Unitarity in Extra-Dimensional Gravity Models

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Introduction
**Let’s quantize gravity:** First, weakly perturb the flat 4D metric by a **massless spin-2 field** \( h_{\mu\nu} \)...

\[
\eta_{\mu\nu} = \text{Diag}(+,-,-,-,-) \quad \mid \quad g_{\mu\nu} = \eta_{\mu\nu} + \kappa_{4D} h_{\mu\nu}
\]

**Expansion Parameter:** \( \kappa_{4D} \) has units \([\text{Energy}]^{-1}\)

- Fix by matching to Newton’s law, via \( M_{\text{Pl}} \)
- \( 2 \to 2 \) Tree-Level Unitarity breaks at energy \( \sim M_{\text{Pl}} \)

\[
\mathcal{L}^{(4DG)} = \frac{2}{\kappa_{4D}^2} \sqrt{\det g} \, R \quad \xrightarrow{\text{WFE}} \quad \mathcal{L}^{(4DG)} = \sum_{H=2}^{+\infty} \mathcal{L}_{h^H}^{(4DG)}
\]
Add a Mass = Fierz-Pauli (Massive) Gravity

Give the 4D graviton $h_{\mu\nu}$ a mass $m$ by hand:

$$\mathcal{L}^{(\text{FPG})} = \frac{2}{\kappa_{4D}^2} \sqrt{\det g} \, R + \frac{1}{2} m^2 \left[ h^2 - h_{\mu\nu} h^{\mu\nu} \right]$$

4D gravity $\mathcal{L}^{(4DG)}$  
Fierz-Pauli mass term

At high energies ($s \gg 4m^2$), helicity polarizations diverge like...

$$\epsilon_{\pm 2}^{\mu\nu} \sim \mathcal{O}(1) \quad \epsilon_{\pm 1}^{\mu\nu} \sim \mathcal{O}(\sqrt{s}) \quad \epsilon_0^{\mu\nu} \sim \mathcal{O}(s)$$

and so, 2 → 2 matrix elements grow fastest when scattering is longitudinal (helicity = 0); unitarity violated according to

$$\mathcal{M} \equiv \begin{array}{c} 1 \\ \begin{array}{c} \begin{array}{c} 1 \end{array} \\ \begin{array}{c} 1 \end{array} \end{array} \end{array} \sim \begin{cases} \text{Naive Dim. Analysis:} & \mathcal{O}(s^7) \\ \text{Explicit Calculation:} & \mathcal{O}(s^5) \end{cases}$$

Note: Symmetries automatically cancel four powers of energy! ... but $\mathcal{O}(s^5)$ implies theory breaks at $\Lambda_5 \sim (m^4 M_{\text{Pl}})^{1/5}$
Consider tree-level longitudinal (helicity = 0) scattering of a massive spin-2 particle “1” at high energies in 4D:

\[ M = \sim \mathcal{O}(s?) \] in large \( s \) regime

- **2002:** Arkani-Hamed, Georgi, and Schwartz argue that \( M \) diverges like \( \mathcal{O}(s^5) \) in Fierz-Pauli gravity, and like \( \mathcal{O}(s^3) \) in carefully tuned spin-2 theories. (hep-th/0210184)

- **2003:** Schwartz argues away \( \mathcal{O}(s^3) \) in the 5D Torus by applying a Stueckelberg-like mechanism to the Lagrangian; does not calculate matrix elements (hep-th/0303114)

- **2019:** We explicitly calculate matrix elements, and demonstrate cancellations such that \( M \sim \mathcal{O}(s) \) in 5D Orbifolded Torus and Randall-Sundrum models.

★ This is consistent with 5D expectations! ★
5D Orbifolded Torus (5DOT) Model
How to Build a 5D Orbifolded Torus

Why 5DOT? = RS model with no warping

★ Step 1: Build a 5D Torus ★

• **Parameter 1:** Add a compact spatial dimension \( y \in [0, \pi r_c] \) to 4D spacetime \( (x^\mu) \), where \( r_c \equiv \) the compactification radius

\[
\eta_{MN}^{(5DOT)} = \begin{pmatrix}
\eta_{\mu\nu} & 0 \\
0 & -1
\end{pmatrix}
\]

\[
ds^2 = (dt^2 - d\vec{x}^2) - dy^2
\]
**How to Build a 5D Orbifolded Torus**

- **Step 2: Orbifold**

**Orbifold Symmetry:** Reflect extra dimension about $y = 0$, demand $ds^2$ be invariant under $y \leftrightarrow -y$, so $y \in [-\pi r_c, +\pi r_c]$.

**Why Orbifold the Torus?**

The 5DOT is a nice limit of the RS model. Also, relative to 5D Torus, 5DOT eliminates the spin-1 **graviphoton** $\rho_{\mu}$.

*Schematically...*

$$\delta G_{MN} \approx \kappa \begin{pmatrix} h_{\mu\nu} & \rho_{\mu} \\ \rho_{\nu} & r \end{pmatrix}$$
How to Perturb a 5D Orbifolded Torus

- **Parameter 2**: 5D exp. parameter $\kappa$

$$G_{MN} = \begin{pmatrix} e^{-\kappa \hat{r}/\sqrt{6}}(\eta_{\mu\nu} + \kappa \hat{h}_{\mu\nu}) & 0 \\ 0 & -(1+\hat{r}/\sqrt{6})^2 \end{pmatrix}$$

where $[\kappa] = [\text{Energy}]^{-3/2}$ (implies $\kappa_{AD}$)

**Particles in 5D Matter-Free Orbifolded Torus:**

- **5D Graviton** = $\hat{h}_{\mu\nu}$, a massless spin-2 5D particle
  ▶ **Origin**: local coordinate invariance of constant $y$ sheets

- **5D Radion** = $\hat{r}$, a massless spin-0 5D particle
  ▶ **Origin**: locally perturbing distance between branes

**Caution**: In realistic theories, radion requires additional external stabilization, e.g. Goldberger-Wise mechanism; radion stabilization plays no role in the present analysis.
5D to 4D: Organizing the 5D Lagrangian

Define $\mathcal{L}_{h^H r^R} \equiv \text{all } \mathcal{L}_{5D} \text{ terms with } H \text{ gravitons and } R \text{ radions:}$

$$\mathcal{L}_{5D} = \sum_{H,R} \mathcal{L}_{h^H r^R} \equiv \sum_{H,R} (\cdots) \hat{h}^H_{\hat{\mu}} \hat{r}^R$$

By construction, each term in this set is either

- **A-Type**: has two 4D derivatives $\partial_\mu \partial_\nu$, or
- **B-Type**: has two extra-dimensional derivatives $\partial_y^2$

\[
\mathcal{L}_{h^H r^R} = \kappa^{(H+R-2)} \left[ \lambda_A(R) \mathcal{L}_{A:h^H r^R} + \lambda_B(R) \mathcal{L}_{B:h^H r^R} \right]
\]

**5D to 4D**: Once we have a 5D WFE theory, we convert it into an effective 4D theory via $y$-integration

$$S = \int d^4x \left( \int dy \; \mathcal{L}_{5D} \right) \implies \mathcal{L}^{(\text{eff})}_{4D} \equiv \int_{-\pi r_c}^{+\pi r_c} dy \; \mathcal{L}_{5D}$$
5D to 4D: Kaluza-Klein (KK) Decomposition

\[ \hat{f}_{\mu}(x, y) = \frac{1}{\sqrt{\pi r_c}} \sum_{n=0}^{+\infty} \hat{f}_{\mu}^{(n)}(x) \psi^{(n)}(y) \]

We utilize extra-dimensional wavefunctions \( \psi^{(n)}(y) \) that satisfy

\[
\begin{cases}
\frac{1}{\pi r_c} \int_{-\pi r_c}^{+\pi r_c} dy \lambda_A(0) \psi^{(m)}(y) \psi^{(n)} = \delta_{m,n} \\
\frac{1}{\pi r_c} \int_{-\pi r_c}^{+\pi r_c} dy \lambda_B(0) (\partial_y \psi^{(m)})(\partial_y \psi^{(n)}) = m_n^2 \delta_{m,n}
\end{cases}
\]

where \( n \equiv \text{KK number} \). Given \( R \) radion fields, we define...

**5DOT:** \[
\begin{align*}
\lambda_A(R) &= 1 \\
\lambda_B(R) &= 1
\end{align*}
\]

**RS:** \[
\begin{align*}
\lambda_A(R) &= e^{k[2(R-1)|y|-R\pi r_c]} \\
\lambda_B(R) &= e^{-2k|y|} \lambda_A(R)
\end{align*}
\]
5D to 4D: KK Decomposition of Quadratic Spin-2

\[ \mathcal{L}_{hh} = \lambda_A(0) \overline{\mathcal{L}}_{A:hh} + \lambda_B(0) \overline{\mathcal{L}}_{B:hh} \]

\[ \overline{\mathcal{L}}_{A:hh} = -\hat{h}_{\mu\nu}(\partial^\mu \partial^\nu \hat{h}) + \hat{h}_{\mu\nu}(\partial^\mu \partial_\rho \hat{h}^{\rho\nu}) - \frac{1}{2} \hat{h}_{\mu\nu}(\Box \hat{h}^{\mu\nu}) + \frac{1}{2} \hat{h}(\Box \hat{h}) \]

\[ \overline{\mathcal{L}}_{B:hh} = -\frac{1}{2} (\partial_y \hat{h}_{\mu\nu})(\partial_y \hat{h}^{\mu\nu}) + \frac{1}{2} (\partial_y \hat{h})^2 \]

such that the 4D effective spin-2 Lagrangian is canonical:

\[ \mathcal{L}_{hh}^{(\text{eff})} = \mathcal{L}_{\text{Kin}}^{(S=2)}(\hat{h}^{(0)}) + \sum_{n=1}^{+\infty} \mathcal{L}_{\text{FP}}(m_n, \hat{h}^{(n)}) \]

5D Graviton \( \hat{h}_{\mu\nu} \) becomes many 4D particles:

- **4D Graviton**: a massless spin-2 4D particle \( \hat{h}_{\mu\nu}^{(0)} \)
- **KK Modes**: massive spin-2 4D particles \( \hat{h}_{\mu\nu}^{(n)} \) for \( n > 0 \)
Similarly, we may decompose the 5D radion:

$$\mathcal{L}_{rr} = \lambda_A(2) \overline{\mathcal{L}}_{A:rr}$$

where

$$\overline{\mathcal{L}}_{A:rr} = \frac{1}{2} (\partial_\mu \hat{r})(\partial^\mu \hat{r})$$

5D diffeomorphism invariance may be used to ensure the 5D radion is flat in the extra dimension (e.g. $\hat{r}(x, y) \rightarrow \hat{r}(x)$). Thus, its KK decomposition has only a zero mode:

$$\hat{r}(x) = \frac{1}{\sqrt{\pi r_c}} \hat{r}^{(0)}_\mu (x) \psi^{(0)} \quad \implies \quad \mathcal{L}^{(\text{eff})}_{rr} = \mathcal{L}^{(S=0)}_{\text{Kin}}(\hat{r}^{(0)})$$

5D Radion $\hat{r}$ becomes a single 4D particle:

- 4D Radion: a massless spin-0 4D particle $\hat{r}^{(0)}$
Per field content, our 4D effective Lagrangian equals...

\[ \mathcal{L}_{hh^H r^R}^{(\text{eff})} = \left[ \frac{\kappa}{\sqrt{\pi r_c}} \right] (H + R - 2) \sum_{\vec{n} = \vec{0}}^{+\infty} \left\{ \bar{a}(R|\vec{n}) \cdot \mathcal{K}(\vec{n}) \left[ \overline{\mathcal{L}}_{A:h^H r^R} \right] \right\} + \bar{b}(R|\vec{n}) \cdot \mathcal{K}(\vec{n}) \left[ \overline{\mathcal{L}}_{B:h^H r^R} \right] \]

where \( \mathcal{K} \) is an operator that maps 5D fields to 4D fields, and

\[ \bar{a}(R|n_1\ldots n_H) \equiv \frac{1}{\pi r_c} \int_{-\pi r_c}^{+\pi r_c} dy \ \lambda_A(R) \psi^{(n_1)} \ldots \psi^{(n_H)} \psi^{(0)} \]^

\[ \bar{b}(R|n_3\ldots n_H|n_1 n_2) \equiv \frac{r_c}{\pi} \int_{-\pi r_c}^{+\pi r_c} dy \ \lambda_B(R) \]

\[ \times (\partial_y \psi^{(n_1)})(\partial_y \psi^{(n_2)})\psi^{(n_3)} \ldots \psi^{(n_H)} \psi^{(0)} \]^

These couplings embody all nontrivial model dependence; they are evaluated analytically for 5DOT & numerically for RS.
5D to 4D: Interactions

\( \mathcal{L}_{5D} \) contains the following important vertices:

\[
\begin{align*}
  &\text{h} &\text{h} &\text{h} \\
  &\text{h} &\text{h} &\text{h} \\
  &\text{h} &\text{h} &\text{h}
\end{align*}
\]

This implies that (after KK decomposition) \( \mathcal{L}^{(\text{eff})}_{4D} \) contains:

\[
\begin{align*}
  &\text{m} &\text{n} &\text{k} \\
  &\text{k} &\text{m} &\text{n} \\
  &\text{m} &\text{l} &\text{n}
\end{align*}
\]

The explicit vertex rules are complicated, but once we have them, we can calculate the desired matrix elements...
High-Energy Behavior of Longitudinal Matrix Elements

\[
M_{sg} = \sim \mathcal{O}(s^5)
\]

\[
M_h(n) = \sim \mathcal{O}(s^5)
\]

\[
M_r = \sim \mathcal{O}(s^3)
\]

We’ve directly confirmed these behaviors, which are consistent with the arguments of Arkani-Hamed, et. al. Define...

\[
M(n_{\text{max}}) = M_{sg} + M_r + \sum_{n=0}^{n_{\text{max}}} M_h(n) = \sum_{k} \overline{M}^{(k)}(n_{\text{max}}) \cdot s^k
\]
The 5D Orbifolded Torus has **KK number conservation**:

\[ \begin{align*}
\begin{array}{c}
1 \\
\downarrow \quad \uparrow \\
n & \Rightarrow \\
1 \\
\downarrow \quad \uparrow \\
1 & 0 \\
\uparrow \quad \downarrow \\
1 \\
\downarrow \quad \uparrow \\
1 & 2
\end{array}
\end{align*} \]

or

Therefore, we may calculate \( M \) exactly with a finite sum:

\[ M = M_{sg} + M_r + M_h(0) + M_h(2) = \sum_{k=-\infty}^{+5} \overline{M}^{(k)} \cdot s^k \]

\[ \overline{M}^{(5)} = \overline{M}^{(4)} = \overline{M}^{(3)} = \overline{M}^{(2)} = 0 \]

\[ \overline{M}^{(1)} = \frac{3 \kappa^2}{256 \pi r_c} [7 + \cos(2\theta)] \csc^2 \theta \]

With warping, we instead have an infinite sum...
Randall-Sundrum (RS) Model
How to Build the 5D RS Model

⋆ **Step 3: Warp the Extra Dimension** ⋆

- **Parameter 3:** Add a warping parameter $k$

\[
\eta^{(RS)}_{MN} = \begin{pmatrix}
    e^{-2k|y|} & 0 \\
    0 & -1
\end{pmatrix} \eta_{\mu\nu} \quad \Rightarrow \quad ds^2 = e^{-2k|y|}(dt^2 - d\vec{x}^2) - dy^2
\]
How to Perturb the 5D RS Model

Many options for perturbing the vacuum. The **Einstein frame parameterization** is automatically canonical in 4D:

\[
G_{MN} = \begin{pmatrix}
  e^{-2(k|y|+\hat{u})} (\eta_{\mu\nu} + \kappa \hat{h}_{\mu\nu}) & 0 \\
  0 & -(1+2\hat{u})^2
\end{pmatrix}
\]

\[
\hat{u} \equiv \frac{\kappa \hat{r}}{2\sqrt{6}} e^{+k(2|y|−\pi r_c)}
\]

Eliminate 4D cosmological constant via 5D CC & tensions:

\[
* \quad \mathcal{L}_{5D}^{(RS)} = \frac{2}{\kappa^2} \sqrt{\det G} \ R + \left[ \text{bulk CC + brane tensions} \right] *
\]
RS: WFE & Fixing Parameters

We then weak field expand $\mathcal{L}_{5D}^{(RS)}$, integrate the extra dimension, and KK decompose according to the procedure described earlier.

... except now we calculate couplings numerically, because of

$$\begin{align*}
\text{RS:} & \\
\lambda_A(R) &= e^{k[2(R-1)|y| - R\pi r_c]} \\
\lambda_B(R) &= e^{-2k|y|} \lambda_A(R)
\end{align*}$$

In order to show you RS plots, there are 3 parameters ($\kappa, r_c, k$) we need to fix. I set...

<table>
<thead>
<tr>
<th>Lightest Spin-2 Mass:</th>
<th>$m_1 = 1 \text{ TeV}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unitless Parameter:</td>
<td>$k r_c = 9.5$</td>
</tr>
<tr>
<td>4D Planck Mass:</td>
<td>$M_{Pl} = 2.435 \times 10^{15} \text{ TeV}$</td>
</tr>
</tbody>
</table>
RS: Cancellations, $\mathcal{O}(s^5) :: kr_c = 9.5 \& m_1 = 1 \text{ TeV}$

\[ O(s^5) \rightarrow \text{TeV}^{-2k} \]

- As we include more modes, $O(s^5)$ tends to 0...

(All plots are in powers of TeV: $O(s^k) \rightarrow \text{TeV}^{-2k}$)
RS: Cancellations, $O(s^4) :: kr_c = 9.5 \& m_1 = 1 \text{ TeV}$

\[ \star \text{.. and so does } O(s^4)\ldots \star \]
RS: Cancellations, $O(s^3)$ :: $kr_c = 9.5$ & $m_1 = 1$ TeV

⋆ ⋯ and (with the radion’s help) so does $O(s^3)$⋯ ⋆
RS: Cancellations, $\mathcal{O}(s^2) :: kr_c = 9.5 \& m_1 = 1 \text{ TeV}$

* .. and so does $\mathcal{O}(s^2)$... *
RS: Cancellations, $\mathcal{O}(s) :: kr_c = 9.5$ & $m_1 = 1$ TeV

$\star \ldots$ until finally $\mathcal{O}(s)$ converges! $\star$

Same behavior found in other KK combos (inc. 14 $\rightarrow$ 23) and for other values of warping ($k \neq 9.5$) too!
All of the preceding calculations are facilitated by my own code. To obtain a useful 5D Lagrangian, we have to perform a weak field expansion; performing WFE and calculating the matrix elements can be computationally intensive.

I developed a diagrammatic method to do WFE, generalized my diagrams to contain arbitrary bosonic content, and coded the whole formalism into Mathematica. For a wide class of (4+)D metrics, my code performs WFE, IBP reduction, and calculates vertex rules in the 4D effective theory.
As promised: we find $\mathcal{M} \sim \mathcal{O}(s)$ in 5DOT & RS!

We’re wrapping up our analysis now (including unitarity bounds & implications of the RS model as an EFT). Both the analysis and my code will be available in Summer 2019.

Thank you for attending! Questions?

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