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Cosmology

Lecture 3

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That was **fun**! Let's see if it works for something else...

Try **proton-antiproton** freeze-out:
what's the **relic** matter **abundance** in a baryon-symmetric Universe?

$$\sigma \sim \Lambda_{\rm QCD}^{-2}$$

$$n\ \sigma = H \rightarrow T^3\ \Lambda^{-2} = T^2/M_P \rightarrow T = \Lambda^2/M_P$$

doesn't quite work, we're way **outside** the regime of validity for **hot relics**, since $T \lll\lll m_p$ ...

Need to work out the case of **cold relics**, which looks nastier by eye

$$n \sim (m_\chi T)^{3/2} \exp\left(-\frac{m_\chi}{T}\right)$$

Here's the trick: **freeze-out** condition gives

$$n_{\rm f.o.} \sim \frac{T^2_{\rm f.o.}}{M_P \cdot \sigma}$$

now define $m_\chi / T \equiv x$ (cold relic: **x>>1**)

Freeze-out condition (x) now reads

$$\frac{m_\chi^3}{x^{3/2}} e^{-x} = \frac{m_\chi^2}{x^2 \cdot M_P \cdot \sigma}$$

...so we gotta **solve** $\sqrt{x} \cdot e^{-x} = \frac{1}{m_\chi \cdot M_P \cdot \sigma}$

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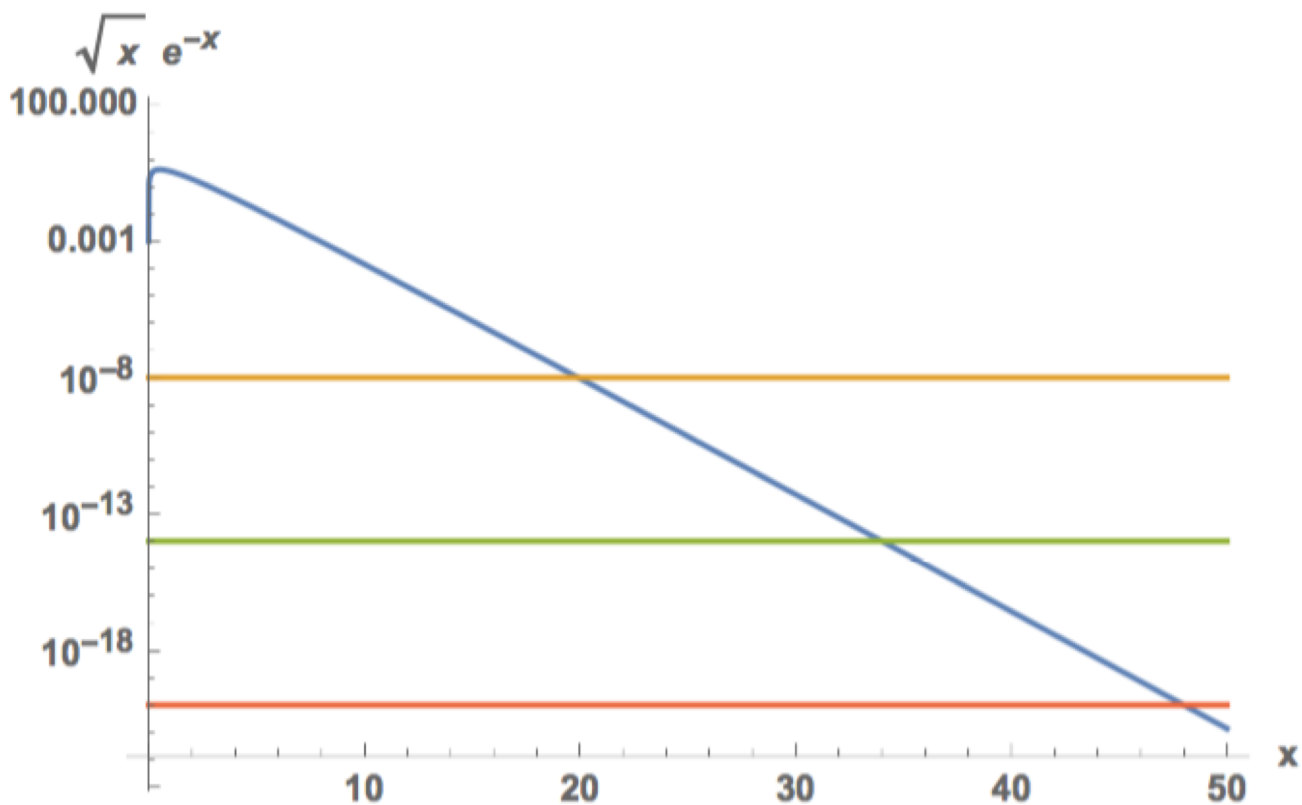
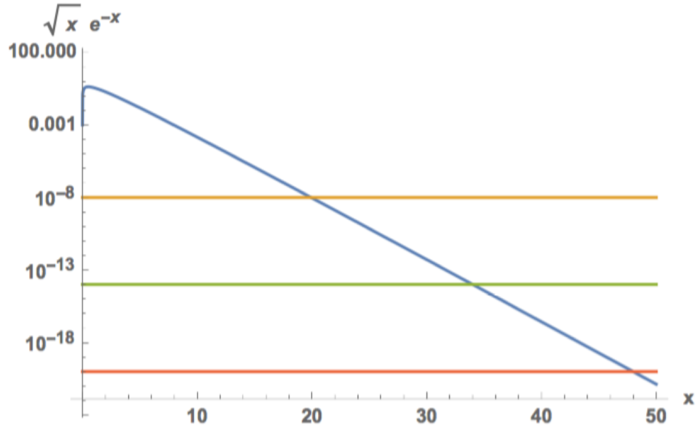




$$\sqrt{x} \cdot e^{-x} = \frac{1}{m_\chi \cdot M_P \cdot \sigma}$$

$$\sigma \sim G_F^2 m_\chi^2$$

Take e.g. a "**weakly interacting massive particle**"

$$m_\chi \sim 10^2 \text{ GeV}.$$

$$\sqrt{x} \cdot e^{-x} = \frac{1}{m_\chi \cdot M_P \cdot \sigma} \sim \frac{1}{10^2 \cdot 10^{18} \cdot 10^{-6}} \sim 10^{-14}.$$

thus ***$x = m_\chi / T \sim 35$***

Off to calculating the **thermal relic density**

$$\Omega_\chi = \frac{m_\chi \cdot n_\chi(T = T_0)}{\rho_c} = \frac{m_\chi\, T_0^3}{\rho_c} \frac{n_0}{T_0^3}$$

iso-entropic universe aT ~const

$$\frac{n_0}{T_0^3} \simeq \frac{n_{\rm f.o.}}{T_{\rm f.o.}^3}$$

$$\Omega_\chi = \frac{m_\chi\, T_0^3}{\rho_c} \frac{n_{\rm f.o.}}{T_{\rm f.o.}^3} = \frac{T_0^3}{\rho_c} x_{\rm f.o.} \left(\frac{n_{\rm f.o.}}{T_{\rm f.o.}^2}\right) = \left(\frac{T_0^3}{\rho_c\, M_P}\right) \frac{x_{\rm f.o.}}{\sigma}$$

$$\left(\frac{\Omega_\chi}{0.2}\right) \simeq \frac{x_{\rm f.o.}}{20} \left(\frac{10^{-8}\ {\rm GeV}^{-2}}{\sigma}\right)$$

Notice we neglected relative **velocity**...
What is the velocity of a cold relic at freeze-out?

$$\frac{3}{2}T = \frac{1}{2}mv^2$$

...just use **equipartition** theorem... $v=(3/x)^{1/2}$ ~ 0.3

Now, back to **relic density**:

$$\left(\frac{\Omega_\chi}{0.2}\right) \simeq \frac{x_{\rm f.o.}}{20}\left(\frac{10^{-8}\ {\rm GeV}^{-2}}{\sigma}\right)$$

$$\sigma_{\rm EW} \sim G_F^2 T_{\rm f.o.}^2 \sim G_F^2 \left(\frac{E_{\rm EW}}{20}\right)^2 \sim 10^{-8}\ {\rm GeV}^{-2},$$

$$\left(\frac{\Omega_\chi}{0.2}\right) \simeq \frac{x_{\rm f.o.}}{20}\left(\frac{10^{-8}\ {\rm GeV}^{-2}}{\sigma}\right)$$

Is this **unique** to **WIMPs**? **No**.

$$\sigma \sim \frac{g^4}{m_\chi^2}$$

"**WIMPless**" miracle... what did we use?

$$m_\chi \cdot \sigma \cdot M_P \gg 1$$

$$\sigma \sim 10^{-8}\ {\rm GeV}^{-2}$$

Substitute and find that ***m_χ >> 0.1 eV*** !

In practice various **constraints** on light thermal relics from structure formation, relativistic degrees of freedom at BBN, CMB... ***m_χ > MeV***

What is the **range** of **masses** expected for cold relics?

Cross section cannot be arbitrarily large: **unitarity** limit

$$\sigma \lesssim \frac{4\pi}{m_\chi^2}$$

$$\frac{\Omega_\chi}{0.2} \gtrsim 10^{-8}\ \mathrm{GeV}^{-2} \cdot \frac{m_\chi^2}{4\pi}$$

$$\left(\frac{m_\chi}{120\ \mathrm{TeV}}\right)^2 \lesssim 1$$

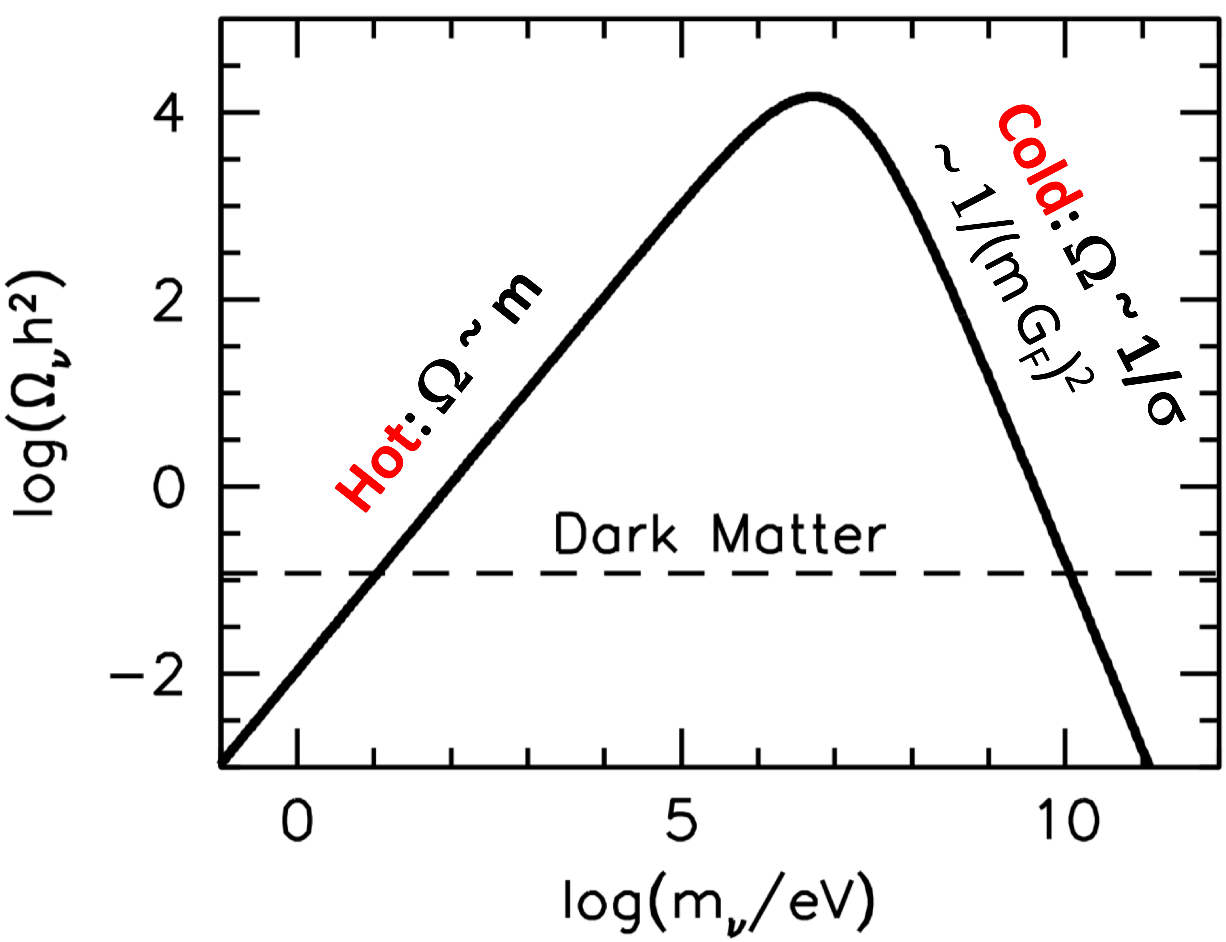
What is the **range** of **masses** expected for cold relics?

If you have a **WIMP**, defined by a cross section $\sigma \sim G_F^2\, m_\chi^2$

$$\Omega_\chi h^2 \sim 0.1 \frac{10^{-8}\ \mathrm{GeV}^{-2}}{G_F^2\, m_\chi^2} \sim 0.1 \left(\frac{10\ \mathrm{GeV}}{m_\chi} \right)^2$$

"**Lee-Weinberg**" limit

WIMP's **thermal** relic density



$\Omega \sim 1/\sigma$ is "**catastrophic**" for baryons

Roughly, **10^{-10} relic protons** and antiprotons per actual proton!

Gives a job to **Baryogenesis** model builders…

Discussion so far OK for a **qualitative** assessment of **relic density**

State of the art much more sophisticated: Solve **Boltzmann equation**

$$\hat{L}[f] = \hat{C}[f]$$

$$\hat{L}_{\rm NR} = \frac{\rm d}{{\rm d}t} + \frac{{\rm d}\vec{x}}{{\rm d}t}\vec{\nabla}_x + \frac{{\rm d}\vec{v}}{{\rm d}t}\vec{\nabla}_v$$

$$\hat{L}_{\rm cov} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma}\, p^\beta\, p^\gamma \frac{\partial}{\partial p^\alpha}$$

Looks ugly, but for the **FRW** metric **phase-space** density simplifies…

$$f(\vec{x}, \vec{p}, t) \rightarrow f(|\vec{p}|, t) \qquad f(E, t)$$

$$\hat{L}[f] = E\frac{\partial f}{\partial t} - \frac{\dot{a}}{a}|\vec{p}|^2\, \frac{\partial f}{\partial E}$$

Now, what we are interested in are **number** densities, which in terms of **phase-space** densities are simply…

$$n(t) = \sum_{\text{spin}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f(E,t)$$

…**integrate** the Liouville operator over **momentum space** and get

$$\int L[f] \cdot g \frac{\mathrm{d}^3 p}{(2\pi)^3} = \frac{\mathrm{d}n}{\mathrm{d}t} + 3H \cdot n,$$

Back to **Boltzmann** equation, suppose a **2-to-2** reaction, with 3, 4 **in eq.**

$$1 + 2 \leftrightarrow 3 + 4$$

Consider the **collision** factor, and again integrate over **momenta**…

$$g_1 \int \hat{C}[f_1] \frac{\mathrm{d}^3 p}{(2\pi)^3} = -\langle \sigma \cdot v_{\mathrm{M\o l}} \rangle \left(n_1 n_2 - n_1^{\mathrm{eq}} n_2^{\mathrm{eq}} \right)$$

…where the **cross section**

$$\sigma = \sum_f \sigma_{12 \to f}$$

$$g_1 \int \hat{C}[f_1] \frac{\mathrm{d}^3 p}{(2\pi)^3} = -\langle \sigma \cdot v_{\mathrm{M\o l}} \rangle \left(n_1 n_2 - n_1^{\mathrm{eq}} n_2^{\mathrm{eq}} \right)$$

let's understand the rest of the equation:

$$v_{\mathrm{M\o l}} \equiv \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1\, E_2}$$

$$\langle \sigma \cdot v_{\mathrm{M\o l}} \rangle = \frac{\int \sigma \cdot v_{\mathrm{M\o l}}\, e^{-E_1/T} e^{-E_2/T}\, \mathrm{d}^3 p_1\, \mathrm{d}^3 p_2}{\int e^{-E_1/T} e^{-E_2/T}\, \mathrm{d}^3 p_1\, \mathrm{d}^3 p_2}$$

Final version of **Boltzmann Eq.**

$$\dot{n} + 3Hn = \langle \sigma v \rangle \left(n_{\mathrm{eq}}^2 - n^2 \right)$$

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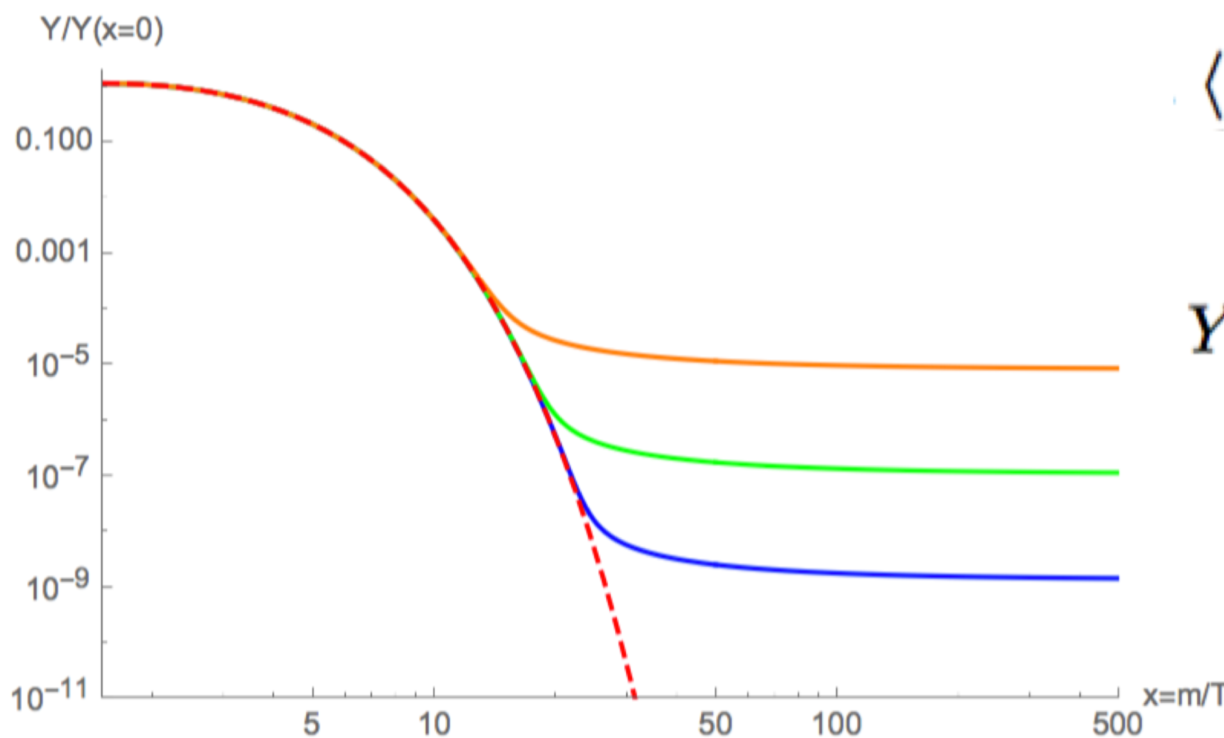
$$\frac{dY(x)}{dx} = -\frac{xs\langle \sigma v \rangle}{H(m)} \left(Y(x)^2 - Y_{\rm eq}^2(x) \right)$$



$$\langle \sigma v \rangle = \langle \sigma v \rangle_0 x^{-n}$$

$$Y_{\rm today} \simeq \frac{n+1}{\lambda} x_{\rm f.o.}^{n+1}$$

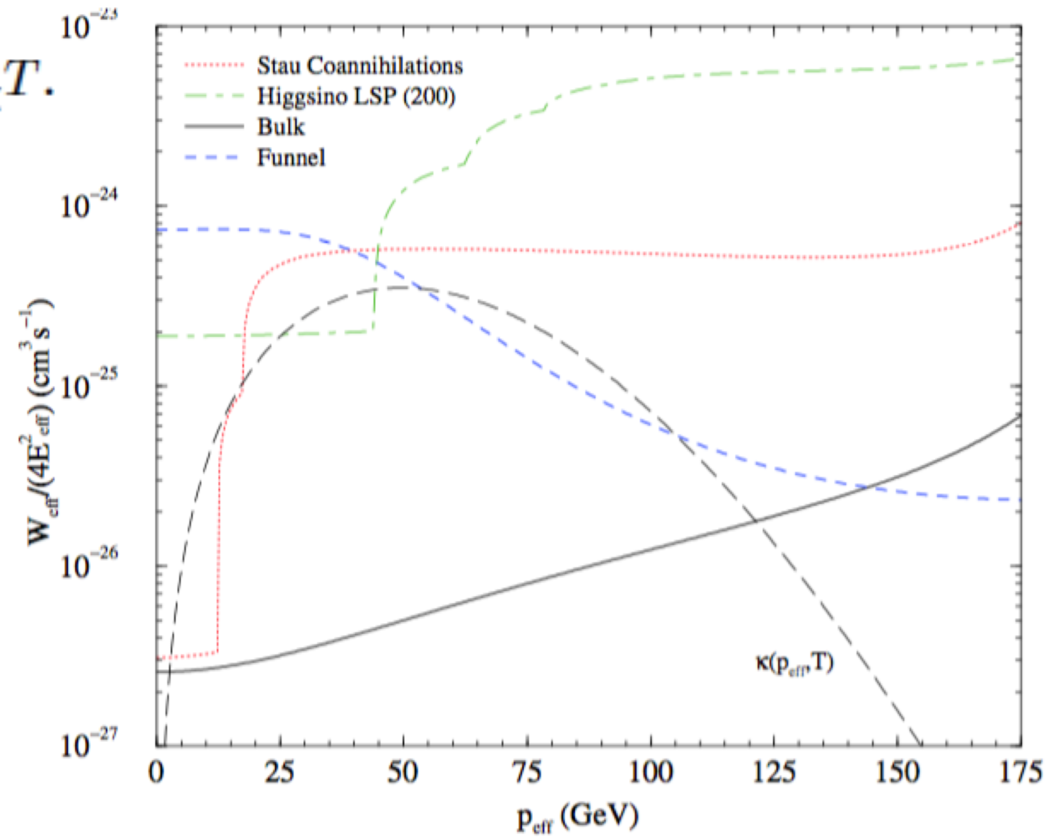
$$\lambda = \frac{\langle \sigma v \rangle_0 s_0}{H(m)}$$

There exist important "**exceptions**" to this standard story:

1. **Resonances**
2. **Thresholds**
3. **Co-annihilation**

$$\langle s\rangle \simeq 4m_\chi^2 + 6m_\chi T.$$

$$\langle\sigma v\rangle \;\rightarrow\; \langle\sigma_{\rm eff} v\rangle = \frac{\sum_{i<j=1}^{N}\sigma_{ij}\exp\left(-\frac{\Delta m_i+\Delta m_j}{T}\right)}{\sum_{i=1}^{N} g_i \exp\left(-\frac{\Delta m_i}{T}\right)}.$$



Affects what the **pair-annihilation** rate **today** is compared to what it was at **freeze-out**!

$$\langle\sigma_{\rm eff} v\rangle = \int_0^\infty dp_{\rm eff}\,\frac{W_{\rm eff}(p_{\rm eff})}{4E_{\rm eff}^2}\,\kappa(p_{\rm eff},T) \qquad E_{\rm eff}^2 = \sqrt{p_{\rm eff}^2 + m^2}.$$

So far we looked into what happens if we fiddle with the left hand side of

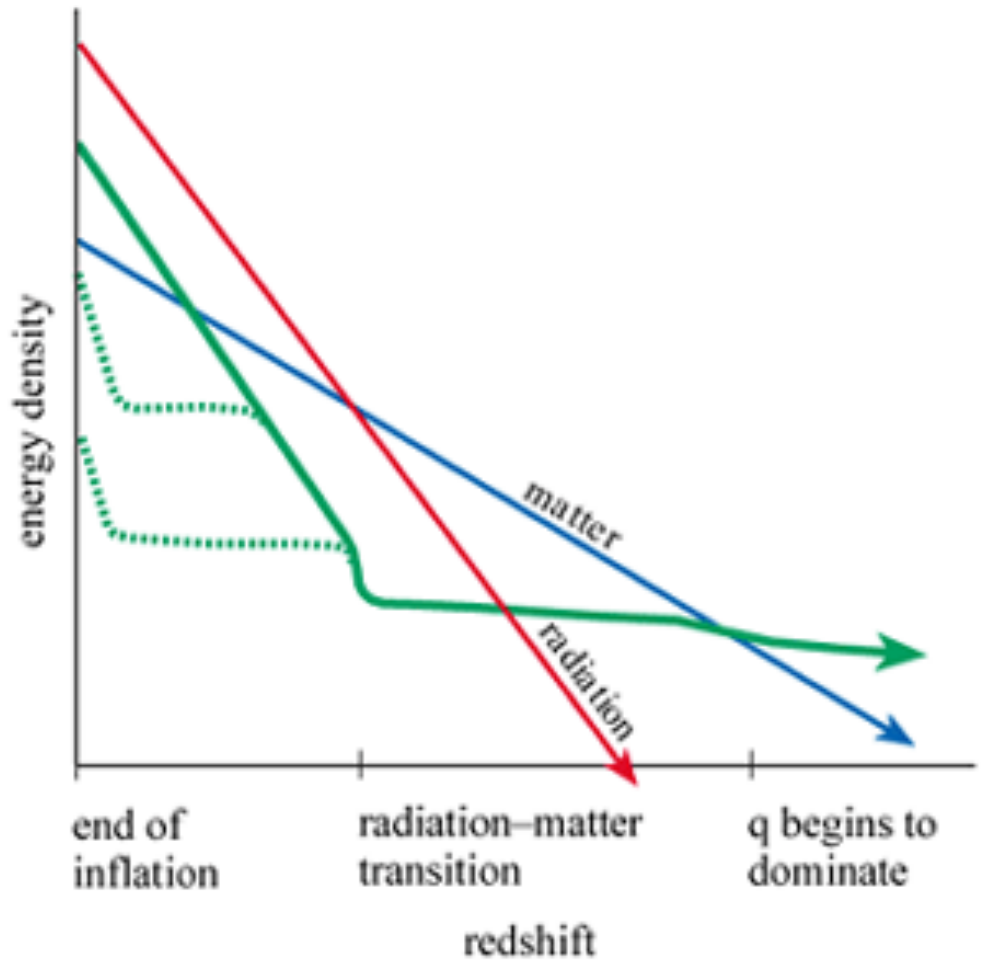
$$\Gamma = n \cdot \sigma \sim H,$$

Consider a "**Quintessence**" dark energy model – homogeneous real scalar field

$$\rho_\phi = \frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 + V(\phi)$$

$$P_\phi = \frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 - V(\phi)$$

$$w = P_\phi / \rho_\phi \qquad \rho_\phi \sim a^{-3(1+w)} \qquad \rho \sim a^{-6}$$



energy density
matter
radiation
end of
inflation
radiation–matter
transition
q begins to
dominate
redshift

$$H \sim \frac{T^2}{M_P}\frac{T}{T_{\rm KRE}} \qquad (T \gtrsim T_{\rm KRE})$$

$$n_{\rm f.o.}\langle\sigma\ v\rangle \sim \frac{T^2}{M_P}\frac{T}{T_{\rm KRE}}.$$

$$\frac{n_{\rm f.o.}}{T^2_{\rm f.o.}} \sim \frac{1}{M_P\ \langle\sigma\ v\rangle}\frac{T_{\rm f.o.}}{T_{\rm KRE}}. \qquad \Omega_\chi^{\rm quint} = \frac{T_0^3}{M_P\cdot\rho_c}x_{\rm f.o.}\left(\frac{n_{\rm f.o.}}{T^2_{\rm f.o.}}\right)$$

$$\frac{\Omega_\chi^{\rm quint}}{\Omega_\chi^{\rm standard}} \sim \frac{T_{\rm f.o.}}{T_{\rm KRE}} \lesssim \frac{m_\chi}{20}\frac{1}{T_{\rm BBN}} \sim 10^4\frac{m_\chi}{100\ {\rm GeV}}.$$

After **chemical** decoupling (number density freezes out),
DM can still be in **kinetic** equilibrium
(i.e. its **velocity** distribution is in equilibrium)

generically, this is the case, since for **cold** relics

$$\chi\chi \leftrightarrow ff \qquad \rightarrow \quad \Gamma = n_{\mathrm{non-rel}} \cdot \sigma$$
$$\chi f \leftrightarrow \chi f \qquad \rightarrow \quad \Gamma = n_{\mathrm{rel}} \cdot \sigma$$

Think of a **prototypical WIMP**:

$$\sigma_{\chi f \leftrightarrow \chi f} \sim G_F^2 T^2$$

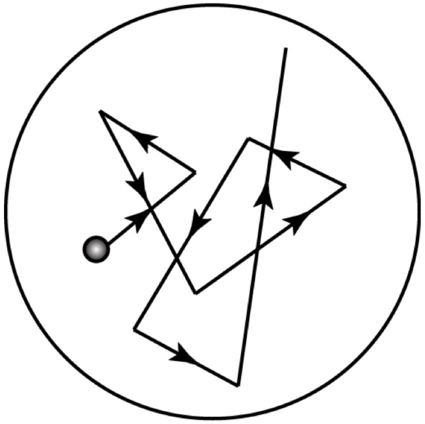
Problem: every collision has a **momentum transfer** $\delta p \sim T$

...but we need to keep the (cold) DM momentum in equilibrium, i.e.

$$\frac{p^2}{2m_\chi} \sim T \qquad p \sim \sqrt{m_\chi T}$$

so ***δp << p***, we need a bunch of kicks!

However, **subtlety**: kicks are in **random directions**!



$$N = \left(\frac{p}{\delta p}\right)^2 \sim \frac{m_\chi T}{T^2} = \frac{m_\chi}{T} \gg x_{\rm f.o.} \gtrsim 20$$

Let's estimate a typical WIMP **kinetic decoupling temperature**

$$n_{\rm rel} \cdot \sigma_{\chi f \leftrightarrow \chi f} \left(\frac{\delta p}{p}\right)^2 \sim T^3 \cdot G_F^2 T^2 \cdot \frac{T}{m_\chi} \sim H \sim \frac{T^2}{M_P}.$$

$$T_{\rm kd} \sim \left(\frac{m_\chi}{M_P \cdot G_F^2}\right)^{1/4} \sim 30\ {\rm MeV}\ \left(\frac{m_\chi}{100\ {\rm GeV}}\right)^{1/4}$$

What does this implies for **structure formation**?

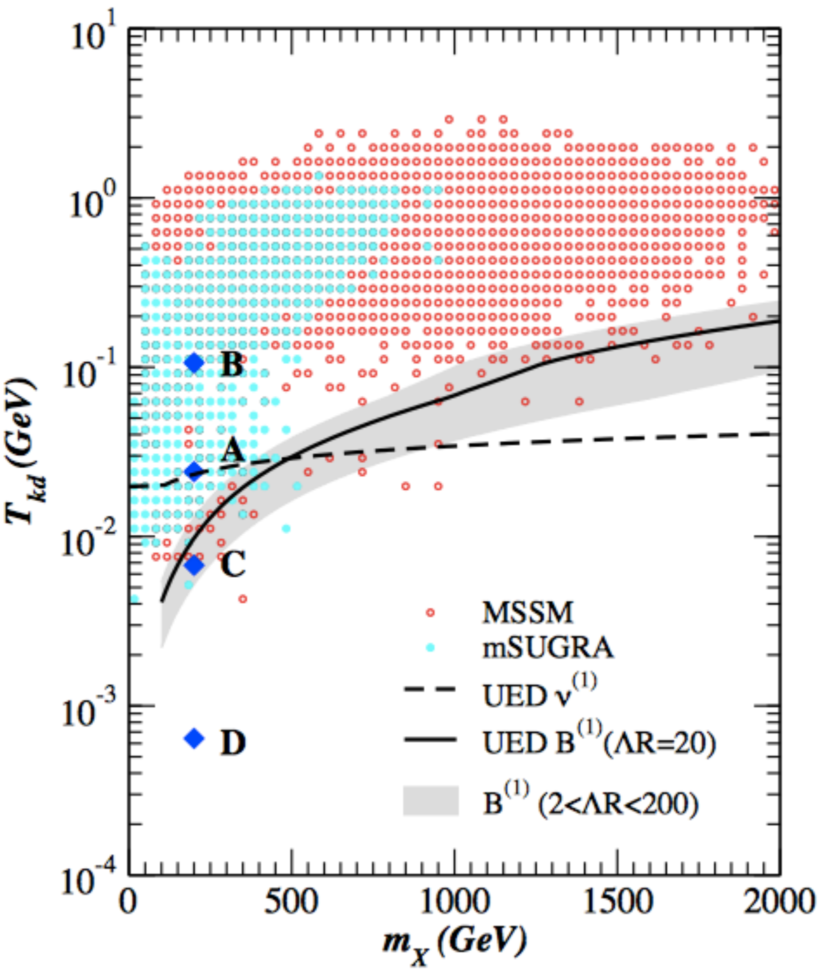
$$M_{\rm ao} \sim \frac{4\pi}{3}\left(\frac{1}{H(T_{\rm kd})}\right)^3 \rho_{\rm DM}(T_{\rm kd}) \sim 30\ M_\oplus \left(\frac{10\ {\rm MeV}}{T_{\rm kd}}\right)^3$$

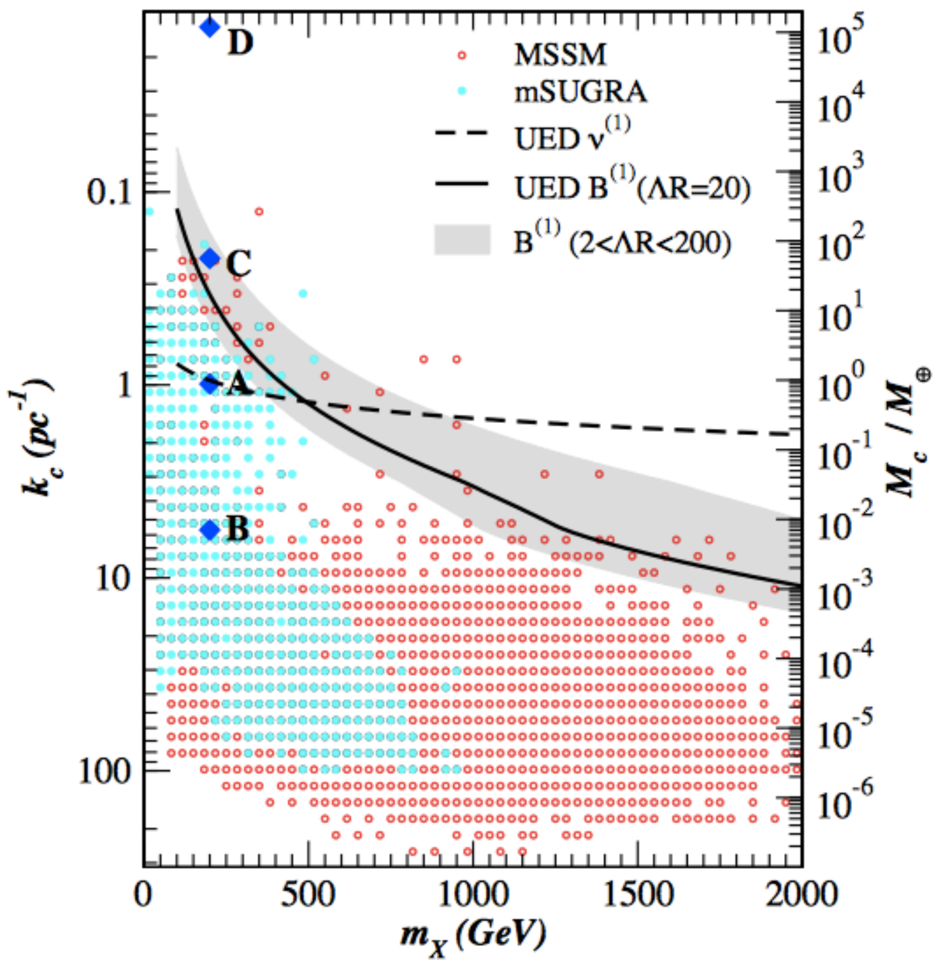
$$M_\oplus \simeq 3 \times 10^{-6} M_\odot$$

First structures that collapse are these tiny **minihalos**
(maybe some survive today?)

Structures then **merge** into bigger and bigger halos
(**bottom-up** structure formation)

Notice that the kinetic decoupling/cutoff scale **varies** significantly even for a selected particle dark matter scenario!
e.g. for **SUSY, UED**





What happens instead for **hot relics**?

They decouple when **$T \gg m_\nu$**

Structures can only collapse when **$T \sim m_\nu$**

(i.e. when things slow down enough for gravitational collapse!)

Structures are cutoff to the **horizon size** at that temperature

$$d_\nu \sim H^{-1}(T \sim m_\nu) \qquad d_\nu \sim \frac{M_P}{m_\nu^2}$$

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$$M_{\text{cutoff, hot}} \sim \left(\frac{1}{H(T = m_\nu)}\right)^3 \rho_\nu(T = m_\nu) \sim \left(\frac{M_P}{m_\nu^2}\right)^3 m_\nu \cdot m_\nu^3 = \frac{M_P^3}{m_\nu^2}$$

$$\frac{M_P^3}{m_\nu^2} \sim 10^{15}\ M_\odot \left(\frac{m_\nu}{30\ \text{eV}}\right)^{-2} \sim 10^{12}\ M_\odot \left(\frac{m_\nu}{1\ \text{keV}}\right)^{-2}$$

How does this compare with **observations**?

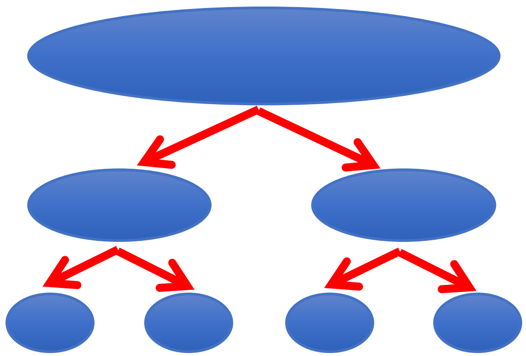
$$\frac{M_P^3}{m_\nu^2} \sim 10^{15}\, M_\odot \left(\frac{m_\nu}{30\ \text{eV}}\right)^{-2} \sim 10^{12}\, M_\odot \left(\frac{m_\nu}{1\ \text{keV}}\right)^{-2}$$

Observational **constraints** give

$$M_{\text{cutoff}} \ll M_{\text{Ly}-\alpha} \simeq 10^{10}\, M_\odot$$

So at best dark matter can be **keV** scale, if produced thermally

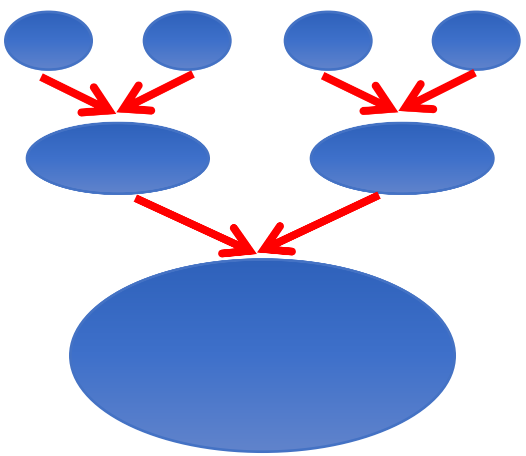
Structure formation looks strikingly different for hot and cold dark matter



Hot Dark Matter

Top-Down

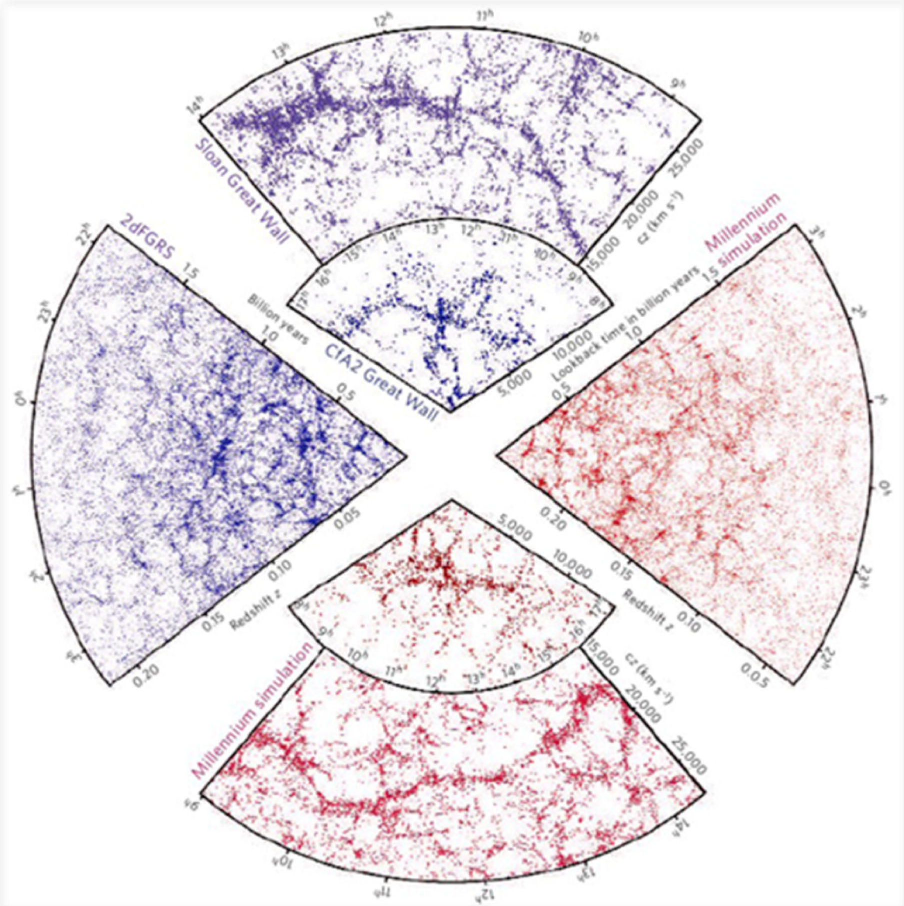
[doesn't work!]



Cold Dark Matter

Bottom-Up

[Yeah!]



1980's: Davis, Efstathiou, Frenk and White show that simulations of structure formation in a universe with **cold dark matter** match observed structure incredibly well!!