Lattice QCD overview: form-factors, $V_{cb}$, $V_{ub}$, R-ratios, ...

Implications of LHCb measurements and future prospects
CERN, October 2018

Andreas Jüttner

UNIVERSITY OF
Southampton
Lattice Flavour Physics and CKM

\[ V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \]
Lattice Flavour Physics and CKM

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

illustrations from L. Lellouch’s Les Houches Lecture arXiv:1104.5484
Assumptions

We assume factorisation of SM:

\[ \Gamma_{\text{exp.}} \equiv V_{\text{CKM}}(\text{WEAK})(\text{EM})(\text{STRONG}) \]

**Strong** contribution given in terms of hadronic form factors (lattice)

Weak & EM & strong treated separately — although in real world all three SM sectors talk to each other

**In particular EM:**
Note that \( O(\alpha_{EM}) \approx 1\% \) — so OK as long as we keep it in mind
Quark Flavour Physics

e.g tree level leptonic $B$ decay:

Assumed factorisation: $\Gamma_{\text{exp.}} = V_{\text{CKM}}(\text{WEAK})(\text{EM})(\text{STRONG})$

$\langle 0|A_\mu|B\rangle_{\text{QCD}}$

$$\Gamma(B \to l\nu_l) = |V_{ub}|^2 \frac{m_B}{8\pi} G_F m^2_l \left(1 - \frac{m^2_l}{m^2_B}\right)^2 f_B^2$$

experiment output theory prediction
Lattice QCD

\[ \mathcal{L}_{\text{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_f \bar{\psi}_f (i\gamma^\mu D_\mu - m_f) \psi_f \]

Free parameters:
- gauge coupling \( g \rightarrow \alpha_s = g^2/4\pi \)
- quark masses \( m_f = u,d,s,c,b,t \)

- Lagrangian of massless gluons and \textit{almost massless quarks}
- What experiment sees are bound states, e.g. \( m_\pi, m_\rho \gg m_{u,d} \)
- Underlying physics non-perturbative
Lattice QCD

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Path integral quantisation:

\[
\langle 0 | O | 0 \rangle = \frac{1}{Z} \int \mathcal{D}[U, \psi, \bar{\psi}] O e^{-iS_{\text{lat}}[U, \psi, \bar{\psi}]} \\
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Euclidean space-time Boltzmann factor
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Euclidean space-time Boltzmann factor

finite volume, space-time grid (IR and UV regulators)

\( \propto L^{-1} \propto a^{-1} \)

→ Well defined, finite dimensional Euclidean path integral
→ From first principles, solve via MCMC
State-of-the-art lattice QCD

What we can do

- simulations of QCD with dynamical (sea) $u,d,s,c$ quarks with masses as found in nature
- bottom as valence quark
- cut-off $a^{-1} \leq 4\text{GeV}$
- volume $L \leq 6\text{fm}$
- $m_u=m_d$, $\alpha_{EM}=0$ — this is changing!
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Parameter tuning
need to fix quark masses and coupling by sacrificing a few hadronic inputs (typically meson/baryon masses and/or simple matrix elements)
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Parameter tuning

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BMW Collaboration
Advertisement:
Flavour Lattice Averaging Group

“What’s currently the best lattice value for a particular quantity?”


WEB UPDATES: http://flag.unibe.ch

- quantities:
  \[ m_{u,d,s,c,b}, f_K/f_\pi, f_+^{K\pi}(0), B_K, SU(2) \text{ and } SU(3) \text{ LECs} \]
  \[ f_{D(s)}, f_{B(s)}, B_{B(s)}, B(s) - \text{ and } D(s) - \text{semileptons} \]
  \[ \alpha_s \]

- summary of results
  - evaluation according to FLAG quality criteria (colour coding)
  - averages of best values where possible
  - detailed summary of properties of individual simulations

FLAG-4 aiming for early 2019
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Challenges for heavy quark on the lattice

A multi-scale problem

\[ a^{-1} \ll \text{physics of interest} \ll L^{-1} \]
finite cutoff
finite box size

finite lattice spacing
• hard to discretise \( b \)-quarks (slowly getting there but need to play and control tricks like effective theory, improvement, extrapolation in \( m_h, \ldots \), which are not needed for light quarks)

finite lattice volume
• physical pion mass ‘expensive’ to reconcile with above bounds — we often dial heavier (cheaper) \( m_\pi \) and then extrapolate (model or EFT)
Extrapolation of lattice data

\[ B \rightarrow Kl\nu \]

lattice data points with ‘heavy’ pion model/EFT/z-parametrisation extrapolate into kinematic region inaccessible by lattice data
Challenges for heavy quark on the lattice

\[
\frac{d\Gamma(B_{(s)} \rightarrow P\ell\nu)}{dq^2} = \frac{G_F^2 |V_{ub}|^2}{24\pi^3} \frac{(q^2 - m_\ell^2)^2 \sqrt{E_P^2 - m_P^2}}{q^4 m_{B(s)}^2} \left[ \left(1 + \frac{m_\ell^2}{2q^2}\right) m_{B(s)}^2 (E_P^2 - m_P^2) |f_+(q^2)|^2 + \frac{3m_\ell^2}{8q^2} (m_{B(s)}^2 - m_P^2)^2 |f_0(q^2)|^2 \right]
\]

\[
\frac{d\Gamma(B^- \rightarrow D^0 \ell^- \bar{\nu})}{dw} = \frac{G_F^2 m_D^3}{48\pi^3} (m_B + m_D)^2 (w^2 - 1)^{3/2} |\eta_{EW}|^2 |V_{cb}|^2 |G(w)|^2,
\]

\[
\frac{d\Gamma(B^- \rightarrow D^{0*} \ell^- \bar{\nu})}{dw} = \frac{G_F^2 m_{D^*}^3}{4\pi^3} (m_B - m_{D^*})^2 (w^2 - 1)^{1/2} |\eta_{EW}|^2 |V_{cb}|^2 \chi(w) |F(w)|^2
\]
Challenges for heavy quark on the lattice

\[ a^{-1} \ll \text{physics of interest} \ll L^{-1} \]

finite cutoff \quad \text{finite box size}

**kinematics**

\[ q^2 = (E_B - E_{\text{light}})^2 - (\vec{p}_B - \vec{p}_{\text{final}})^2 \]

lattice does best with mesons at rest (statistical error and cutoff effects smaller)

this is at tension with the suppression of the decay rate at large momenta

Kinematical reach limited in lattice QCD \( \rightarrow \) extract value of \( V_{CKM} \) from simultaneous analysis of exp. and lattice data
And now some results...

The following is an incomplete list of mostly recent results (for comprehensive list see e.g. FLAG 2017 web update)

Activity has picked up lately and more collaborations are now contributing on $B_s$ decays

That’s great since different lattice techniques are being used, each with slightly different systematic effects

In the following as per request:
concentrate on $b \rightarrow u$ and $b \rightarrow c$ form factors, CKM MEs and R-ratios
$b \rightarrow u$
Current activity on $B \to \pi l\nu$

RBC/UKQCD is working on update

Gelzer Lattice 2018

$2+1+1$, HISQ, Fermilab HQ
$B \rightarrow \pi$ comparison

Overlay plot by Colquhoun showing (preliminary) agreement

JLQCD Preliminary
**Introduction**

$(2+1)$-flavor asqtad

$B_s \rightarrow K$

$(2+1+1)$-flavor HISQ

**Conclusion**

Analysis

Form factors for $B_s \rightarrow K$

Current activity on

$B_s \rightarrow Kl\nu$

Gelzer Lattice 2018

2+1 AsqTad

Gelzer Lattice 2018

2+1+1, HISQ, Fermilab HQ
$B_s \rightarrow K$ comparison

Overlay plot of RBC/UKQCD and Gelzer (Fermilab MILC)
a bit of tension maybe — need to wait for final error budget
tematic uncertainties, discussed in Sec.

In this latter case, it orange bands display the preferred

where the first uncertainty comes from the lattice computatio

semileptonic

element

Note that the

The same work provides

Let us summarize the lattice input that satisfies FLAG require

Note that generating synthetic data is a trivial task but less s

\[ V^{\nu}(s) = 3 \text{ BCL fit (five parameters) to the plotted data with errors.} \]

\[ \text{FLAG2017} \]

\[ B \to \pi l \nu \]

\[ f_0 \text{ average} \]

\[ f_+ \text{ HPQCD 06} \]

\[ f_+ \text{ FNAL/MILC 15} \]

\[ f_0 \text{ FNAL/MILC 15} \]

\[ f_+ \text{ RBC/UKQCD 15} \]

\[ f_0 \text{ RBC/UKQCD 15} \]

\[ \text{BCL3} \]

\[ B \to K l \nu \]

\[ f_0 \text{ average} \]

\[ f_+ \text{ HPQCD 14} \]

\[ f_+ \text{ RBC/UKQCD 15} \]

\[ f_0 \text{ HPQCD 14} \]

\[ f_0 \text{ RBC/UKQCD 15} \]

\[ \text{BCL3} \]

\[ \text{Updated Jul./Nov. 2017} \]

\[ \text{HFAG quotes} \]

\[ B \to \pi l \nu \}

\[ |V_{ub}| \times 10^3 \]

\[ |V_{ub}| = 3.73(14) \times 10^{-3} \]

\[ \text{FLAG 2017} \]

\[ \text{FLAG estimate for } N_t = 2+1 \]

\[ B \to \pi l \nu \]

\[ B \to \tau l \nu \text{ (BaBar)} \]

\[ B \to \tau l \nu \text{ (Belle)} \]

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<table>
<thead>
<tr>
<th>Non-latt. $N_t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>HFAG Inclusive</td>
</tr>
</tbody>
</table>

\[ B \to \pi l \nu \]

\[ B \to \tau l \nu \text{ (BaBar)} \]

\[ B \to \tau l \nu \text{ (Belle)} \]
$B \to K$

Current activity on $B \to K\ell\ell$

HPQCD Lattice 2018
2+1, NRQCD

Fermilab/MILC Lattice 2018
2+1+1, HISQ, Fermilab HQ
**B → K**

Current activity on $B \to K\pi$

![Graph](image1)

![Graph](image2)
b → c
Hard:
b, c and light valence quarks

hopefully more activity here in the near future
B → D^*

lattice data away from zero recoil

Fermilab/MILC’s
Vaquero’s CKM slides
2+1, Fermilab HQ

BGL
Preliminary

Preliminary

“soon results on R(D*)”

- experimentally precise
- D* assumed stable on in lattice simulations
- issue with parametrisation

CLN vs BGL?

Gambino, Bigi, PRD 94, 094008 (2016)

There is more activity…
(RBC/UKQCD, JLQCD, Bailey et al....)
\[ \mathbf{B}_s \rightarrow \mathbf{D}_s \]

Fair bit of activity here mostly preliminary but advanced

---

**2+1+1 HISQ, NRQCD b**

- HPQCD Preliminary
  - arXiv:1711.03487

---

**2+1+1 HISQ, HISQ b**

- Lattice 2018
  - HPQCD Preliminary
$B_s \rightarrow D_s$ comparison

Overlay plot - there is a bit of a spread let’s wait until analyses finalised will lead to prediction for $R(D_s)$
$B_s \rightarrow D_s$ comparison

Overlay plot - there is a bit of a spread let’s wait until analyses finalised will lead to prediction for $R(D_s)$
$B_s \rightarrow D_s^*$

- $h_{A_1}(q_{\text{max}}^2)$
- $a \approx 0.09\,\text{fm}$
- $a \approx 0.06\,\text{fm}$
- NRQCD[1711.11013]

Graph showing $h_{A_1}(q_{\text{max}}^2)$ vs. $M_{\eta_b}$.

HPQCD Preliminary

McLean Lattice 2018
$V_{cb}$

$B \rightarrow D^* \nu \quad |V_{cb}| \times 10^3 = 39.27(56)(49)$

$B \rightarrow D \nu \quad |V_{cb}| \times 10^3 = 40.1(1.0)$
Ratios

Ratios are great!

Cancellations of
- statistical errors (from Markov Chain Monte Carlo)
- systematic errors
  - renormalisation factors
  - discretisation effects
  - finite volume effects
  - quark-mass extrapolation (in particular heavy quark mass)
  - ...

Ratios of branching fractions, differential decay rates, asymmetries…
$$V_{ub}/V_{cb}$$

Well known result from ratio of baryonic decay rates (exp.+lattice)

$$\Lambda_b^0 \rightarrow \Lambda_c^+ \mu \bar{\nu}$$
$$\Lambda_b^0 \rightarrow p \mu \bar{\nu}$$

LHCb PRD96, 112005 (2017), 1709.01920  

alternative: ratio of SL $B_s$ decays  
Monahan et al. arXiv:1808.09285

$$B_s \rightarrow K \ell \nu$$
$$B_s \rightarrow D_s \ell \nu$$

Results assuming above $|V_{ub}/V_{cb}|^2$

ratio of diff. decay rates  
for./backw. asymmetry  
polarisation asymm.

RBC/UKQCD is finalising their analysis
What’s next?
Beyond %-level precision

With a sub-percent precision goal we can't ignore isospin breaking effects:

- strong isospin breaking

\[
\frac{m_u - m_d}{\Lambda_{\text{QCD}}} \sim O(1\%)
\]

- QED

\[
\alpha \approx \frac{1}{137} \sim O(1\%)
\]
Beyond %-level precision

With a sub-percent precision goal we can't ignore isospin breaking effects:

• strong isospin breaking

\begin{align*}
m_u &= 2.46(24)\text{MeV} \quad m_d = 5.03(26)\text{MeV} \\
\frac{m_u - m_d}{\Lambda_{\text{QCD}}} &\sim O(1\%) \\
\alpha &\approx \frac{1}{137} \sim O(1\%)
\end{align*}

• QED

lattice QCD+QED
Isospin breaking: EM effects

Factorisation $\Gamma = \text{Weak} \times \text{EM} \times \text{Strong}$

Lattice will no longer compute decay constants and form factors but aim directly at decay rate.

Two major technical difficulties:
- Photon is massless and induces large finite-size effects.
- IR singularities (Bloch-Nordsieck) need to be dealt with.
EM corrections to leptonic decay

EM corrections to leptonic decay

- Leptonic decay at $O(\alpha^0)$:

$$\Gamma(\pi^+ \rightarrow l^+ \nu_l) = \frac{G_F^2 |V_{ud}|^2 f_\pi^2}{8\pi} m_\pi m_l^2 \left(1 - \frac{m_l^2}{m_\pi^2}\right)^2$$

EM corrections to leptonic decay

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  \]

- including elm. effects @ $O(\alpha)$:
  \[
  \Gamma(\pi^+ \to l^+ \nu_l(\gamma)) = \Gamma(\pi^+ \to l^+ \nu_l) + \Gamma(\pi^+ \to l^+ \nu_l\gamma)
  \]

EM corrections to leptonic decay


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$$\Gamma(\pi^+ \to l^+ \nu_l(\gamma)) = \Gamma(\pi^+ \to l^+ \nu_l) + \Gamma(\pi^+ \to l^+ \nu_l \gamma)$$

$$\equiv \Gamma_0 + \Gamma_1$$

IR div. cancel between terms on r.h.s. between virtual and real photons
(Bloch Nordsieck)
Summary

- Increased activity in $B_{(s)}$SL decay calculations, mostly preliminary but anomaly-hype has created some momentum

- more results — more independent systematics

- most results based on effective theory for $b$-quark — this will continue for a while but increasingly extrapolations towards $b$

- we are learning how to include QED effects which become relevant at %-level. First applications leptonic decay under way. This will change the way analyses are done…
Backup - Baryon SL decays

Early work on $\Lambda_b \to \Lambda_c$ (quenched, focused on Isgur-Wise function):

S. Gottlieb and S. Tamhankar, arXiv:hep-lat/0301022/Lattice 2002

Our work, using RBC/UKQCD 2 + 1 flavor ensembles:

<table>
<thead>
<tr>
<th>Transition</th>
<th>$m_b$</th>
<th>$a$ [fm]</th>
<th>$m_\pi$ [MeV]</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda_b \to \Lambda$</td>
<td>$\infty$</td>
<td>0.11, 0.08</td>
<td>230–360</td>
<td>WD, DL, SM, MW, arXiv:1212.4827/PRD 2013</td>
</tr>
<tr>
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<td>WD, DL, SM, MW, arXiv:1306.0446/PRD 2013</td>
</tr>
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<td>$\Lambda_b \to p$</td>
<td>phys.</td>
<td>0.11, 0.08</td>
<td>230–360</td>
<td>WD, CL, SM, arXiv:1503.01421/PRD 2015</td>
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<td>0.11, 0.08</td>
<td>230–360</td>
<td>WD, CL, SM, arXiv:1503.01421/PRD 2015; AD, SK, SM, AR, arXiv:1702.02243/JHEP 2017</td>
</tr>
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<td>$\Lambda_b \to \Lambda$</td>
<td>phys.</td>
<td>0.11, 0.08</td>
<td>230–360</td>
<td>WD, SM, arXiv:1602.01399/PRD 2016</td>
</tr>
<tr>
<td>$\Lambda_b \to \Lambda^*$</td>
<td>phys.</td>
<td>0.11</td>
<td>340</td>
<td>SM, GR, arXiv:1608.08110/Lattice 2016</td>
</tr>
<tr>
<td>$\Lambda_b \to \Lambda_c^*$</td>
<td>phys.</td>
<td>0.11, 0.08</td>
<td>300–430</td>
<td>SM, GR, Later in this talk</td>
</tr>
<tr>
<td>$\Lambda_c \to \Lambda$</td>
<td>0.11, 0.08</td>
<td>$\mathbf{140}$–360</td>
<td>SM, arXiv:1611.09696/PRL 2017</td>
<td></td>
</tr>
<tr>
<td>$\Lambda_c \to p$</td>
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<td>230–360</td>
<td>SM, arXiv:1712.05783/PRD 2018</td>
<td></td>
</tr>
</tbody>
</table>

WD = William Detmold
DL = C.-J. David Lin
SM = Stefan Meinel
MW = Matthew Wingate
CL = Christoph Lehner
AD = Alakabha Datta
SK = Saeed Kamali
AR = Ahmed Rashed
GR = Gumaro Rendon (graduate student at U of A)

Meinel’s slide, CKM 2018
EM corrections to leptonic decay

- cut on small photon momentum $< \Delta E \rightarrow \gamma$ sees point-like $\pi$
  $\Delta E \approx 20\text{MeV}$ experimentally accessible and $\pi$ point like

\[
\Gamma(\Delta E) = \lim_{V \to \infty} (\Gamma_0 - \Gamma_0^{pt}) + \lim_{V \to \infty} (\Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E))
\]

\[
\Gamma(\pi^+ \rightarrow l^+\nu_l)
\]

lattice and analytical finite $V$

\[
\Gamma(\pi^+ \rightarrow l^+\nu_l\gamma(\Delta E))
\]

analytically in $V \to \infty$

both terms separately IR finite, gauge invariant on its own

Sub precent precision requires inclusion of IB breaking effects for which new techniques in QFT are being developed