Lattice results for $\alpha_s(m_Z)$ by the ALPHA collaboration

(part I; part II by M. Dalla Brida)

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ALPHA Collaboration

workshop on precision measurements of α_s,
ECT* Trento, February 11-15, 2019

References:

- "**Determination of the QCD Λ-parameter and the accuracy of perturbation theory at high energies**,"
  Mattia Dalla Brida, Patrick Fritzsch, Tomasz Korzec, Alberto Ramos, S.S., Rainer Sommer [ALPHA Collaboration],
  Phys. Rev. Lett. **117**, no. 18, 182001 (2016) arXiv:1604.06193 [hep-ph].
- "**A non-perturbative exploration of the high energy regime in $N_\mathrm{f} = 3$ QCD** ,"
  Mattia Dalla Brida, Patrick Fritzsch, Tomasz Korzec, Alberto Ramos, S.S., Rainer Sommer [ALPHA Collaboration],
  Eur. Phys. J. C **78** (2018) no.5, 372 arXiv:1803.10230 [hep-lat].
- "**Slow running of the Gradient Flow coupling from 200 MeV to 4 GeV in $N_\mathrm{f} = 3$ QCD,**"
  Mattia Dalla Brida, Patrick Fritzsch, Tomasz Korzec, Alberto Ramos, S.S, Rainer Sommer [ALPHA Collaboration],
  Phys. Rev. D **95**, no. 1, 014507 (2017), arXiv:1607.06423 [hep-lat].

⇒ "**QCD Coupling from a Nonperturbative Determination of the Three-Flavor Λ Parameter**,"
Mattia Bruno, Mattia Dalla Brida, Patrick Fritzsch, Tomasz Korzec, Alberto Ramos, Stefan Schaefer, S. S., Hubert Simma Rainer Sommer [ALPHA Collaboration],
Phys. Rev. Lett. **119**, no. 10, 102001 (2017), arXiv:1706.03821 [hep-lat].

Recent overview talks at conferences:

- **Tomasz Korzec** @ Lattice 2017: "Determination of the Strong Coupling Constant by the ALPHA Collaboration," EPJ Web Conf. **175** (2018) 01018, doi:10.1051/epjconf/201817501018 [arXiv:1711.01084 [hep-lat]].
- **Mattia Dalla Brida** @ ICNFP 2018: "Precision Determination of α_s from Lattice QCD," Universe **4** (2018) no.12, 148 doi:10.3390/universe4120148 [arXiv:1812.06692 [hep-ph]].
- **Alberto Ramos** @ "Quark confinement and the hadron spectrum 2018", to appear in the proceedings (PoS)

Introduction

Central result by the ALPHA collaboration

$$\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f}=3)} = 341(12)\mathrm{MeV}$$

- includes a *rather conservative* assessment of all errors!
- very good control of perturbative truncation errors; perturbation theory avoided up to the electro-weak scale!
- valuable tests of perturbation theory at high energies.

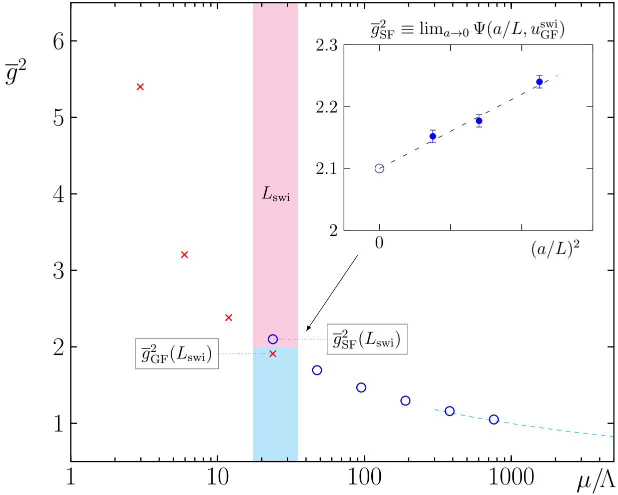
Perturbative decoupling across charm and bottom thresholds leads to

$$\alpha_s(m_Z) = 0.11852(80)(25) = 0.11852(84)$$

Some questions hopefully answered in talk:

1. How to cover such a wide range of scales using lattice QCD?
2. How to control the continuum limit?
3. How to check for perturbative truncation errors?

Sketch of the strategy (predating the data!)



(courtesy Patrick Fritzsch 2014)

Topics in this talk:

- Non-perturbatively defined couplings and the Λ-parameter
- Recursive step-scaling techniques to bridge large scale differences
- Results for the SF coupling between $1/L_0 \approx 4\mathrm{GeV}$ and O(100) GeV
- Extraction of $L_0\Lambda^{(3)}$ & tests of perturbation theory
- Conclusions

Mattia Dalla Brida's talk (Tuesday):

- GF coupling & non-perturbative matching of SF and GF schemes at $1/L_0$
- Scale evolution from $\approx 4\mathrm{GeV}$ to 200 MeV
- precise scale setting in physical units using CLS gauge configurations [Bruno et al, JHEP 1502 (2015) 043]
- Synthesis

The QCD Λ-parameter vs. $\alpha_s(\mu) = \bar{g}^2(\mu)/4\pi$

The coupling $\alpha_s(\mu)$ can be traded for its associated Λ-parameter:

$$\Lambda = \mu\varphi(\bar{g}(\mu)) = \mu\left[b_0\bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} dg\left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

- exact solution of Callan-Symanzik equation: $\left(\mu\frac{\partial}{\partial\mu} + \beta(\bar{g})\frac{\partial}{\partial\bar{g}}\right)\Lambda = 0$
- Number N_f of massless quarks is fixed.
- If the coupling $\bar{g}(\mu)$ non-perturbatively defined so is its β-function!
- $\beta(g)$ has asymptotic expansion $\beta(g) = -b_0 g^3 - b_1 g^5 - b_2 g^7$..

  $$b_0 = (11 - \tfrac{2}{3}N_\mathrm{f})/(4\pi)^2, \qquad b_1 = (102 - \tfrac{38}{3}N_\mathrm{f})/(4\pi)^4, \quad \ldots$$

  $b_{0,1}$ are universal, scheme-dependence starts with 3-loop coefficient b_2.
- Scheme dependence of Λ almost trivial:

  $$g_\mathrm{X}^2(\mu) = g_\mathrm{Y}^2(\mu) + c_\mathrm{XY}g_\mathrm{Y}^4(\mu) + \ldots \quad \Rightarrow \quad \frac{\Lambda_\mathrm{X}}{\Lambda_\mathrm{Y}} = \mathrm{e}^{c_\mathrm{XY}/2b_0}$$

$\Rightarrow$ can use $\Lambda_{\overline{\mathrm{MS}}}$ as reference (even though the $\overline{\mathrm{MS}}$-scheme is purely perturbative!)

The QCD Λ-parameter and $\alpha_s(\mu) = \bar{g}^2(\mu)/4\pi$

$$\Lambda = \mu\varphi(\bar{g}(\mu)) = \mu\left[b_0\bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} dg\left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

- Continuum relation, exact at any scale μ:
  - require large μ to evaluate integral perturbatively
  - require small μ to match hadronic scale

$\Rightarrow$ problem of large scale differences:

- The scale μ must reach the perturbative regime: $\mu \gg \Lambda$
- lattice cutoff must still be larger: $\mu \ll a^{-1}$
- spatial volume must be large enough to contain pions: $L \gg 1/m_\pi$
- Taken together a naive estimate gives

$$L/a \gg \mu L \gg m_\pi L \gg 1 \quad \Rightarrow L/a \simeq \mathrm{O}(10^3)$$

$\Rightarrow$ widely different scales cannot be resolved simultaneously on a single lattice!

Step scaling function

- Widely different scales cannot be resolved simultaneously on a *single* lattice

$\Rightarrow$ break calculation up in steps [Lüscher, Weisz, Wolff '91; Jansen et al. '95]:

1. define $\bar{g}^2(L)$ that runs with the space-time volume, i.e. $\mu = 1/L$
2. construct the step-scaling function

$$\sigma(u) = \bar{g}^2(2L)\Big|_{u=\bar{g}^2(L)}$$

   for a range of values $u \in [u_{\text{min}}, u_{\text{max}}]$

3. iteratively step up/down in scale by factors of 2:

$$\bar{g}^2(L_{\text{max}}) = u_{\text{max}} \equiv u_0, \quad u_k = \sigma(u_{k+1}) = \bar{g}^2(2^{-k}L_{\text{max}}), \quad k = 0, 1, \ldots$$

4. match to hadronic input at a hadronic scale L_{max}, i.e. $F_K L_{\max} = \mathrm{O}(1)$
5. once arrived in the perturbative regime $L_{\text{pert}} = 2^{-n}L_{\text{max}}$ one now knows $u_n = \bar{g}^2(L_{\text{pert}})$; determine $L_{\text{pert}}\Lambda$ and combine to obtain Λ/F_K.

Lattice approximants $\Sigma(u, a/L)$ for $\sigma(u)$

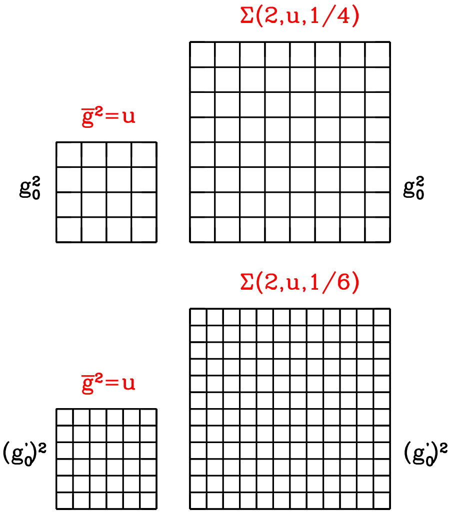
- choose g_0 and $L/a = 4$, measure $\bar{g}^2(L) = u$ (this sets the value of u)
- double the lattice and measure

$$\Sigma(u, 1/4) = \bar{g}^2(2L)$$

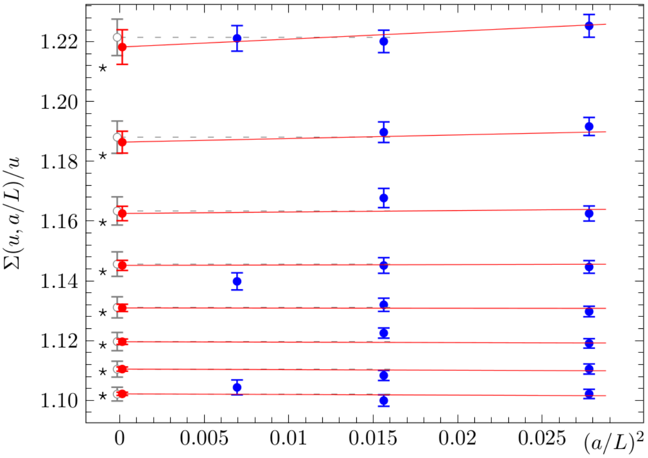
- now choose $L/a = 6$ and tune g_0' such that $\bar{g}^2(L) = u$ is satisfied
- double the lattice and measure

$$\Sigma(u, 1/6) = \bar{g}^2(2L)$$

- $\sigma(u) = \lim_{a/L \to 0} \Sigma(u, a/L)$.
- change u and repeat...



Continuum limit $\sigma(u) = \lim_{a/L \to 0} \Sigma(u, a/L)$

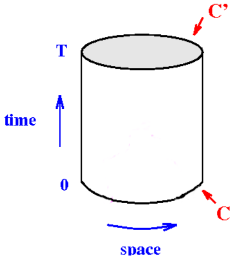


Wanted: renormalized finite volume coupling, which...

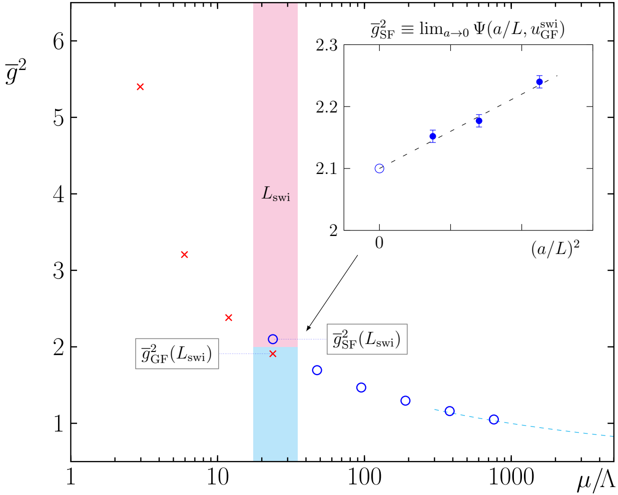
- is non-perturbatively defined in a finite space-time volume;
- can be expanded in perturbation theory with reasonable effort;
- is gauge invariant;
- is quark mass-independent (defined in the chiral limit).
- can be evaluated by MC simulation with good statistical precision

$\Rightarrow$ not easy to satisfy! Here:

1. impose Schrödinger functional (SF) boundary conditions: periodic in space, Dirichlet in time
2. use 2 definitions of the coupling with complementary properties at low/high scales:
   - traditional SF coupling [Lüscher, Narayanan, Weisz & Wolff '92]
   - gradient flow (GF) coupling & SF b.c.'s [Fritzsch & Ramos '13]



Sketch of the strategy (predating the data)



(courtesy Patrick Fritzsch 2014)

A family of SF couplings I

Dirichlet b.c.'s in Euclidean time, abelian boundary values C_k, C'_k:

$$A_k(x)|_{x_0=0} = C_k(\eta,\nu), \qquad A_k(x)|_{x_0=L} = C'_k(\eta,\nu)$$

$$C_k = \frac{i}{L}\begin{pmatrix} \eta - \frac{\pi}{3} & 0 & 0 \\ 0 & \eta\nu - \frac{\eta}{2} & 0 \\ 0 & 0 & -\eta\nu - \frac{\eta}{2} + \frac{\pi}{3} \end{pmatrix}, \quad C'_k = \frac{i}{L}\begin{pmatrix} -\eta - \pi & 0 & 0 \\ 0 & \eta\nu + \frac{\eta}{2} + \frac{\pi}{3} & 0 \\ 0 & 0 & \frac{\eta}{2} - \eta\nu + \frac{2\pi}{3} \end{pmatrix}$$

$\Rightarrow$ induce family of abelian, spatially constant background fields B_μ with parameters η, ν ($\rightarrow$ 2 abelian generators of SU(3)):

$$B_k(x) = C_k(\eta,\nu) + \frac{x_0}{L}\left(C'_k(\eta,\nu) - C_k(\eta,\nu)\right), \qquad B_0 = 0.$$

- Induced background field is unique up to gauge equivalence
- Effective action

$$\mathrm{e}^{-\Gamma[B]} = \int D[A,\psi,\overline{\psi}]\mathrm{e}^{-S[A,\psi,\overline{\psi}]}, \qquad \Gamma[B] = \frac{1}{g_0^2}\Gamma_0[B] + \Gamma_1[B] + \mathrm{O}(g_0^2)$$

- Define

$$\frac{1}{\bar{g}_\nu^2(L)} = \left.\frac{\partial_\eta \Gamma[B]}{\partial_\eta \Gamma_0[B]}\right|_{\eta=0} = \left.\frac{\langle \partial_\eta S\rangle}{\partial_\eta \Gamma_0[B]}\right|_{\eta=0}$$

$\Rightarrow$ 1-parameter family of SF couplings as response of the system to a change of a colour-electric background field. [Lüscher et al. '92]

A family of SF couplings II

- ν-dependence is explicit, obtained by computing $\bar{g}^2 \equiv \bar{g}^2_{\nu=0}$ and $\bar{v}$ at $\nu = 0$:

$$\frac{1}{\bar{g}^2_\nu} = \frac{1}{\bar{g}^2} - \nu \bar{v}$$

- relation between couplings at ν and $\nu = 0$ gives exact ratio:

$$r_\nu = \Lambda/\Lambda_\nu = \exp(-\nu \times 1.25516)$$

- The β-function is known to 3-loops:

$$(4\pi)^3 \times b_{2,\nu} = -0.06(3) - \nu \times 1.26$$

  N.B.: values ν of O(1) look perfectly fine!

- infrared cutoff (finite volume) $\Rightarrow$ no renormalons; secondary minimum of the action:

$$\exp(-2.62/\alpha) \simeq (\Lambda/\mu)^{3.8}$$

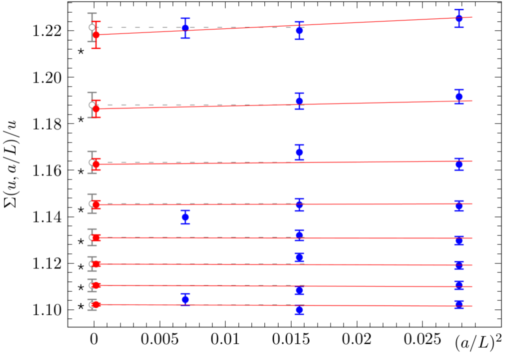
- Cutoff effects: O(a^4) at tree-level, but O(a) effects from the boundaries:
  - subtracted perturbatively
  - variation of coefficients treated as systematic error, continuum extrapolations $\propto a^2$

Extrapolating the SSF to the continuum limit

- u-values $\in [1, 2.012]$ lattice sizes $L/a = 4, 6, 8, 12$.
- Subtract lattice effects up to 2-loop order [Bode, Weisz & Wolff '99];
  $\Sigma^{(2)}(u, a/L) = \Sigma(u, a/L)/(1 + \delta_1(L/a)u + \delta_2(L/a)u^2)$

Global fit ansatz:

$$\Sigma^{(2)}(u, a/L) = u + s_0 u^2 + s_1 u^3 + c_1 u^4 + c_2 u^5 + \rho_1 u^4 \frac{a^2}{L^2} + \rho_2 u^5 \frac{a^2}{L^2}$$

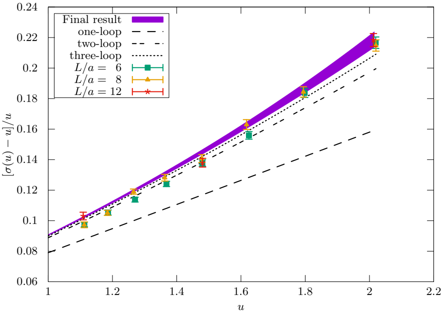


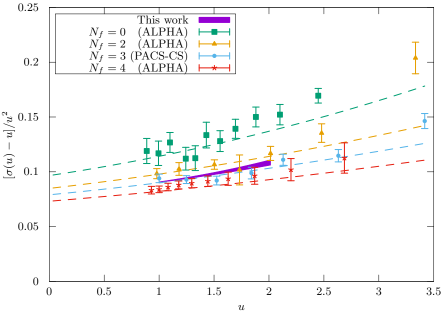
- s_0, s_1 fixed to perturbative values:

  $s_0 = 2b_0 \ln 2, \quad s_1 = s_0^2 + 2b_1 \ln 2$

- 4 parameters: c_1, c_2, ρ_1, ρ_2;
  19 data points, χ^2/d.o.f. ≈ 1, continuum limit is well-controlled!

SSF in the continuum limit





$\Rightarrow$ Significantly improved precision compared to previous work with $N_\mathrm{f} = 0, 2, 3, 4$

Computation of $L_0\Lambda$

- Define $L_{\text{swi}} \equiv L_0$ implicitly by

$$\bar{g}^2(L_0) = 2.012 = u_0$$

- Use the non-perturbative continuum step scaling function $\sigma(u)$:

$$u_{n-1} = \sigma(u_n), \quad n = 1, \ldots, \qquad \Rightarrow \quad u_n = \bar{g}^2\left(2^{-n}L_0\right)$$

- At scale $2^{-n}L_0$ obtain $L_0\Lambda$ using the perturbative β-function:

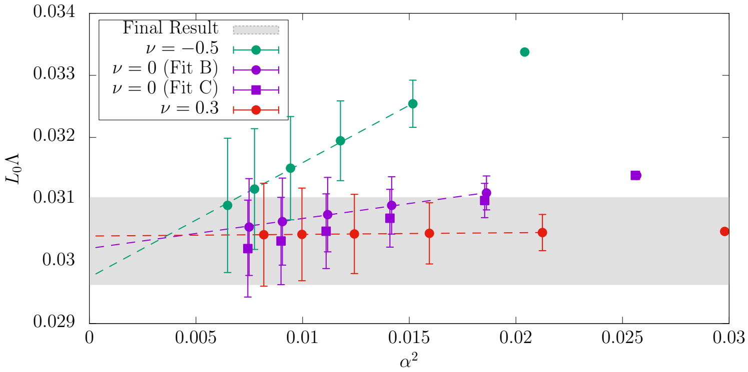
$$\begin{aligned} L_0\Lambda \quad &= \quad 2^n\left[b_0\bar{g}^2(2^{-n}L_0)\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2(2^{-n}L_0)}} \\ &\times \exp\left\{-\int_0^{\bar{g}(2^{-n}L_0)} dg\left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\} \end{aligned}$$

- Do the same for schemes $\nu \neq 0$ using the continuum relation:

$$\frac{1}{\bar{g}_\nu^2(L_0)} = \frac{1}{2.012} - \nu \times 0.1199(10)$$

$\Rightarrow$ check accuracy of perturbation theory: $L_0\Lambda$ must be independent of ν and number of steps, n !

Result for $L_0\Lambda$

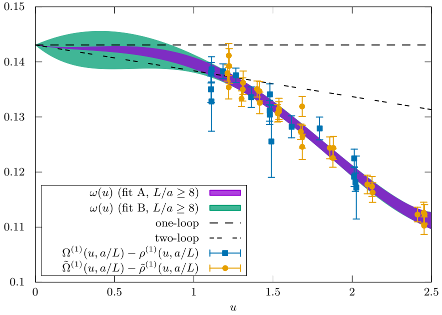


- All results agree around $\alpha = 0.1$, we quote

$$L_0\Lambda = 0.0303(7) \quad \Rightarrow \quad L_0\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3} = 0.0791(19) \qquad \text{(error } 2.4\%\text{)}$$

Recall $L_0 \equiv L_{\mathrm{swi}}$ is defined implicitly by $\bar{g}^2(L_0) = 2.012$.

Continuum result for $\bar{v} = \omega(u)$



- Continuum extrapolation analogous to $\sigma(u)$, but much more data between $L/a = 6$ to $L/a = 24$ covering a factor 4 in resolution!
- consider 2 continuum parameterizations (v_1, v_2 are known from PT):

$$\begin{aligned} \text{fit A:} \quad \omega(u) &= v_1 + v_2 u + d_1 u^2 + d_2 u^3 + d_3 u^4 \\ \text{fit B:} \quad \omega(u) &= v_1 + d_1 u + d_2 u^2 + d_3 u^3 + d_4 u^4 \end{aligned}$$

- Observe large deviation from perturbation theory at $\alpha = 0.19$:

$$\left(\omega(\bar{g}^2) - v_1 - v_2 \bar{g}^2\right) / v_1 = -3.7(2)\alpha^2$$

- too large for PT to be trustworthy at these couplings!

Alternative test via the $\overline{\text{MS}}$-scheme I

Idea: Perturbatively match the SF coupling to the $\overline{\text{MS}}$-coupling then evaluate the Λ-parameter using the 5-loop β-function

- Relation between couplings, allowing for a scale factor s:

$$4\pi\alpha_{\overline{\text{MS}}}(s/L) \equiv \bar{g}^2_{\overline{\text{MS}}}(L/s) = \bar{g}^2_\nu(L) + p^\nu_1(s)\bar{g}^4_\nu(L) + p^\nu_2(s)\bar{g}^6_\nu(L) + \text{O}(\bar{g}^8)$$

- Same as earlier, except now in the $\overline{\text{MS}}$ scheme:

$$\Lambda_{\overline{\text{MS}}} L_0 = \frac{sL_0}{L}\varphi_{\overline{\text{MS}}}\left[\bar{g}_{\overline{\text{MS}}}(L/s)\right] = s\,2^n \varphi_{\overline{\text{MS}}}\left[\sqrt{\bar{g}^2_\nu(L) + p^\nu_1(s)\bar{g}^4_\nu(L) + p^\nu_2(s)\bar{g}^6_\nu(L)}\right],$$

- expect to see independence of the number of steps n, scale factor s and parameter ν.
- Look at $\nu = 0$, depdendence on n and s.
- Note: The neglected order for Λ:

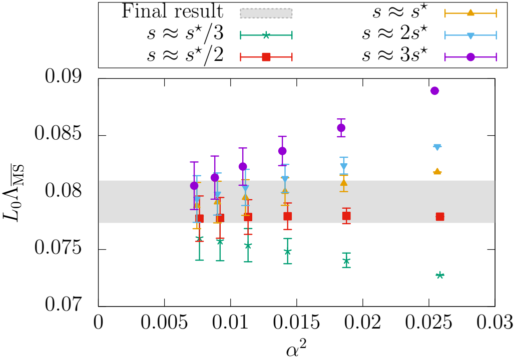
$$\Delta g^2 \frac{d\varphi}{dg^2} \propto \Delta g^2 \left\{g\beta(g)\right\}^{-1} = \Delta g^2 \times \text{O}(g^{-4})$$

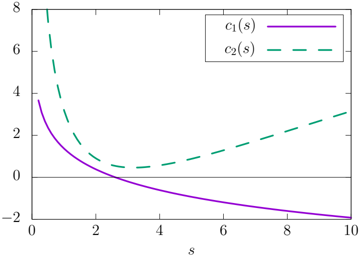
$\Rightarrow$ truncation error: $\text{O}(g^8) \times \text{O}(g^{-4}) = \text{O}(g^4) = \text{O}(\alpha^2)$.

Alternative test via the $\overline{\text{MS}}$-scheme II

$$\alpha(sq) = \alpha_\nu(q) + c_1^\nu(s)\alpha_\nu^2 + c_2^\nu(s)\alpha_\nu^3(q) + \ldots, \qquad p_i^\nu = c_i^\nu/(4\pi)^i$$

parameters: $\nu = 0,\ s^* \approx 3$



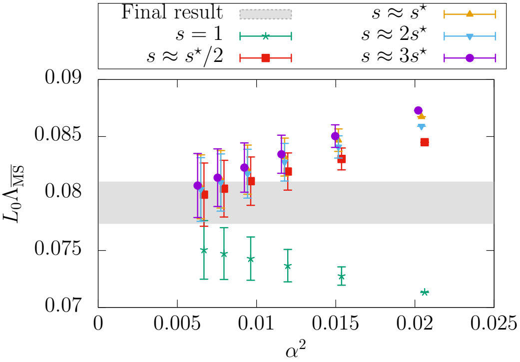


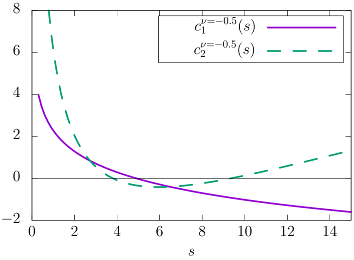
- Choice of scale factor is important, coefficients can get large.
- "fastest apparent convergence" principle: $c_1(s^*) = 0$ which means $s^* = \Lambda_{\overline{\text{MS}}}/\Lambda = 2.612 \approx 3$ seems like a good idea.

Alternative test via the $\overline{\mathrm{MS}}$-scheme III

$$\alpha(sq) = \alpha_\nu(q) + c_1^\nu(s)\alpha_\nu^2 + c_2^\nu(s)\alpha_\nu^3(q) + \ldots, \qquad p_i^\nu = c_i^\nu/(4\pi)^i$$

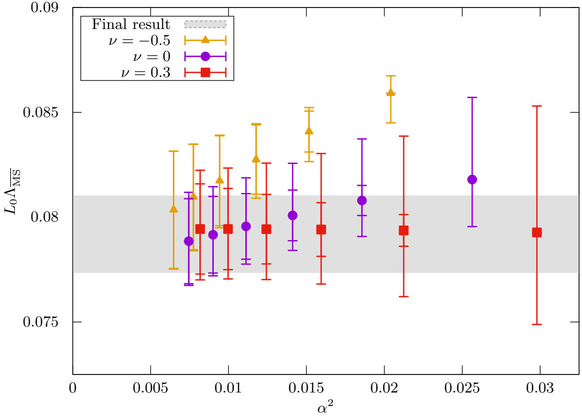
parameters: $\nu = -0.5,\ s^* \approx 5$





Alternative test via the $\overline{\mathrm{MS}}$-scheme IV

variation of the scale factor $s \in [s^*/2, 2s^*]$



$\Rightarrow$ may significantly underestimate the systematic error!

Conclusions & Outlook

- The determination of α_s is well-suited for the lattice approach;
- The systematics can be well controlled by combining technical tools developed over the last 25 years:
  - finite volume renormalization schemes and recursive step-scaling methods
  - non-perturbative Symanzik improvement
  - perturbation theory adapted to finite volume; relation between SF and $\overline{\mathrm{MS}}$-coupling known to 2-loop order!
  - gradient flow couplings and scales (cf. Mattia Dalla Brida's talk)

$\Rightarrow$ Completely solves the problem of large scale differences; perturbation theory at low energies can be avoided!

- Turning this around: many opportunities to test perturbation theory at high energies!

$\Rightarrow$ with hindsight: estimates of perturbative truncation errors require some luck!

Intermediate result presented here

$$\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3} = 0.0791(19)/L_0 \qquad 1/L_0 \approx 4\,\mathrm{GeV}$$

still requires precise connection to a hadronic scale (cf. M. Dalla Brida).