_I(x,\bar{x})]},$$

where

$$\Upsilon_I(x, \bar{x}) = e^{i \int_{\bar{x}}^{x} H_0(t) \; dt} \Upsilon(x) e^{-i \int_{\bar{x}}^{x} H_0(t) dt}.$$

Then, the evolution matrix in the original basis equals

$$S(x,x_0) = \bar{U}_I(x) S_I(x,x_0) \bar{U}_I(x_0)^\dagger,$$

or explicitly for evolution from x_0 to x_f:

$$S(x_f,x_0) = e^{-i\int_{\bar{x}}^{x_f} dt H_0(t)} e^{-iC[\Upsilon_I(x,\bar{x})]} e^{-i\int_{x_0}^{\bar{x}} dt H_0(t)}.$$

$$\begin{aligned}
\bar{C}_1 \equiv C_1[\Upsilon_I(x,\bar{x})] &= \int_{x_0}^{x_f} dx \Upsilon_I(x), \\
\bar{C}_2 \equiv C_2[\Upsilon_I(x,\bar{x})] &= -i\frac{1}{2}\int_{x_0}^{x_f} dx \int_{x_0}^{x} dy \left[\Upsilon_I(x), \quad \Upsilon_I(y)\right],
\end{aligned}$$

etc.. Here bar indicates that C_i have been calculated in the interaction representation with $\bar{U}_I$-matrix integrated from the middle point of trajectory.

$$\bar{C}_1 = 2\int_0^L dr \mathrm{Re}\Upsilon_I(r),$$
$$\bar{C}_2 = 2\int_0^L dr \int_0^r dp \left[\mathrm{Im}\Upsilon_I(r), \quad \mathrm{Re}\Upsilon_I(p)\right]$$

from which we immediately conclude that $\bar{C}_i$ are real.

$$S(x_f, x_0) = e^{-i\Phi_0(L)} e^{-iC[\Upsilon_I(x,\bar{x})]} e^{-i\Phi_0(L)}.$$

Oscillation probabilities

In applications of the Magnus expansion, adjusting the formalism to specific physical situation we can select

- propagation basis, that is, the basis of neutrino states in which we consider evolution;
- split of the Hamiltonian into self-commuting and non-commuting parts;
- perturbation terms.

In what follows we will consider for simplicity the two neutrino mixing case and a symmetric density profile keeping in mind applications to the neutrino propagation inside the Earth.

Low energy and low density limit

In the low energy or/and low density case it is convenient to consider evolution in the mass eigenstates basis, $\nu_{mass} = (\nu_1, \nu_2)$. In this basis the Hamiltonian can be written as

$$H(x) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta m^2/2E \end{pmatrix} + U^\dagger \begin{pmatrix} V(x) & 0 \\ 0 & 0 \end{pmatrix} U,$$

where U is the vacuum mixing matrix . The self-commuting part can be chosen as

$$H_0(x) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta^m(x) \end{pmatrix},$$

$$\Delta^m(x) \equiv \frac{\Delta m^2}{2E} \sqrt{\left(\cos 2\theta - \frac{2EV(x)}{\Delta m^2}\right)^2 + \sin^2 2\theta}\ .$$

$$\Upsilon(x) = A(x)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + B(x)\begin{pmatrix} 1 & 0 \\ 0 & \text{-1} \end{pmatrix},$$

where

$$\begin{aligned} A(x) &= \frac{1}{2}\sin 2\theta\ V(x), \\ B(x) &= \frac{1}{2}\left[\Delta^m(x) - \frac{\Delta m^2}{2E} + V(x)\cos 2\theta\right]. \end{aligned}$$

V: $V \ll \Delta m^2/2E$, we have

$$B(x) \approx A^2(x)\frac{2E}{\Delta m^2}.$$

$$\bar{C} \simeq \bar{C}_1 + \bar{C}_2 \approx I_V \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$I_V \equiv \sin 2\theta \int_0^L dr\ V(r) \cos \phi(r).$$

$$\phi(x) \equiv \int_0^x \Delta^m(r)\ dr$$

The $S-$matrix in the mass-eigenstates basis

$$S = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} \cos I_V & -i \sin I_V \\ -i \sin I_V & \cos I_V \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{pmatrix}.$$

$$\phi \equiv \phi_{\bar{x} \to x_f} = \phi_{x_0 \to \bar{x}} = \phi(L).$$

For transition probabilities:

$$P_{\nu_2 \to \nu_1} = |S_{21}|^2 = \sin^2 I_V.$$

$$P_{\nu_i \to \nu_\alpha} = |(U \cdot S)_{\alpha i}|^2.$$

$$P_{\nu_2 \to \nu_e} = \sin^2\theta + \frac{1}{2}\sin 2\theta \;\, \sin 2I_V \;\, \sin\phi + \cos 2\theta \;\, \sin^2 I_V,$$

where the first term is simply projection squared of ν_2 state onto ν_e.

$$S_{flavor-flavor} = U \cdot S \cdot U^\dagger.$$

$$P_{\nu_e \to \nu_\alpha} = (\cos I_V \;\, \sin 2\theta \;\, \sin\phi + \sin I_V \;\, \cos 2\theta)^2.$$

In the limit $V \to 0$, we have $I_V \to 0$ and the first term reproduces the standard vacuum oscillation probability.

Perturbation around average potential V_0

Let us perform expansion with respect to an average potential V_0. This means that we use another propagation basis:

$$\nu_f = U(\theta_0^m)\nu_0^m,$$

$\theta_0^m = \theta^m(V_0)$

$$\sin 2\theta^m(V) = \frac{\sin 2\theta}{\sqrt{(\cos 2\theta - 2EV/\Delta m^2)^2 + \sin^2 2\theta}}.$$

In the ν_0^m basis the Hamiltonian equals

$$H(x) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta_0^m \end{pmatrix} + U^\dagger(\theta_0^m)\begin{pmatrix} \Delta V(x) & 0 \\ 0 & 0 \end{pmatrix} U(\theta_0^m),$$

Δ_0^m is the difference of the eigenvalues in matter with potential V_0

$$\Delta V(x) \equiv V(x) - V_0.$$

$$I'_V = \sin 2\theta_0^m \int_0^L \Delta V(x)\ \cos\phi(x)\ dx.$$

$$P_{\nu_2\to\nu_e} = \cos^2 I'_V [\sin^2\theta + \sin 2\theta_0^m \sin 2(\theta_0^m - \theta)\sin^2\phi] + \\ + \frac{1}{2}\sin 2I'_V \sin 2(2\theta_0^m - \theta)\sin\phi + \sin^2 I'_V \cos^2(2\theta_0^m - \theta).$$

$$P_{\nu_2\to\nu_1} = [\cos I'_V\ \ \sin 2(\theta_0^m - \theta)\ \ \sin\phi + \sin I'_V\ \ \cos 2(\theta_0^m - \theta)]^2 .$$

$$P_{\nu_e\to\nu_\alpha} = (\cos I'_V\ \ \sin 2\theta_0^m\ \ \sin\phi + \sin I'_V\ \ \cos 2\theta_0^m)^2 .$$

Adiabatic perturbation theory in Magnus expansion

Let us again consider symmetric density profile. As the propagation basis we take the basis of the eigenstates of instantaneous Hamiltonian, $\nu^m \equiv (\nu_{1m}, \nu_{2m})$:

$$\nu_f = U(\theta^m(x))\nu^m$$

and $\theta^m(x)$ is the instantaneous mixing angle in matter . The Hamiltonian for the eigenstates equals $H(x) = H_0 + \Upsilon_\theta(x)$, where

$$H_0(x) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta^m(x) \end{pmatrix} , \quad \Upsilon_\theta(x) = \dot{\theta}^m(x) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ,$$

and

$$\dot{\theta}^m(x) \equiv \frac{d\theta^m(x)}{dx} = \frac{\sin 2\theta^m(x)}{2\Delta^m(x)} \, \frac{dV(x)}{dx} .$$

Straightforward calculations give

$$\bar{C}_1 = I_\theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \bar{C}_2 = I_{\theta\theta} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

where

$$
\begin{aligned}
I_\theta &= -2 \int_{\bar{x}}^{x_f} \dot{\theta}^m(x) \; \sin\phi_{\bar{x}\to x} dx = \\
&= 2 \int_{\bar{x}}^{x_f} [\theta^m(x) - \theta_s^m] \; \Delta^m(x) \; \cos\phi_{\bar{x}\to x} \; dx,
\end{aligned}
$$

$$
\begin{aligned}
I_{\theta\theta} &= -\int_{x_0}^{x_f} dx \int_{x_0}^{x} \dot{\theta}^m(x)\dot{\theta}^m(y) \sin\phi_{y\to x} dy = \\
&= 4 \int_{\bar{x}}^{x_f} dx \int_{\bar{x}}^{x} \dot{\theta}^m(x)\dot{\theta}^m(y) \sin\phi_{\bar{x}\to y} \cos\phi_{\bar{x}\to x} dy.
\end{aligned}
$$

$$
\begin{aligned}
P_{\nu_2\to\nu_e} &= \cos^2 I_\theta \; \sin^2\theta \; + \cos^2 I_\theta \; \sin 2\theta_s^m \; \sin 2(\theta_s^m - \theta) \; \sin^2\phi \; + \\
&+ \frac{1}{2} \sin 2I_\theta \; \sin 2(2\theta_s^m - \theta) \; \sin\phi \; + \sin^2 I_\theta \; \cos^2(2\theta_s^m - \theta),
\end{aligned}
$$

$$P_{\nu_2\to\nu_1} = \cos^2 I_\theta \ \sin^2 2(\theta_s^m - \theta) \ \sin^2\phi + \frac{1}{2}\sin 2I_\theta \ \sin 4(\theta_s^m - \theta) \ \sin\phi + \sin^2 I_\theta \ \cos^2 2(\theta_s^m - \theta) \ ,$$

$$P_{\nu_e\to\nu_\alpha} = \cos^2 I_\theta \ \sin^2 2\theta_s^m \ \sin^2\phi + \frac{1}{2}\sin 2I_\theta \ \sin 4\theta_s^m \ \sin\phi + + \sin^2 I_\theta \ \cos^2 2\theta_s^m .$$

Notice that $I_\theta \approx I'_V$, when $\theta^m - \theta_s^m \ll 1$.

Let us take into account the second order of the Magnus expansion.

$$P_{\nu_e\to\nu_\alpha} = \left[\sin 2\theta_s^m \ \cos I_t \ \sin\phi \ + \ \frac{\sin I_t}{I_t}(I_\theta \ \cos 2\theta_s^m - I_{\theta\theta} \ \sin 2\theta_s^m \ \cos\phi)\right]^2 .$$

$$I_t \equiv \sqrt{I_\theta^2 + I_{\theta\theta}^2},$$

$$
\begin{aligned}
P_{\nu_1 \to \nu_e} &= c_{13}^2 c_{12}^2 - \frac{1}{2} c_{13}^2 \sin 2\theta_{12} \sin 2I_{21} \sin\phi - c_{13}^2 \cos 2\theta_{12} \sin^2 I_{21} \\
P_{\nu_2 \to \nu_e} &= c_{13}^2 s_{12}^2 + \frac{1}{2} c_{13}^2 \sin 2\theta_{12} \sin 2I_{21} \sin\phi + c_{13}^2 \cos 2\theta_{12} \sin^2 I_{21} \\
P_{\nu_3 \to \nu_e} &= s_{13}^2
\end{aligned}
$$

$$
\begin{aligned}
P_{\nu_e\to\nu_e} &= 1-(\cos d\ \ \sin 2\theta_{13}^m\ \ \sin\psi\ \ +\ \ \sin d\ \ \cos 2\theta_{13}^m)^2+\delta P^{(2)}_{\nu_e\to\nu_e} \\
P_{\nu_\mu\to\nu_e} &= s_{23}^2\ (\cos d\ \ \sin 2\theta_{13}^m\ \ \sin\psi\ \ +\ \ \sin d\ \ \cos 2\theta_{13}^m)^2+\delta P^{(1)}_{\nu_\mu\to\nu_e}+\delta P^{(2)}_{\nu_\mu\to\nu_e} \\
P_{\nu_\mu\to\nu_\tau} &= \sin^2 2\theta_{23}\ \Big(\cos d\ (\cos^2\theta_{13}^m\ \ \sin^2(\psi-\phi)+\sin^2\theta_{13}^m\ \ \sin^2\phi-\frac{\sin^2 2\theta_{13}^m}{4} \\
&\quad -\frac{\sin d}{2}\sin 2\theta_{13}^m(\sin(\psi-2\phi)+\cos 2\theta_{13}^m\cos d\ \ \sin\psi)+\sin^2\frac{d}{2}-\frac{1}{4}\sin^2
\end{aligned}
$$

$$
P_{\nu_\mu\to\nu_e}-P_{\nu_e\to\nu_\mu} = -\sin\delta_{cp}\ \Delta_\odot\ L\ \ \sin 2\theta_{12}\ \ \sin 2\theta_{23}\ \ \sin\phi\ \ \sin(\psi-\phi)\cdot
$$

$$
\cdot\left(\frac{\cos\theta_{13}^m\ \ \cos(\theta_{13}^m+\theta_{13})}{\phi}-\frac{\sin\theta_{13}^m\ \ \sin(\theta_{13}^m+\theta_{13})}{\psi-\phi}\right)(\cos d\ \ \sin 2\theta_{13}^m\ \ \sin\psi+\sin d\ \ \cos
$$

For the constant potential (in our approximation $d^{(1)}=0$)

$$P_{\nu_e \to \nu_e} = 1 - \sin^2 2\theta_{13}^m \ \sin^2 \psi$$

$$P_{\nu_\mu \to \nu_e} = s_{23}^2 \ \sin^2 2\theta_{13}^m \ \sin^2 \psi + \frac{1}{4} \Delta_\odot \ L \ \sin 2\theta_{12} \ \sin 2\theta_{23} \ \sin 2\theta_{13}^m \ \sin \psi \cdot$$

$$\cdot (\cos \theta_{13}^m \ \cos(\theta_{13}^m + \theta_{13}) \frac{\sin(2\phi - \psi - \delta_{cp}) + \sin(\psi + \delta_{cp})}{\phi} +$$

$$+ \sin \theta_{13}^m \ \sin(\theta_{13}^m + \theta_{13}) \frac{-\sin(2\phi - \psi - \delta_{cp}) + \sin(\psi - \delta_{cp})}{\psi - \phi})$$

$$P_{\nu_\mu \to \nu_\tau} = \sin^2 2\theta_{23} \ \left(c_{13}^{m\,2} \sin^2(\psi - \phi) + s_{13}^{m\,2} \sin^2 \phi - \frac{\sin^2 2\theta_{13}^m}{4} \sin^2 \psi \right) + O(\frac{\Delta}{\Delta}$$

$$P_{\nu_\mu \to \nu_e} - P_{\nu_e \to \nu_\mu} = -\sin \delta_{cp} \ \Delta_\odot \ L \ \sin 2\theta_{12} \ \sin 2\theta_{23} \ \sin 2\theta_{13}^m \ \sin \psi \ \sin \phi \ \sin($$

$$\cdot \left(\frac{\cos \theta_{13}^m \ \cos(\theta_{13}^m + \theta_{13})}{\phi} - \frac{\sin \theta_{13}^m \ \sin(\theta_{13}^m + \theta_{13})}{\psi - \phi} \right)$$

$$V(x) = V^0(x) + v(x), \qquad H(x) = H^0(x) + \Upsilon(x)$$

$$H^0(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Delta m^2_\odot}{2E} & 0 \\ 0 & 0 & \frac{\Delta m^2_{atm}}{2E} \end{pmatrix} + U^\dagger \begin{pmatrix} V^0(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U$$

$$\Upsilon(x) = \frac{1}{3} U^\dagger \begin{pmatrix} 2\ \mathrm{v(x)} & 0 & 0 \\ 0 & \text{-v(x)} & 0 \\ 0 & 0 & \text{-v(x)} \end{pmatrix} U$$

$$\begin{aligned} S_{ji} &= Te^{-i\int_0^L H(x)\ dx} \\ &= Te^{-i\int_0^L H^0(x)\ dx} \cdot Te^{-i\int_0^L Te^{i\int_0^x H^0(y)\ dy}\Upsilon(x)\ Te^{-i\int_0^x H^0(y)\ dy}\ dx} \\ &= S^0_{ji} \cdot e^{-i\bar{C}} \end{aligned}$$

$$Te^{i\int_0^x H^0(y)\ dy} \cdot \Upsilon(x) \cdot Te^{-i\int_0^x H^0(y)\ dy} = S^{0\dagger}(x)\Upsilon(x)S^0(x)$$

$$\begin{aligned}
\bar{C} &= \bar{C}_1 + \bar{C}_2 + \cdots \\
\bar{C}_1 &= \int_0^L dx\ S^{0\dagger}(x)\Upsilon(x)S^0(x) \\
\bar{C}_2 &= -i\int_0^L dx\int_0^x dy\ [S^{0\dagger}(x)\Upsilon(x)S^0(x),\ S^{0\dagger}(y)\Upsilon(y)S^0(y)]
\end{aligned}$$

$$\begin{aligned}
S &\equiv S(L) = S^0(L)\cdot e^{-i\bar{C}} \approx S^0(L)\cdot(1 - i\bar{C}_1) \\
&= S^0(L) - i\int_0^L S^o(x \to L)\Upsilon(x)S^0(x)\ dx \\
&= S^0(L) - iS^0(b \to L)\int_a^b S^0(x \to b)\Upsilon(x)S^0(a \to x)\ dx\ S(0 \to a)
\end{aligned}$$

$$S^0(a \to x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i(\frac{\Delta m^2_\odot}{2E} - \cos 2\theta_{12}\bar{V})(x-a)} & 0 \\ 0 & 0 & e^{-i\frac{\Delta m^2_{atm}}{2E}(x-a)} \end{pmatrix}$$

$$P = P^0 + \Delta P$$

$$P_{\nu_1 \to \nu_e} = \cos^2\theta - \frac{1}{2}\sin^2 2\theta \int_{x_0}^{x_f} dx V(x) \sin\phi_{x \to x_f}$$

$$\phi_{x \to x_f} \simeq \left(\frac{\Delta m^2}{2E} - \cos 2\theta\bar{V}\right)(x_f - x)$$

SuperK

$N_\nu = 20000$

$$A = \frac{N_{day} - N_{night}}{N_{day} - N_{night}}$$

$$0.063 \pm 0.042(stat) \pm 0.037(sys) \quad SKII$$

$$0.021 \pm 0.020(stat) \pm 0.012(sys) \quad SK1$$

Averaging over energy

$$\overline{P}_{\nu_1 \to \nu_e} = \int dE' f(E', E) \; P_{\nu_1 \to \nu_e}.$$

attenuation factor $F(d)$

$$\overline{P}_{\nu_1 \to \nu_e} = \; \cos^2\theta - \frac{1}{2}\sin^2 2\theta \int_{x_0}^{x_f} dx V(x) F(x_f - x) \sin\phi_{x \to x_f}$$

the absence of averaging $\to \quad F = 1.$
For simplicity the box like resolution function $f(E', E)$

$$\overline{P}_{\nu_1 \to \nu_e} = \frac{1}{\Delta E}\int_{E - \frac{\Delta E}{2}}^{E + \frac{\Delta E}{2}} dE P_{\nu_1 \to \nu_e}.$$

$\Delta E \ll E$ and $\Delta_m \simeq \Delta m^2 [1 - \epsilon \cos 2\theta]/2E$

$$F(d) = \frac{1}{Q(d)} \sin Q(d), \qquad Q(d) \equiv \frac{\pi d \Delta E}{l_\nu E} \;,$$

Conclusion

We have developed new formalism for computations of the oscillation probabilities in matter with varying density. It is based on the Magnus expansion and has a virtue to be unitary in each order of the expansion. Using the Magnus expansion one can develop different perturbation theories, and in particular the improved adiabatic perturbation theory.

We have obtained the semi-analytical formulas for various oscillation probabilities in the second order of the Magnus expansion. The Magnus expansion (apart from being unitary) leads also to better convergence of series. The developed unitary formalism gives new insight into previously obtained results and their limitations.

Using several explicit examples we show that restoration of unitarity gives better approximation to results of exact numerical calculations. The results obtained here can be immediately used for description of propagation of the solar and supernova neutrinos inside the Earth. They also can be used to describe the flavor oscillations of the atmospheric and accelerator neutrinos.

When detector has not sufficient energy resolution it is 'short-sighted' and 'see' ONLY nearby structures.
The energy resolution of the water-cherencov detectors is about 3 MeV for neutrinos with 10 MeV energy and according to our analytical results these detectors ONLY see about 1500 km far from the detector.

When detector has good energy resolution
($\Delta E \leq 0.5$MeV) it can detect the core.

Neutrino tomography is only way for searching the core of the earth. (the usual technique (sonic) is failed since sound ways are reflecting from the core).