The Stueckelberg Superfield in Supergravity

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based on work done in collaboration with:
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Making contact between supergravity and nature

- **Problem:** (unbroken) supersymmetry doesn't allow de Sitter solutions (i.e. positive cosmological constant).

Easy to see from algebra

- Attempt to embed (A)dS algebra

$$[P_\mu, P_\nu] = s\frac{1}{4L^2}M_{\mu\nu}, \quad \text{with} \quad s = \begin{cases} -1 & \text{dS} \\ +1 & \text{AdS} \end{cases}$$

into SUSY algebra:

$$[P_\mu, Q_\alpha] = \frac{1}{4L}(\gamma_\mu Q)_\alpha$$
$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}(\gamma^\mu)_{\alpha\beta} P_\mu - \frac{1}{8L}(\gamma^{\mu\nu})_{\alpha\beta} M_{\mu\nu}$$

then Jacobi identity $[P_\mu, P_\nu, Q] = 0$ fixes

$$s = +1$$

Making contact between supergravity and nature

- **Problem:** (unbroken) supersymmetry doesn't allow de Sitter solutions (i.e. positive cosmological constant).
- **Solution:**SUSY is broken on dS solution.
- Requires the introduction of the *goldstino*, on which supersymmetry is realised non-linearly.
- Very difficult to achieve in pure supergravity theory.

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Constrained superfields

- Would like to incorporate the goldstino into a superfield (or supermultiplet), such that we can make full use of the machinery of SUSY/sugra, make contact with $\mathcal{N} > 1$, etc.
- Proposed solution is that the goldstino is part of a *nilpotent supefield X* [Rocek '78,Lindstrom, Rocek '79,Samuel, Wess '83,Casalbuoni,De Curtis,Dominici,Feruglio,Gatto '89,Komargodski, Seiberg '09,Kuzenko, Tyler '11]

$$X = \varphi + \sqrt{2}\theta G + \theta^2 F^X$$

$$X^2 = 0 \quad \Rightarrow \quad \varphi = \frac{G^2}{2F^X}$$

Constrained superfields

- Couple to supergravity via superpotential

$$\mathcal{W} = a(X^0)^3 + b(X^0)^2 X$$

with a and b arbitrary constants and X^0 is the compensator. This gives solutions with cosmological constant and gravitino mass

$$\Lambda \propto |b|^2 - |a|^2, \qquad m_\psi \propto a$$

- Much recent activity related to dS solutions in supergravity[Bandos,Bergshoeff,Dudas,Farakos,Ferrara,Freedman,Hasegawa,Kallosh,Kehagias,Porrati,Van Proeyen,Sagnotti,Scalisi,Yamada,...'14-18], inflationary models[Dall Agata,Antoniadis,Dudas,Farakos,Ferrara,Kahn,Kallosh,Linde,Roberts,Sagnotti,Thaler,Zavala, Zwirner,...'14-18], brane models[Angelantonj,Antoniadis,Dudas,Mourad, Pradisi,Riccioni,Sagnotti,Uranga,Vercnocke,Zavala...'99-'18]

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Stueckelberg: gentle introduction

- Start with a Lagrangian with an explicitly broken symmetry:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} A_\mu A^\mu$$

where the gauge field A_μ lacks the usual gauge symmetry $\delta A_\mu = \partial_\mu \alpha$ due to the mass term. This can be restored by introducing an extra scalar field s, and writing

$$A_\mu = \tilde{A}_\mu + \partial_\mu s, \quad \begin{cases} \delta \tilde{A}_\mu = \partial_\mu \alpha \\ \delta s = -\alpha \end{cases}$$

and plugging back into the Lagrangian

$$\mathcal{L} = -\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{m^2}{2} \partial_\mu s \partial^\mu s - \frac{m^2}{2} \left(\tilde{A}_\mu \tilde{A}^\mu + 2 A_\mu \partial^\mu s \right)$$

- The procedure can be interpreted as performing a gauge transformation on the field and promoting the parameter to a field (the Stueckelberg field).

Unimodular gravity

- Original formulation of unimodular gravity

$$S = \int d^4x \left[\sqrt{-g}R - \Lambda\left(\sqrt{-g} - \epsilon_0\right)\right]$$

where Λ is a Lagrange multiplier field which imposes the unimodularity condition

$$\sqrt{-g} = \epsilon_0$$

with ϵ_0 being a constant (which could be set to 1 if we want). This version of the Lagrangian is not invariant under the full group of diffeomorphisms, but only under a subgroup of transformations calledd TDiffs, whose parameter satisfies $\nabla_\mu \xi^\mu = 0$. It is obvious that the problematic term is

$$\int d^4x \Lambda \epsilon_0$$

where Λ transforms as a scalar.

Unimodular gravity with Stueckelberg

- One can restore the full general coordinate invariance through the Stueckelberg trick of introducing an extra field which transforms appropriately. We do this by performing a g.c.t. $x^\mu \to y^\mu(x)$ on the problematic term

$$\int d^4x \Lambda(x)\epsilon_0 \qquad \to \qquad \int d^4y \Lambda'(y)\epsilon_0 = \int d^4x |J| \Lambda(x)\epsilon_0$$

we note that the Jacobian can be rewritten as a total derivative

$$\begin{aligned} |J| &= -\tfrac{1}{4!}\varepsilon^{\alpha\beta\gamma\delta} J^\mu_\alpha J^\nu_\beta J^\rho_\gamma J^\sigma_\delta \varepsilon_{\mu\nu\rho\sigma} \\ &= -\tfrac{1}{4!}\varepsilon^{\alpha\beta\gamma\delta} \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta}\frac{\partial y^\rho}{\partial x^\gamma}\frac{\partial y^\sigma}{\partial x^\delta}\varepsilon_{\mu\nu\rho\sigma} \\ &= \frac{\partial}{\partial x^\alpha}\left[-\tfrac{1}{4!}\varepsilon^{\alpha\beta\gamma\delta} y^\mu \frac{\partial y^\nu}{\partial x^\beta}\frac{\partial y^\rho}{\partial x^\gamma}\frac{\partial y^\sigma}{\partial x^\delta}\varepsilon_{\mu\nu\rho\sigma}\right] \end{aligned}$$

Unimodular gravity with Stueckelberg

- We define the new fields as

$$\phi^\alpha \equiv -\frac{1}{4!}\varepsilon^{\alpha\beta\gamma\delta} y^\mu \frac{\partial y^\nu}{\partial x^\beta}\frac{\partial y^\rho}{\partial x^\gamma}\frac{\partial y^\sigma}{\partial x^\delta}\varepsilon_{\mu\nu\rho\sigma}$$

and then the new action

$$S = \int d^4x \left[\sqrt{-g}R - \Lambda\left(\sqrt{-g} - \epsilon_0 \partial_\mu \phi^\mu\right)\right]$$

is invariant under the full group of g.c.t. We can see this by noting that $\partial^\mu \tau_\mu$ transforms as a scalar density (i.e. the same way as $\sqrt{-g}$)- note that ϕ^μ transform as scalars. It is also possible to perform the Stueckelberg trick infinitessimally

$$x^\mu \to x^\mu + f^\mu$$

and work order by order in f^μ.

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Superspace

- Extend ST to superspace $(x^\mu, \theta, \bar{\theta})$; work with superfields $\Phi(x^\mu, \theta, \bar{\theta})$
- SUSY transformations cast as coordinate transformations in superspace:

$$x^\mu \to x^\mu + i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}$$
$$\theta \to \theta + \epsilon$$
$$\bar{\theta} \to \bar{\theta} + \bar{\epsilon}$$

- Note that SUSY transformations are only a *subset* of the full g.c.t. in superspace !

(Global) Chiral superspace

- Often useful to work with chiral superfields

$$\bar{D}_{\dot{a}}\Phi = 0$$

where we defined

$$D_a = \frac{\partial}{\partial\theta^a} + i\sigma^\mu_{a\dot{a}}\bar{\theta}^{\dot{a}}\partial_\mu, \qquad \bar{D}_{\dot{a}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{a}}} + i\theta^a\sigma^\mu_{a\dot{a}}\partial_\mu$$

The components of the chiral multiplet are then obtained by projection:

$$A = \Phi|_{\theta=\bar{\theta}=0}$$
$$\chi_a = D_a\Phi|_{\theta=\bar{\theta}=0}$$
$$F = D^2\Phi|_{\theta=\bar{\theta}=0}$$

(Global) Chiral superspace

- Convenient to do a change of coordinates

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$$

which allows us to write the chiral superfield as

$$\Phi = A(y) + \sqrt{2}\theta\chi(y) + F\theta^2$$

Note

$$D_a = \frac{\partial}{\partial\theta^a} + 2i\sigma^\mu_{a\dot{a}}\bar{\theta}^{\dot{a}}\frac{\partial}{\partial y^\mu}, \qquad \bar{D}_{\dot{a}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{a}}}$$

Chiral superfields in curved space

- Generalise the flat space definition

$$\mathcal{D}_{\dot{a}}\Phi = 0$$

where $\mathcal{D}_{\dot{a}}$ is now a covariant derivative (inducing a coupling to the supergravity multiplet). Again, we can project:

$$A = \Phi|_{\theta=\bar{\theta}=0}$$
$$\chi_a = \frac{1}{\sqrt{2}}\mathcal{D}_a\Phi|_{\theta=\bar{\theta}=0}$$
$$F = -\frac{1}{4}\mathcal{D}^2\Phi|_{\theta=\bar{\theta}=0}$$

Chiral superfields in curved space

- The chiral superfield transforms as

$$\delta\phi = -\xi^A \mathcal{D}_A \Phi$$

with

$$\xi^\mu(X) = 2i\left[\theta\sigma^\mu\bar{\xi}(x) - \xi(x)\sigma^\mu\bar{\theta}\right]$$
$$\xi^a(X) = \xi^a(x)$$
$$\bar{\xi}^{\dot{a}}(X) = \bar{\xi}^{\dot{a}}(x)$$

This can be thought of as the superspace generalisation of a scalar field, with coordinate transformations

$$x^\mu \to x^\mu + \xi^\mu(x), \quad \theta^a \to \theta^a + \xi^a(x), \quad \bar{\theta}^{\dot{a}} \to \bar{\theta}^{\dot{a}} + \bar{\xi}^{\dot{a}}(x)$$

Chiral superspace

- It is possible to define chiral coordinates in curved superspace such that

$$\Phi = A + \sqrt{2}\Theta\chi + \Theta^2 F$$

where the Θ variables now carry local Lorenz, rather than Einstein indices. In these coordinates, the transformation is given by

$$\delta\Phi = \eta^M(x,\Theta)\partial_M\Phi$$

with

$$\begin{aligned}
\eta^\mu =& 2i\Theta\sigma^\mu\bar{\epsilon} + \bar{\psi}_\nu\bar{\sigma}^\mu\sigma^\nu\bar{\epsilon}\Theta^2 \\
\eta^m =& \epsilon^m - i\Theta\sigma^\mu\bar{\epsilon}\psi_\mu^m + \Theta^2\Big[\tfrac{1}{3}M^*\epsilon^m + \tfrac{1}{6}b_\alpha\left(\varepsilon\sigma^\alpha\bar{\epsilon}\right)^m \\
& - i\omega_\mu^{mn}\left(\sigma^\mu\bar{\epsilon}\right)_n - \tfrac{1}{2}\psi_\nu^a\left(\bar{\psi}_\mu\bar{\sigma}^\nu\sigma^\mu\bar{\epsilon}\right)\Big]
\end{aligned}$$

Chiral superspace

- The fat thetas allow us to construct invariant actions in supergravity. We define a chiral density superfield Δ via its transformation property

$$\delta\Delta = -\partial_M\left[(-1)^M\eta^M\Delta\right]$$

- It naturally generalises scalar densities (such as $\sqrt{-g}$).
- The product of a chiral density with a chiral superfield is a chiral density

$$\begin{aligned}\delta\Delta\Phi &= -\partial_M\left[(-1)^M\eta^M\Delta\right]\Phi - \Delta\eta^M\partial_M\Phi \\ &= -\partial_M\left[(-1)^M\eta^M\Delta\Phi\right]\end{aligned}$$

Chiral superspace

- We can thus construct invariant actions

$$\delta S = \delta \int d^4x d^2\Theta \Delta g(\Phi)$$
$$= - \int d^4x d^2\Theta \partial_M \left[(-1)^M \eta^M \Delta g\left(\Phi\right) \right]$$

  with g a chiral function of Φ.

- There exists a special chiral density

$$\mathcal{E} = a \; + \; \sqrt{2}\Theta\rho \; + \; \Theta\Theta f, \qquad \text{with}$$
$$a = \frac{1}{2}e$$
$$\rho = \frac{i\sqrt{2}}{4} e\sigma^\mu \bar{\psi}_\mu$$
$$f = -\frac{1}{2}eM^* - \frac{1}{8}e\bar{\psi}_\mu \left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)\bar{\psi}_\nu$$

Chiral superspace

- Then we can write the pure supergravity action as

$$S = \int d^4x d^2\Theta \mathcal{E} R + h.c.$$

with R a chiral superfield with components

$$R| = -\frac{1}{6} M$$
$$\mathcal{D}_a R| = -\frac{1}{6}\left(\sigma^\alpha \bar{\sigma}^\beta \psi_{\alpha\beta} + i b^\alpha \psi_\alpha - i \sigma^\alpha \bar{\psi}_\alpha M\right)_a$$
$$\mathcal{D}^2 R| = -\frac{1}{3} e_a^\mu e_b^\nu \mathcal{R}_{\mu\nu}{}^{ab} + \frac{2}{3} i \bar{\psi}^\mu \bar{\sigma}^\nu \psi_{\mu\nu} + \dots$$

Chiral superspace

- Then we can write the pure supergravity action as

$$S = \int d^4 x d^2 \Theta \mathcal{E} R + h.c.$$

- In components

$$\mathcal{L}_{pure_{SG}} = -\frac{1}{2} e\mathcal{R} - \frac{1}{3} eM^* M + \frac{1}{3} eb^\mu b_\mu + \frac{1}{2} e\varepsilon^{\mu\nu\rho\sigma} \left(\bar{\psi}_\mu \bar{\sigma}_\nu \tilde{\mathcal{D}}_\rho \psi_\sigma - \psi_\mu \sigma_\nu \tilde{\mathcal{D}}_\rho \bar{\psi}_\sigma \right)$$

Unimodular

- The unimodular sugra action can be written as

$$S = \int d^4x d^2\Theta \left[\mathcal{E} R - \Lambda \left(\mathcal{E} - \mathcal{E}_0\right)\right] + h.c.$$

where Λ is now a lagrange multiplier chiral superfield, and $\mathcal{E}_0$ is a *constant superspace object*. Varying over Λ, we get the SUSY analogue of the unimodularity condition:

$\boxed{\mathcal{E} = \mathcal{E}_0}$, or in components:

$$a = \epsilon_0$$
$$\rho = 0$$
$$f = m$$

with ϵ_0 and m some constants. The action is invariant under a restricted set of SUSY and diffeomorphism transformations, such that they preserve the conditions above.

Unimodular Stueckelberg

- To restore full invariance, we perform a (restricted) coordinate transformation:

$$x^M \to y^M(x), \quad \text{with} \quad x^M = (x^\mu, \Theta^m), \quad y^M = (y^\mu, \Gamma^m)$$

where the index M now runs over both space-time and superspace indices (thus including both diffeos and susy transformations). The transformation are (in the chiral case):

$$dy^\mu = dx^\alpha \frac{\partial y^\mu}{\partial x^\alpha} + d\Theta^a \frac{\partial y^\mu}{\partial \Theta^a}$$
$$d\Gamma^m = dx^\alpha \frac{\partial \Gamma^m}{\partial x^\alpha} + d\Theta^a \frac{\partial \Gamma^m}{\partial \Theta^a}$$

so the super-Jacobian is

$$sJ = \frac{\partial y^M}{\partial x^A} = \begin{pmatrix} \frac{\partial y^\mu}{\partial x^\alpha} & \frac{\partial y^\mu}{\partial \Theta^a} \\ \frac{\partial \Gamma^m}{\partial x^\alpha} & \frac{\partial \Gamma^m}{\partial \Theta^a} \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and similarly for the conjugate superspace.

Unimodular Stueckelberg

- Under a superspace g.c.t., the volume element transforms as

$$\int d^4x d^2\Theta \quad \rightarrow \quad \int d^4x d^2\Theta |sJ|$$

with

$$|sJ| = Ber\,(sJ) = det\left(A - BD^{-1}C\right) Det^{-1}(D)$$

where we introduced the Berezinian, the superspace generalization of the determinant. We now define the Stueckelberg superfield

$$S \equiv Ber(sJ)$$

and the Stueckelberg action is just

$$S_{stueck} = \int d^4x d^2\Theta \left[\mathcal{E}R - \Lambda\left(\mathcal{E} - S\mathcal{E}_0\right)\right] + h.c.$$

Stueckelberg action

- The stueckelberg transformation that we perform to restore (infinitessimally) SUSY and diffeos is

$$y^\mu = x^\mu + f^\mu + \eta^\mu$$
$$\Gamma^m = \Theta^m + \eta^m$$

with

$$\eta^\mu = 2i\Theta\sigma^\mu\bar{\zeta} + \bar{\psi}_\nu\bar{\sigma}^\mu\sigma^\nu\bar{\zeta}\Theta^2$$
$$\eta^m = \zeta^m - i\Theta\sigma^\mu\bar{\zeta}\psi^m_\mu + \Theta^2\Big[\tfrac{1}{3}M^*\zeta^m + \tfrac{1}{6}b_\alpha\left(\varepsilon\sigma^\alpha\bar{\zeta}\right)^m - i\omega^{mn}_\mu\left(\sigma^\mu\bar{\zeta}\right)_n - \tfrac{1}{2}\psi^a_\nu\left(\bar{\psi}_\mu\bar{\sigma}^\nu\sigma^\mu\bar{\zeta}\right)\Big]$$

Stueckelberg action in components

Then, the full action will be (setting $\epsilon_0 = \frac{1}{2}$)

$$S = \frac{1}{2\kappa^2}\int \Big[\sqrt{-g}\,\Big[R - \tfrac{2}{3}M^*M + \tfrac{2}{3}b^\mu b_\mu + \varepsilon^{\mu\nu\rho\sigma}\left(\bar\psi_\mu\bar\sigma_\nu\tilde{\mathcal{D}}_\rho\psi_\sigma - \psi_\mu\sigma_\nu\tilde{\mathcal{D}}_\rho\bar\psi_\sigma\right)\Big]$$
$$+\tfrac{1}{2}\sqrt{-g}\,\Big[-2\Lambda_2 + \sqrt{2}i\Lambda_1\sigma^\mu\bar\psi_\mu + 2\Lambda_0\left(\bar\psi_\mu\bar\sigma^{\mu\nu}\bar\psi_\nu + M^*\right)\Big] + h.c.$$
$$+\Big[\Lambda_0\left(\partial_\mu y_2^\mu - m - m\partial_\mu f^\mu\right) - \Lambda_1^a\left(\tfrac{\sqrt{2}}{2}\partial_\mu y_{a1}^\mu + \sqrt{2}\Gamma_{a2} - \sqrt{2}m\zeta_a\right)$$
$$+\Lambda_2\left(1 + \partial_\mu f^\mu - \Gamma_{m1}^m\right)\Big] + h.c. + \mathcal{O}^2$$

with :

$$y_{a1}^\mu = 2i\left(\sigma^\mu\bar\zeta\right)_a, \quad y_2^\mu = \bar\psi_\nu\bar\sigma^\mu\sigma^\nu\bar\zeta, \quad \Gamma_{a1}^m = -i\left(\sigma^\mu\bar\zeta\right)_a\psi_\mu^m, \quad \Gamma_2^m = -i\omega_\mu^{mn}\left(\sigma^\mu\bar\zeta\right)_n + \tfrac{1}{3}M^*\zeta^m$$

Transformations

- The Stueckelberg fields will transform as:

$$\delta f^{\mu} = -\xi^{\mu} + \frac{1}{2}\left[\zeta^{r}\tilde{y}^{\mu}_{r1} - \epsilon^{r} y^{\mu}_{r1}\right] + \ldots$$
$$\delta\zeta^{m} = -\epsilon^{m} + \frac{1}{2} f^{\rho}\partial_{\rho}\epsilon^{m} + \ldots$$

  Then S transforms as required for a chiral density superfield

$$\delta S = -\partial_{M}\left[\eta^{M} S(-1)^{M}\right]$$

- We have restored full SUSY invariance to the action !

E.o.M and deSitter solution

- Write e.o.m. for all our fields (supergravity multiplet components, Λ components and Stueckelberg superfields f^μ and ζ) we see that our theory admits solutions with c.c. of arbitrary sign:

$$\Lambda \propto \langle \Lambda_2 \rangle - m^2$$

$$m_\psi \propto m$$

In progress

- Extend to non-perturbative constructions, via conformal construction.
- Consequences for proposed solutions to c.c. problem.
- Relation to constrained superfields.
- Relation to brane constructions for goldstino.