Unitarity in Extra-Dimensional Gravity Models

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⋆ **SUSY 2019** ⋆

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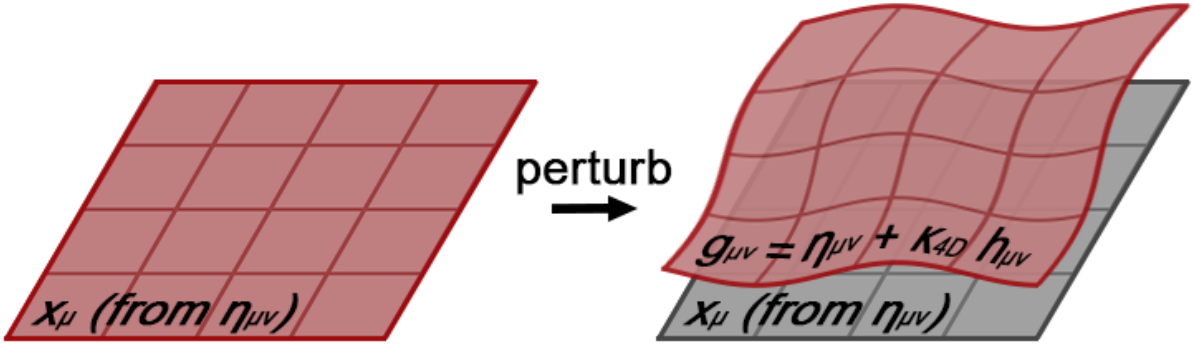
Introduction

(Massless) 4D Gravity

Let's quantize gravity: First, weakly perturb the flat 4D metric by a **massless spin-2 field** $h_{\mu\nu}$...



$$\eta_{\mu\nu} = \mathrm{Diag}(+,-,-,-) \quad \Big| \quad g_{\mu\nu} = \eta_{\mu\nu} + \kappa_{4\mathrm{D}}\, h_{\mu\nu}$$

Expansion Parameter: $\kappa_{4\mathrm{D}}$ has units $[\mathrm{Energy}]^{-1}$

- Fix by matching to Newton's law, via M_{Pl}
- $2 \to 2$ Tree-Level Unitarity breaks at energy $\sim M_{\mathrm{Pl}}$

$$\boxed{\mathcal{L}^{(4\mathrm{DG})} = \frac{2}{\kappa_{4D}^2}\sqrt{\det g}\; R} \quad \overset{\mathrm{WFE}}{\Longrightarrow} \quad \boxed{\mathcal{L}^{(4\mathrm{DG})} = \sum_{H=2}^{+\infty} \mathcal{L}^{(4\mathrm{DG})}_{h^H}}$$

Add a Mass = Fierz-Pauli (Massive) Gravity

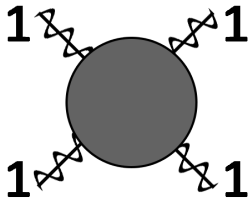
Give the **4D graviton** $h_{\mu\nu}$ a mass m by hand:

$$\mathcal{L}^{(\mathrm{FPG})} = \underbrace{\frac{2}{\kappa_{4D}^2}\sqrt{\det g}\, R}_{\text{4D gravity } \mathcal{L}^{(\mathrm{4DG})}} + \underbrace{\frac{1}{2}m^2\left[h^2 - h_{\mu\nu}h^{\mu\nu}\right]}_{\text{Fierz-Pauli mass term}}$$

At high energies ($s \gg 4m^2$), helicity polarizations diverge like...

$$\epsilon_{\pm 2}^{\mu\nu} \sim \mathcal{O}(1) \qquad \epsilon_{\pm 1}^{\mu\nu} \sim \mathcal{O}(\sqrt{s}) \qquad \boxed{\epsilon_{0}^{\mu\nu} \sim \mathcal{O}(s)}$$

and so, $2 \to 2$ matrix elements grow fastest when scattering is longitudinal (helicity = 0); unitarity violated according to

$\mathcal{M} \equiv$ 

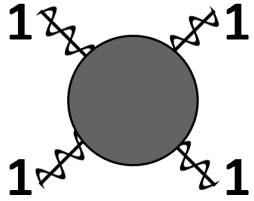
$\sim \begin{cases} \text{Naive Dim. Analysis:} & \mathcal{O}(s^7) \\ \text{Explicit Calculation:} & \mathcal{O}(s^5) \end{cases}$

Note: *Symmetries automatically cancel four powers of energy!*

... but $\mathcal{O}(s^5)$ implies theory breaks at $\Lambda_5 \sim (m^4 M_{\mathbf{Pl}})^{1/5}$

Unitarity Violation in Massive Spin-2 State Theories

Consider tree-level longitudinal (helicity = 0) scattering of a massive spin-2 particle "1" at high energies in 4D:

$\mathcal{M} =$ 

$\sim \mathcal{O}(s^{???})$ in large s regime

- **2002:** Arkani-Hamed, Georgi, and Schwartz argue that $\mathcal{M}$ diverges like $\mathcal{O}(s^5)$ in Fierz-Pauli gravity, and like $\mathcal{O}(s^3)$ in carefully tuned spin-2 theories. (hep-th/0210184)
- **2003:** Schwartz argues away $\mathcal{O}(s^3)$ in the 5D Torus by applying a Stueckelberg-like mechanism to the Lagrangian; does not calculate matrix elements (hep-th/0303114)
- **2019:** We explicitly calculate matrix elements, and demonstrate cancellations such that $\mathcal{M} \sim \mathcal{O}(s)$ in 5D Orbifolded Torus and Randall-Sundrum models.
  ⋆ **This is consistent with 5D expectations!** ⋆

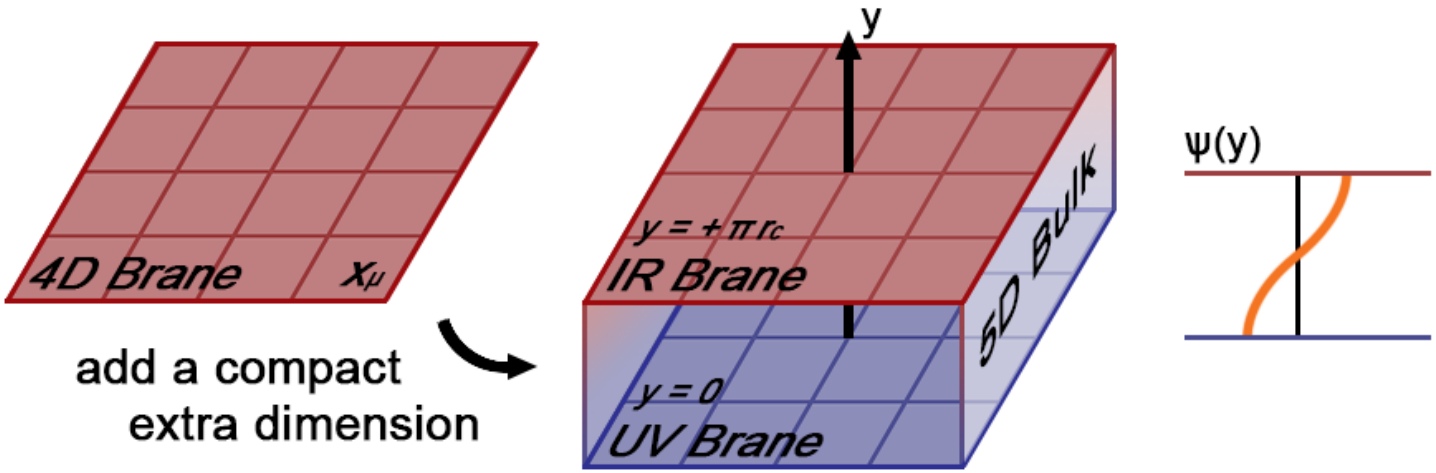
5D Orbifolded Torus (5DOT) Model

How to Build a 5D Orbifolded Torus

Why 5DOT? = RS model with no warping

⋆ Step 1: Build a 5D Torus ⋆

- **Parameter 1:** Add a compact spatial dimension $y \in [0, \pi r_c]$ to 4D spacetime (x^μ), where $r_c \equiv$ the **compactification radius**

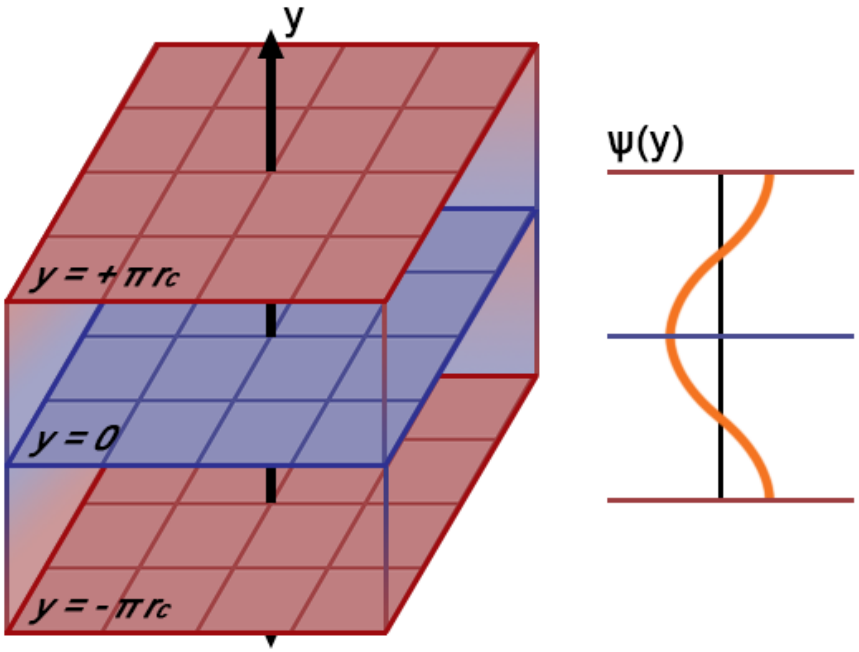


$$\eta_{MN}^{(5\text{DOT})} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix} \implies \boxed{ds^2 = (dt^2 - d\vec{x}^{\,2}) - dy^2}$$

How to Build a 5D Orbifolded Torus

⋆ **Step 2: Orbifold** ⋆

Orbifold Symmetry: Reflect extra dimension about $y = 0$, demand ds^2 be invariant under $y \leftrightarrow -y$, so $y \in [-\pi r_c, +\pi r_c]$.

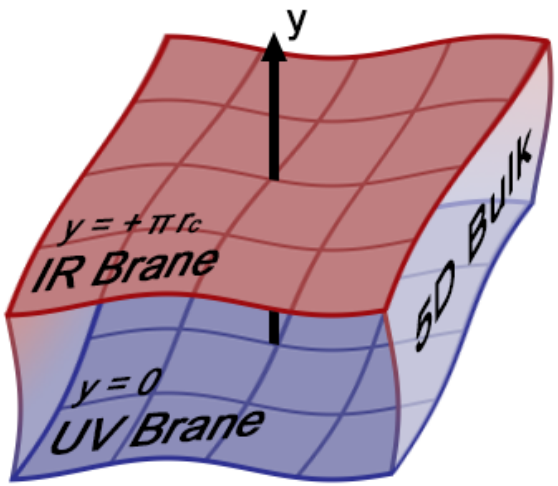


Why Orbifold the Torus?
The 5DOT is a nice limit of the RS model. Also, relative to 5D Torus, 5DOT eliminates the spin-1 **graviphoton** ρ_μ.

Schematically...

$$\delta G_{MN} \approx \kappa \begin{pmatrix} h_{\mu\nu} & \cancel{\rho_\mu} \\ \cancel{\rho_\nu} & r \end{pmatrix}$$

How to Perturb a 5D Orbifolded Torus



- **Parameter 2:** 5D exp. parameter κ

$$G_{MN} = \begin{pmatrix} e^{-\kappa\hat{r}/\sqrt{6}}(\eta_{\mu\nu}+\kappa\hat{h}_{\mu\nu}) & 0 \\ 0 & -(1+\hat{r}/\sqrt{6})^2 \end{pmatrix}$$

where $[\kappa] = [\text{Energy}]^{-3/2}$ (implies κ_{4D})

Particles in 5D Matter-Free Orbifolded Torus:

- **5D Graviton** $= \hat{h}_{\mu\nu}$, a massless spin-2 5D particle
  ▷ **Origin:** local coordinate invariance of constant y sheets

- **5D Radion** $= \hat{r}$, a massless spin-0 5D particle
  ▷ **Origin:** locally perturbing distance between branes
  Caution: In realistic theories, radion requires additional external stabilization, e.g. Goldberger-Wise mechanism; radion stabilization plays no role in the present analysis.

5D to 4D: Organizing the 5D Lagrangian

Define $\mathcal{L}_{h^H r^R} \equiv$ all $\mathcal{L}_{5\mathrm{D}}$ terms with H gravitons and R radions:

$$\mathcal{L}_{5\mathrm{D}} = \sum_{H,R} \mathcal{L}_{h^H r^R} \equiv \sum_{H,R} (\cdots)^{\vec{\mu}}\, \hat{h}^H_{\vec{\mu}}\, \hat{r}^R$$

By construction, each term in this set is either

- **A-Type:** has two 4D derivatives $\partial_\mu \partial_\nu$, or
- **B-Type:** has two extra-dimensional derivatives ∂_y^2

$$\boxed{\mathcal{L}_{h^H r^R} = \kappa^{(H+R-2)} \left[\lambda_A(R)\, \overline{\mathcal{L}}_{A:h^H r^R} + \lambda_B(R)\, \overline{\mathcal{L}}_{B:h^H r^R} \right]}$$

<u>5D to 4D:</u> Once we have a 5D WFE theory, we convert it into an effective 4D theory via y-integration

$$S = \int d^4x \left(\int dy\ \mathcal{L}_{5\mathrm{D}} \right) \quad \Longrightarrow \quad \boxed{\mathcal{L}_{4\mathrm{D}}^{(\mathrm{eff})} \equiv \int_{-\pi r_c}^{+\pi r_c} dy\ \mathcal{L}_{5\mathrm{D}}}$$

5D to 4D: Kaluza-Klein (KK) Decomposition

$$\boxed{\underbrace{\hat{f}_{\vec{\mu}}(x,y)}_{\text{5D field}} = \frac{1}{\sqrt{\pi r_c}} \sum_{n=0}^{+\infty} \underbrace{\hat{f}_{\vec{\mu}}^{(n)}(x)}_{\text{4D fields}} \underbrace{\psi^{(n)}(y)}_{\text{wfxns}}}$$

KK Decomposition

We utilize extra-dimensional wavefunctions $\psi^{(n)}(y)$ that satisfy

$$\begin{cases} \dfrac{1}{\pi r_c}\displaystyle\int_{-\pi r_c}^{+\pi r_c} dy \;\; \lambda_A(0)\; \psi^{(m)}\psi^{(n)} = \delta_{m,n} \\[2ex] \dfrac{1}{\pi r_c}\displaystyle\int_{-\pi r_c}^{+\pi r_c} dy \;\; \lambda_B(0)\; (\partial_y\psi^{(m)})(\partial_y\psi^{(n)}) = m_n^2\; \delta_{m,n} \end{cases}$$

where $n \equiv$ **KK number**. Given R radion fields, we define...

<u>5DOT:</u> $\begin{cases} \lambda_A(R) = 1 \\ \lambda_B(R) = 1 \end{cases}$ $\left(\textbf{<u>RS:</u>} \begin{cases} \lambda_A(R) = e^{k[2(R-1)|y| - R\pi r_c]} \\ \lambda_B(R) = e^{-2k|y|}\lambda_A(R) \end{cases} \right)$

5D to 4D: KK Decomposition of Quadratic Spin-2

$$\boxed{\mathcal{L}_{hh} = \lambda_A(0)\,\overline{\mathcal{L}}_{A:hh} + \lambda_B(0)\,\overline{\mathcal{L}}_{B:hh}}$$

$$\overline{\mathcal{L}}_{A:hh} = -\hat{h}_{\mu\nu}(\partial^\mu\partial^\nu\hat{h}) + \hat{h}_{\mu\nu}(\partial^\mu\partial_\rho\hat{h}^{\rho\nu}) - \frac{1}{2}\hat{h}_{\mu\nu}(\Box\hat{h}^{\mu\nu}) + \frac{1}{2}\hat{h}(\Box\hat{h})$$

$$\overline{\mathcal{L}}_{B:hh} = -\frac{1}{2}(\partial_y\hat{h}_{\mu\nu})(\partial_y\hat{h}^{\mu\nu}) + \frac{1}{2}(\partial_y\hat{h})^2$$

such that the 4D effective spin-2 Lagrangian is canonical:

$$\boxed{\mathcal{L}_{hh}^{(\text{eff})} = \mathcal{L}_{\text{Kin}}^{(S=2)}(\hat{h}^{(0)}) + \sum_{n=1}^{+\infty}\mathcal{L}_{\text{FP}}(m_n, \hat{h}^{(n)})}$$

<u>5D Graviton</u> $\hat{h}_{\mu\nu}$ becomes many 4D particles:

- **4D Graviton:** a massless spin-2 4D particle $\hat{h}_{\mu\nu}^{(0)}$
- **KK Modes:** massive spin-2 4D particles $\hat{h}_{\mu\nu}^{(n)}$ for $n > 0$

5D to 4D: KK Decomposition of Quadratic Spin-0

Similarly, we may decompose the 5D radion:

$$\boxed{\mathcal{L}_{rr} = \lambda_A(2)\, \overline{\mathcal{L}}_{A:rr}} \qquad \text{where} \qquad \overline{\mathcal{L}}_{A:rr} = \frac{1}{2}(\partial_\mu \hat{r})(\partial^\mu \hat{r})$$

5D diffeomorphism invariance may used to ensure **the 5D radion is flat in the extra dimension** (e.g. $\hat{r}(x,y) \to \hat{r}(x)$). Thus, its KK decomposition has only a zero mode:

$$\hat{r}(x) = \frac{1}{\sqrt{\pi r_c}} \hat{r}^{(0)}_{\mu\nu}(x)\psi^{(0)} \qquad \Longrightarrow \qquad \boxed{\mathcal{L}^{(\text{eff})}_{rr} = \mathcal{L}^{(S=0)}_{\text{Kin}}(\hat{r}^{(0)})}$$

<u>**5D Radion**</u> $\hat{r}$ becomes a single 4D particle:

- **4D Radion:** a massless spin-0 4D particle $\hat{r}^{(0)}$

4D Theory: Effective Lagrangian

Per field content, our 4D effective Lagrangian equals...

$$\mathcal{L}^{(\text{eff})}_{h^H r^R} = \left[\frac{\kappa}{\sqrt{\pi r_c}}\right]^{(H+R-2)} \sum_{\vec{n}=\vec{0}}^{+\infty} \Big\{ \overline{a}_{(R|\vec{n})} \cdot \mathcal{K}_{(\vec{n})}\Big[\,\overline{\mathcal{L}}_{A:h^H r^R}\Big] + \overline{b}_{(R|\vec{n})} \cdot \mathcal{K}_{(\vec{n})}\Big[\,\overline{\mathcal{L}}_{B:h^H r^R}\Big] \Big\}$$

where $\mathcal{K}$ is an operator that maps 5D fields to 4D fields, and

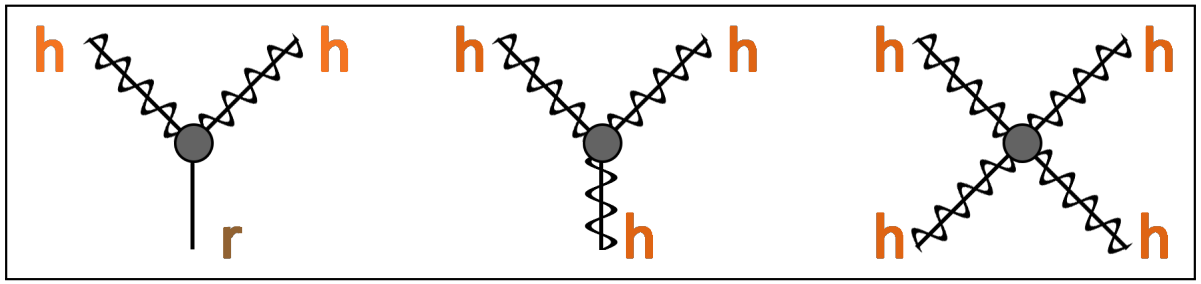
$$\overline{a}_{(R|n_1\cdots n_H)} \equiv \frac{1}{\pi r_c}\int_{-\pi r_c}^{+\pi r_c} dy\ \lambda_A(R)\ \psi^{(n_1)}\cdots\psi^{(n_H)}\left[\psi^{(0)}\right]^R$$

$$\overline{b}_{(R|n_3\cdots n_H|n_1 n_2)} \equiv \frac{r_c}{\pi}\int_{-\pi r_c}^{+\pi r_c} dy\ \lambda_B(R) \times(\partial_y\psi^{(n_1)})(\partial_y\psi^{(n_2)})\psi^{(n_3)}\cdots\psi^{(n_H)}\left[\psi^{(0)}\right]^R$$

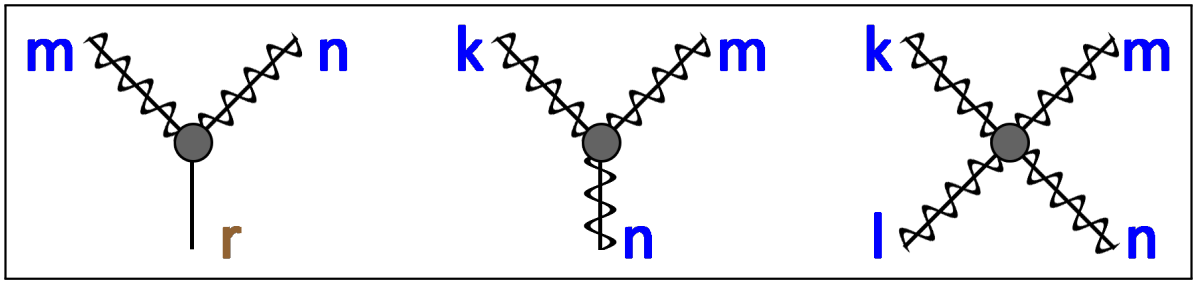
These couplings *embody all nontrivial model dependence*; they are evaluated analytically for 5DOT & numerically for RS.

5D to 4D: Interactions

$\mathcal{L}_{5\mathrm{D}}$ contains the following important vertices:

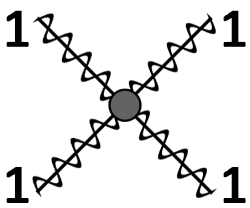


This implies that (after KK decomposition) $\mathcal{L}_{4D}^{(\mathrm{eff})}$ contains:

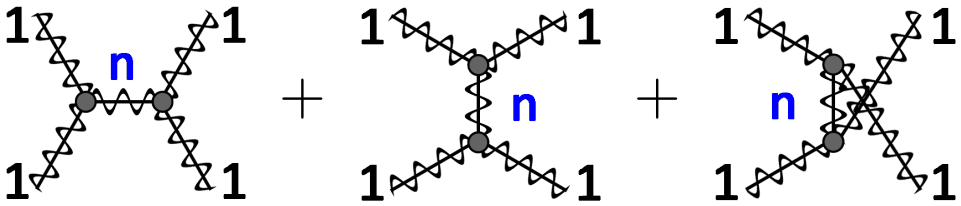


The explicit vertex rules are complicated, but once we have them, we can calculate the desired matrix elements...

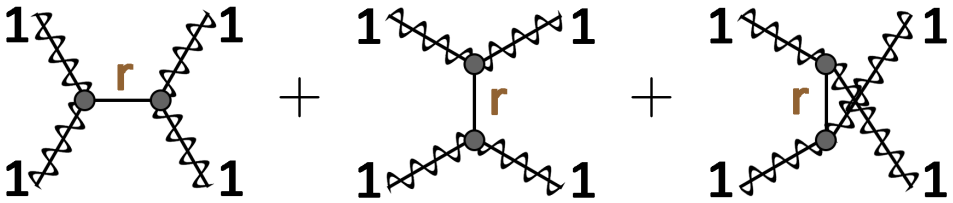
High-Energy Behavior of Longitudinal Matrix Elements

$\mathcal{M}_{sg} =$ 

$\sim \mathcal{O}(s^5)$

$\mathcal{M}_h(n) =$ 

$\sim \mathcal{O}(s^5)$

$\mathcal{M}_r =$ 

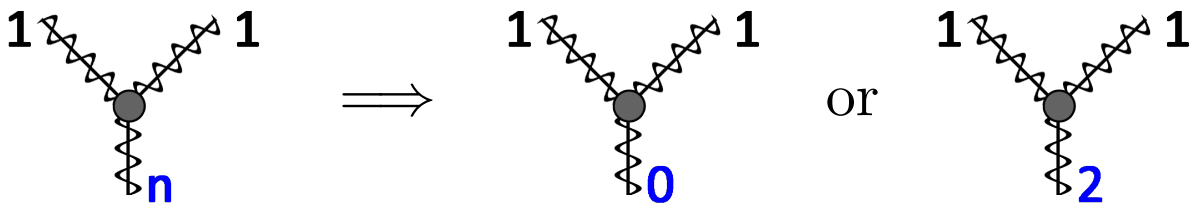
$\sim \mathcal{O}(s^3)$

We've directly confirmed these behaviors, which are consistent with the arguments of Arkani-Hamed, et. al. Define...

$$\boxed{\mathcal{M}(n_{\max}) = \mathcal{M}_{sg} + \mathcal{M}_r + \sum_{n=0}^{n_{\max}} \mathcal{M}_h(n) = \sum_k \overline{\mathcal{M}}^{(k)}(n_{\max}) \cdot s^k}$$

5DOT: Fully-Analytic Result

The 5D Orbifolded Torus has **KK number conservation:**



Therefore, we may calculate $\mathcal{M}$ exactly with a finite sum:

$$\mathcal{M} = \mathcal{M}_{sg} + \mathcal{M}_r + \mathcal{M}_h(0) + \mathcal{M}_h(2) = \sum_{k=-\infty}^{+5} \overline{\mathcal{M}}^{(k)} \cdot s^k$$

$$\overline{\mathcal{M}}^{(5)} = \overline{\mathcal{M}}^{(4)} = \overline{\mathcal{M}}^{(3)} = \overline{\mathcal{M}}^{(2)} = 0$$

$$\overline{\mathcal{M}}^{(1)} = \frac{3\kappa^2}{256\pi r_c}\left[7 + \cos(2\theta)\right]\csc^2\theta$$

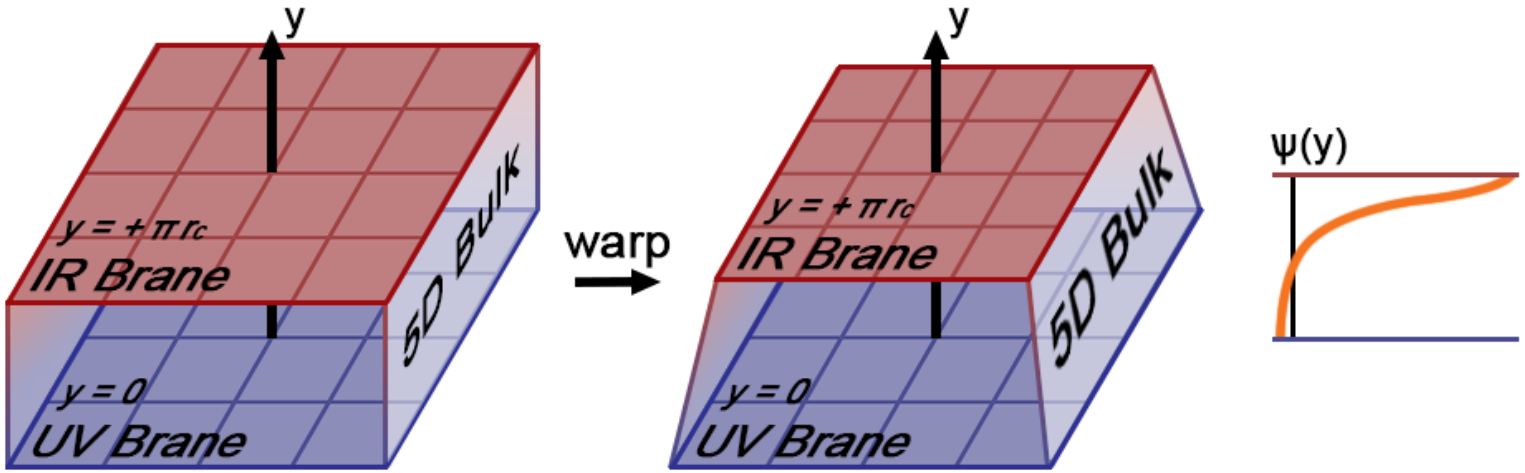
With warping, we instead have an infinite sum...

Randall-Sundrum (RS) Model

How to Build the 5D RS Model

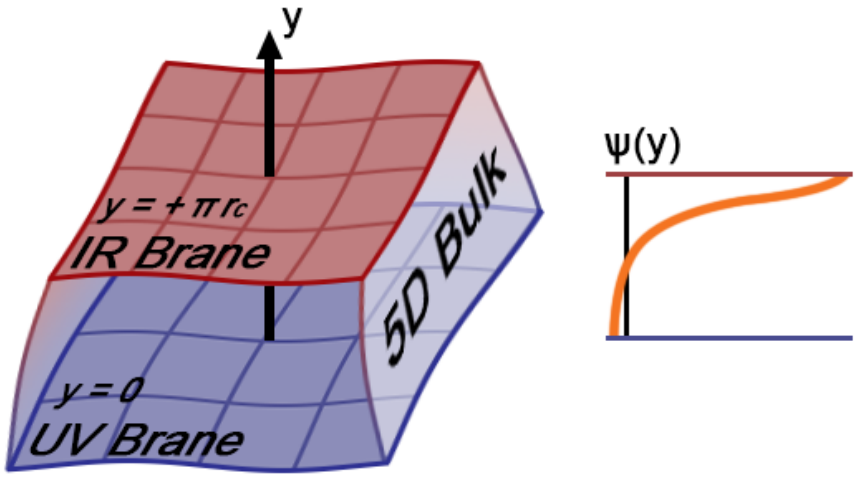
⋆ **Step 3: Warp the Extra Dimension** ⋆

- **Parameter 3:** Add a **warping parameter** k



$$\eta_{MN}^{(\mathrm{RS})} = \begin{pmatrix} e^{-2k|y|}\eta_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix} \implies \boxed{ds^2 = e^{-2k|y|}(dt^2 - d\vec{x}^{\,2}) - dy^2}$$

How to Perturb the 5D RS Model



Many options for perturbing the vacuum. The **Einstein frame parameterization** is automatically canonical in 4D:

$$G_{MN} = \begin{pmatrix} e^{-2(k|y|+\hat{u})}(\eta_{\mu\nu}+\kappa\hat{h}_{\mu\nu}) & 0 \\ 0 & -(1+2\hat{u})^2 \end{pmatrix} \qquad \hat{u} \equiv \frac{\kappa\,\hat{r}}{2\sqrt{6}} e^{+k(2|y|-\pi r_c)}$$

Eliminate 4D cosmological constant via 5D CC & tensions:

$$\star \qquad \boxed{\mathcal{L}_{5\mathrm{D}}^{(\mathrm{RS})} = \frac{2}{\kappa^2}\sqrt{\det G}\; R + \Big[\text{bulk CC} + \text{brane tensions}\Big]} \qquad \star$$

RS: WFE & Fixing Parameters

We then weak field expand $\mathcal{L}_{5D}^{(\mathrm{RS})}$, integrate the extra dimension, and KK decompose according to the procedure described earlier.

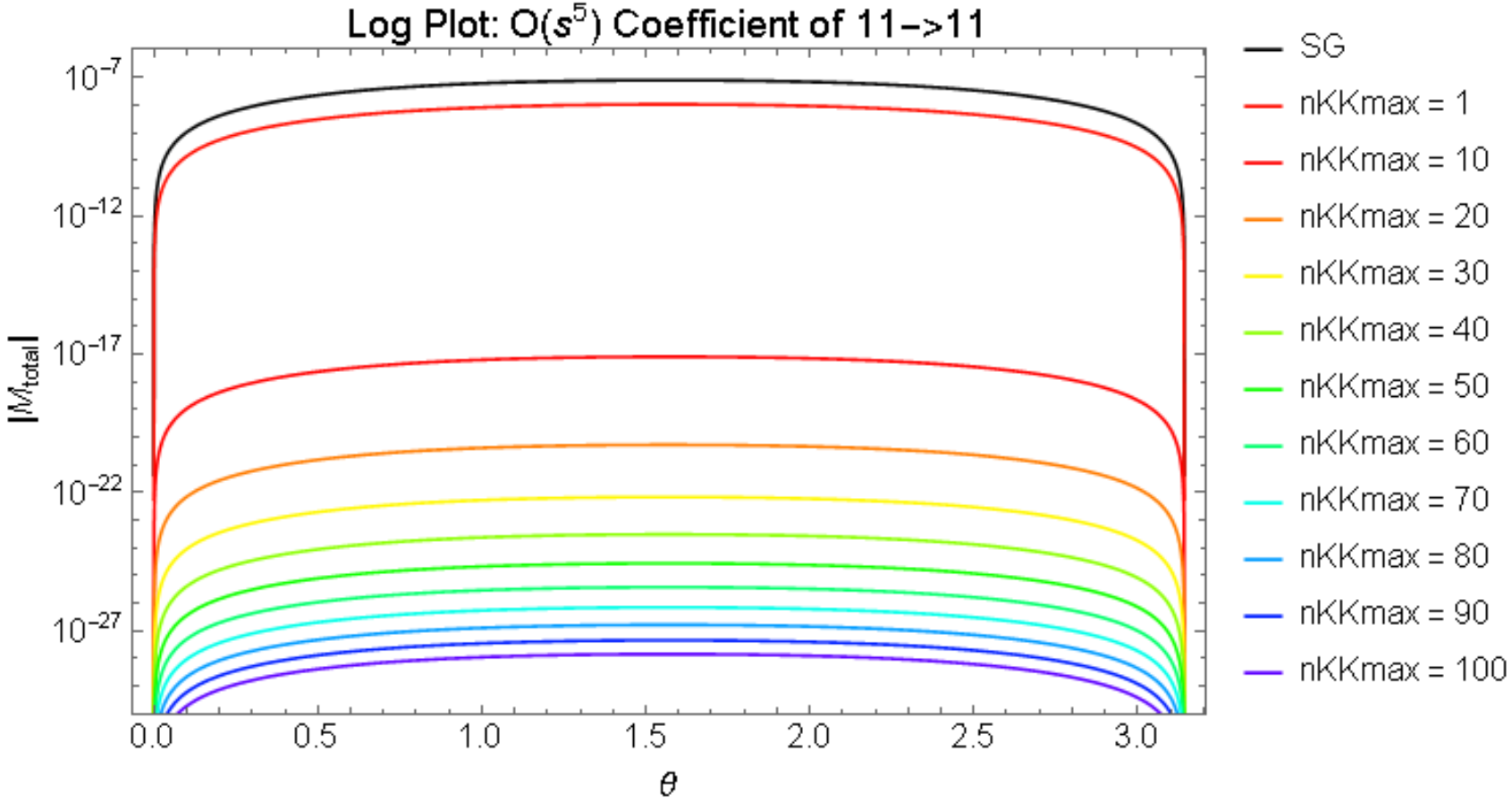
... except now we calculate couplings numerically, because of

$$\underline{\textbf{RS:}} \begin{cases} \lambda_A(R) = e^{k[2(R-1)|y| - R\pi r_c]} \\ \lambda_B(R) = e^{-2k|y|}\lambda_A(R) \end{cases}$$

In order to show you RS plots, there are 3 parameters (κ,r_c,k) we need to fix. I set...

Lightest Spin-2 Mass:	$m_1 = 1$ TeV
Unitless Parameter:	$kr_c = 9.5$
4D Planck Mass:	$M_{\mathrm{Pl}} = 2.435 \times 10^{15}$ TeV

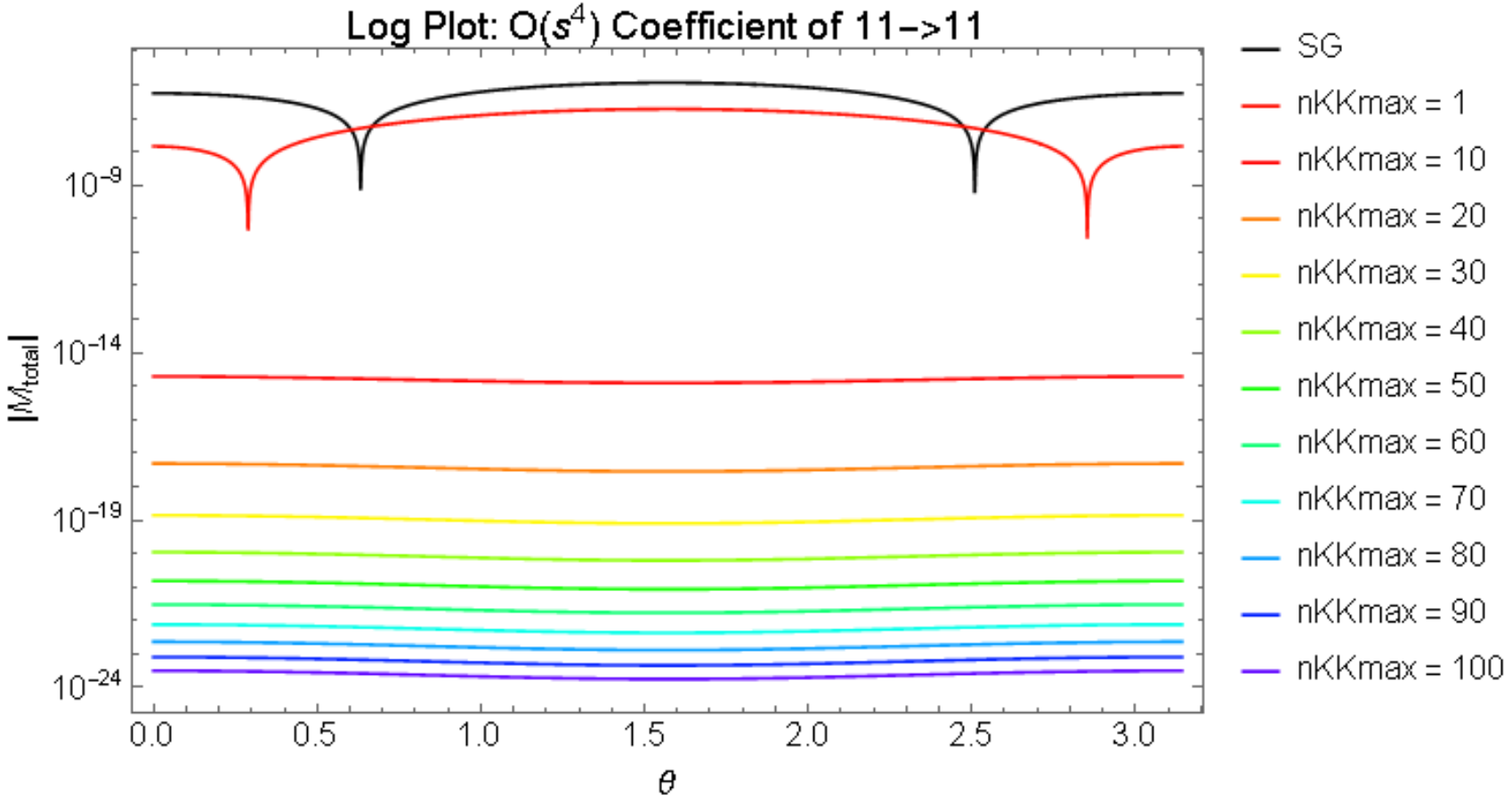
RS: Cancellations, $\mathcal{O}(s^5)$ :: $kr_c = 9.5$ & $m_1 = 1$ TeV



⋆ **As we include more modes, $\mathcal{O}(s^5)$ tends to 0...** ⋆

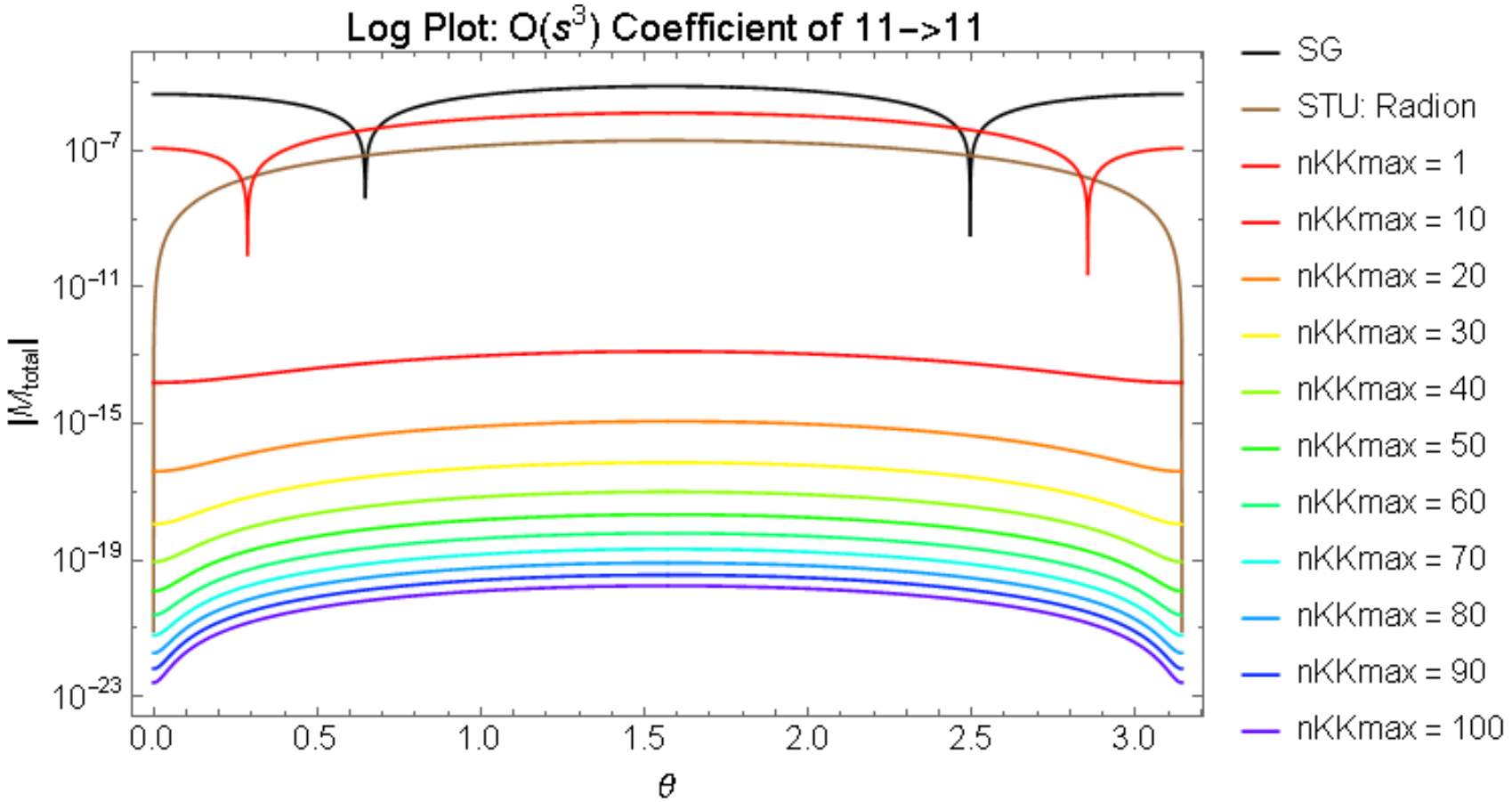
(All plots are in powers of TeV: $\mathcal{O}(s^k) \to \text{TeV}^{-2k}$)

RS: Cancellations, $\mathcal{O}(s^4)$ :: $kr_c = 9.5$ & $m_1 = 1$ TeV



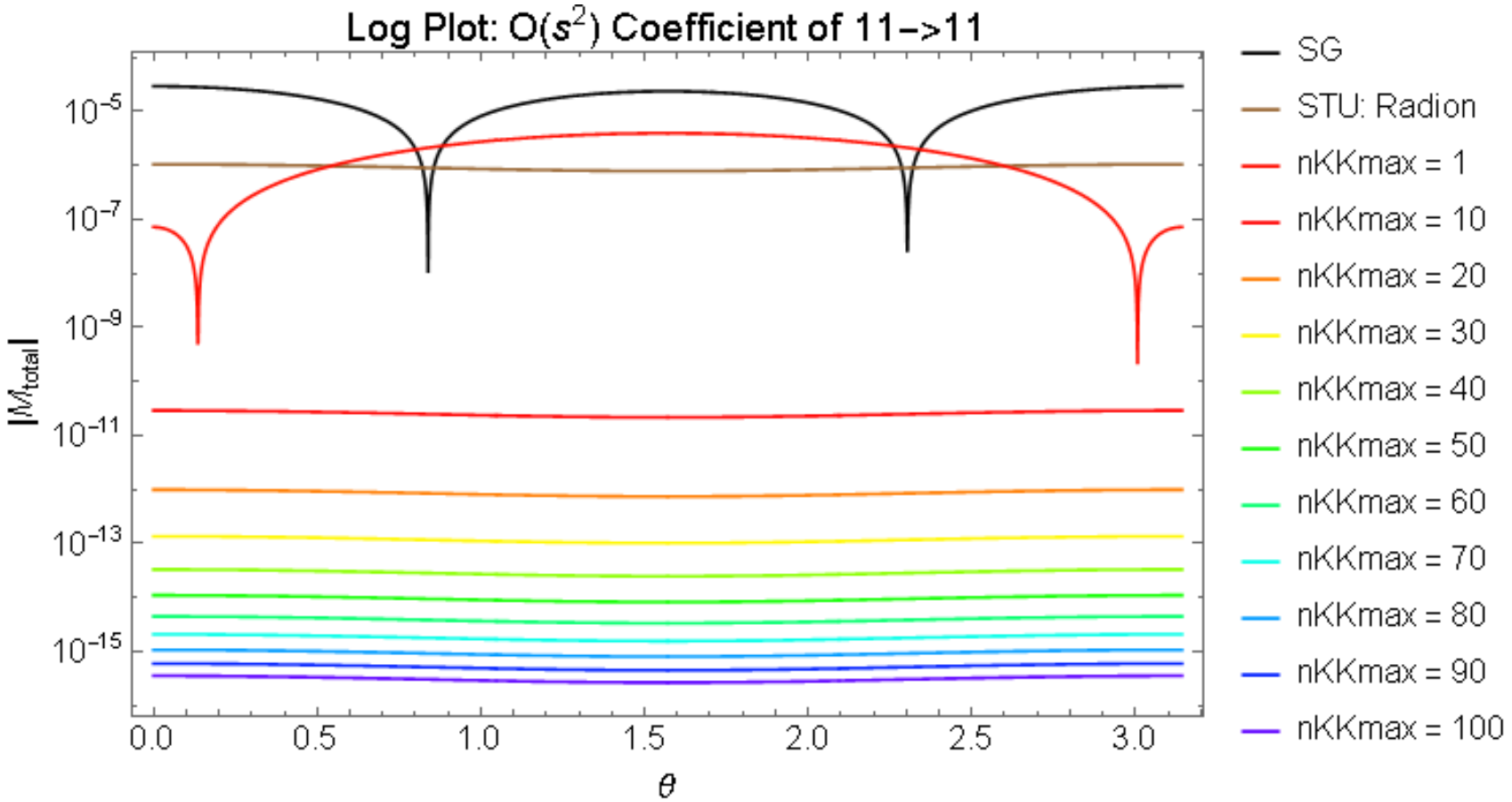
⋆ **.. and so does $\mathcal{O}(s^4)$...** ⋆

RS: Cancellations, $\mathcal{O}(s^3)$ :: $kr_c = 9.5$ & $m_1 = 1$ TeV



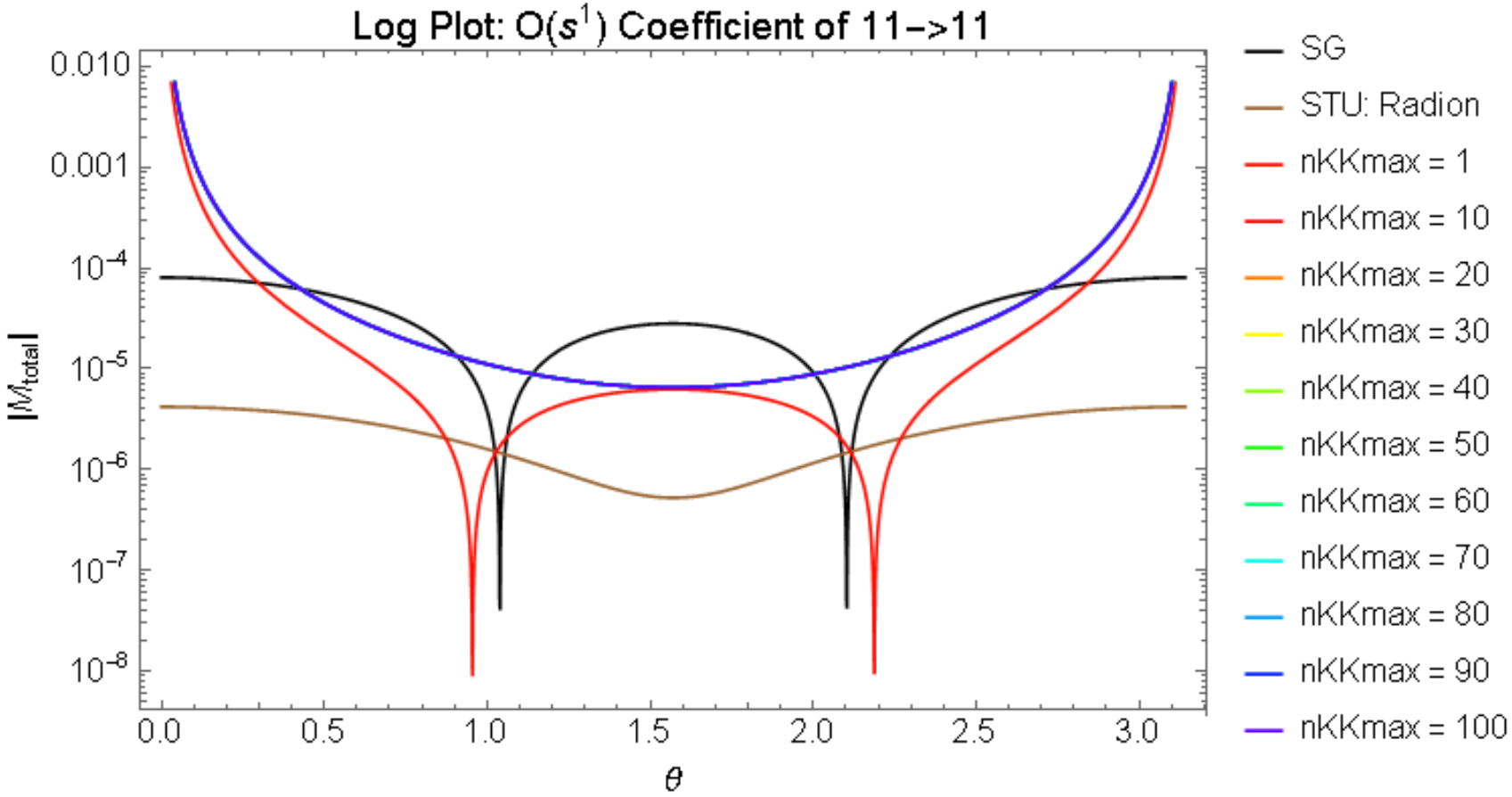
⋆ **.. and (with the radion's help) so does $\mathcal{O}(s^3)$...** ⋆

RS: Cancellations, $\mathcal{O}(s^2)$ :: $kr_c = 9.5$ & $m_1 = 1$ TeV



⋆ **.. and so does $\mathcal{O}(s^2)$...** ⋆

RS: Cancellations, $\mathcal{O}(s)$ :: $kr_c = 9.5$ & $m_1 = 1$ TeV



⋆ **.. until finally $\mathcal{O}(s)$ converges!** ⋆

Same behavior found in other KK combos (inc. $14 \to 23$) and for other values of warping ($k \neq 9.5$) too!

My Code – Available Soon!

All of the preceding calculations are facilitated by my own code. To obtain a useful 5D Lagrangian, we have to perform a weak field expansion; **performing WFE and calculating the matrix elements can be computationally intensive.**

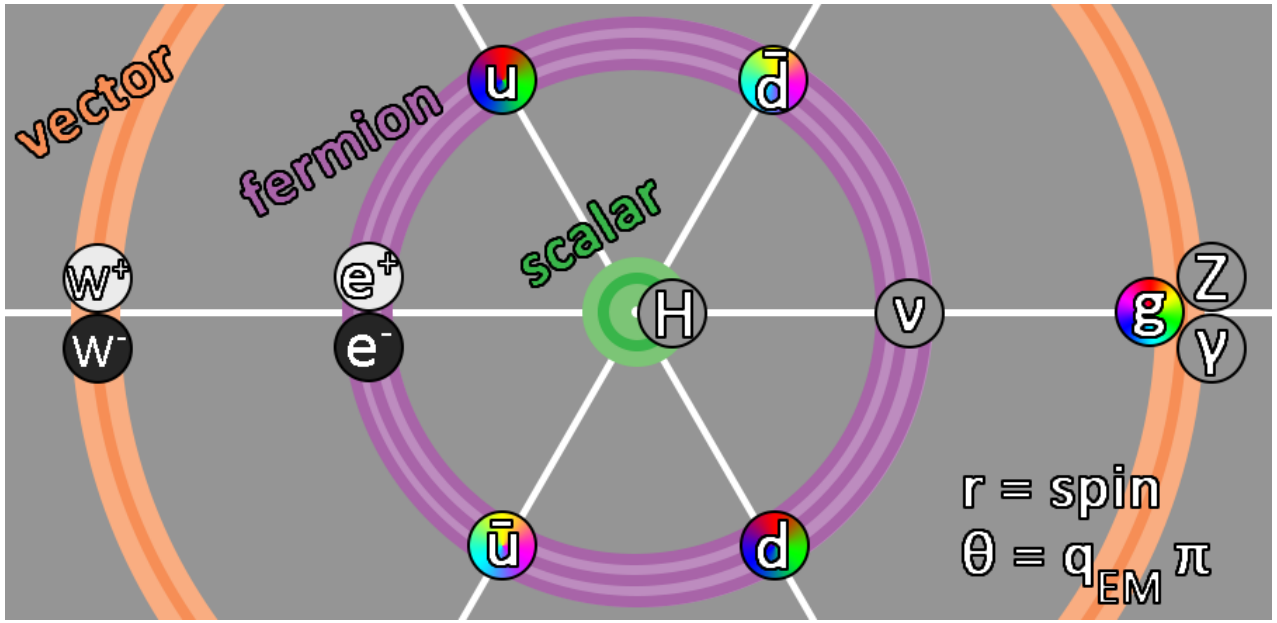
I developed a diagrammatic method to do WFE, generalized my diagrams to contain arbitrary bosonic content, and coded the whole formalism into Mathematica. **For a wide class of (4+)D metrics, my code performs WFE, IBP reduction, and calculates vertex rules in the 4D effective theory.**

Conclusion

⋆ **As promised: we find $\mathcal{M} \sim \mathcal{O}(s)$ in 5DOT & RS!** ⋆

We're wrapping up our analysis now (including unitarity bounds & implications of the RS model as an EFT). Both the analysis and my code will be available in Summer 2019.

Thank you for attending! Questions?



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