α_s from the femto-universe
(Part II)

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ALPHA Collaboration

Precision measurements of the strong coupling constant,
$12^{\rm th}$ of February 2019, ECT*, Trento

Introduction

Where do we stand and where do we go?

Master formula

$$\alpha_{\overline{\rm MS}}(M_Z) \leftarrow \Lambda_{\overline{\rm MS}}^{N_{\rm f}=5}$$

High-energy [S. Sint talk]

- The running of $\bar{g}^2_{\rm SF}(\mu)$ allowed the precise determination

$$\frac{\Lambda_{\overline{\rm MS}}^{N_{\rm f}=3}}{\mu_0} = 0.0791(19) \quad \text{with} \quad \bar{g}^2_{\rm SF}(\mu_0) = 2.012 \qquad \left[\, \mu_0 = L_0^{-1} \approx 4\,{\rm GeV} \,\right]$$

- Now μ_0 must be related to some **physical** quantity, e.g.,

$$f_\pi\,,\ f_K\,,\ m_p\,,\ \ldots$$

- Reaching low-energies is difficult following the same strategy
  - $\mathrm{var}(\bar{g}^2_{\rm SF})\,/\,\bar{g}^4_{\rm SF} \propto \bar{g}^4_{\rm SF} \Rightarrow$ **expensive** for large couplings!
  - $\mathrm{var}(\bar{g}^2_{\rm SF})$ is large, in general, and $\mathrm{var}(\bar{g}^2_{\rm SF}) \overset{a\to 0}{\propto} L/a$

Question: Can we do better than simply brute force?

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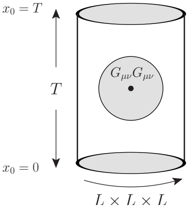
Introduction

The gradient flow coupling

(Lüscher '10; Fodor et. al. '12; Fritzsch, Ramos '13)

Solution: We introduce a new coupling definition:

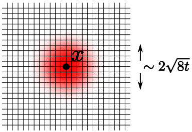
1. Finite volume with Schrödinger functional (SF) bc.'s
2. $\overline{m}_{u,d,s} = 0 \rightarrow$ mass-independent scheme
3. $\overline{g}^2_{\mathrm{GF}}(L^{-1}) \propto t^2 \langle \mathrm{tr}\{G_{\mu\nu}(t,x) G_{\mu\nu}(t,x)\}\rangle_{\mathrm{SF}} \Big|_{\sqrt{8t}=0.3\times L}$



Gradient flow (GF)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x), \quad B_\mu(0,x) = A_\mu(x)$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu], \quad D_\mu = \partial_\mu + [B_\mu, \cdot\,]$$

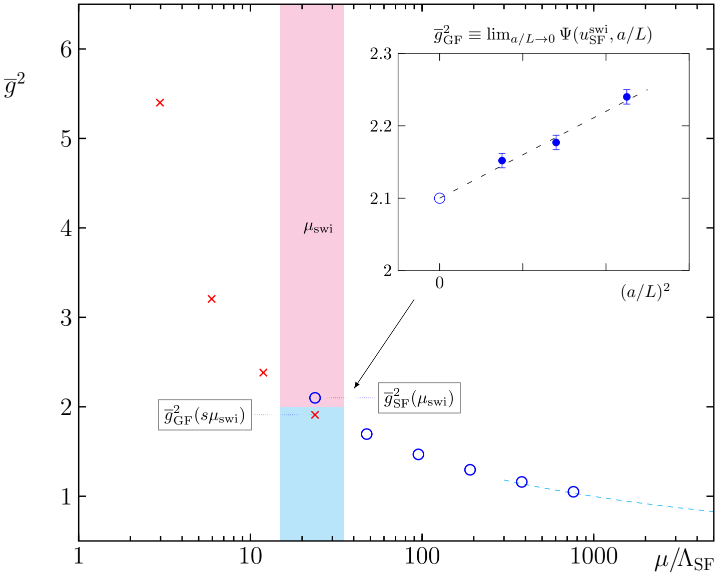


(Lüscher, Weisz '11)

- ✓ Gauge invariant fields are finite
- ✓ Simple to evaluate in simulations
- ✓ $\mathrm{var}(\overline{g}^2_{\mathrm{GF}})$ is small, and $\mathrm{var}(\overline{g}^2_{\mathrm{GF}}) \, / \, \overline{g}^4_{\mathrm{GF}} \overset{a\to 0}{\propto} \mathrm{const.}$
- ✗ Largish lattice artefacts, requires largish L/a
- ✗ PT is quite involved

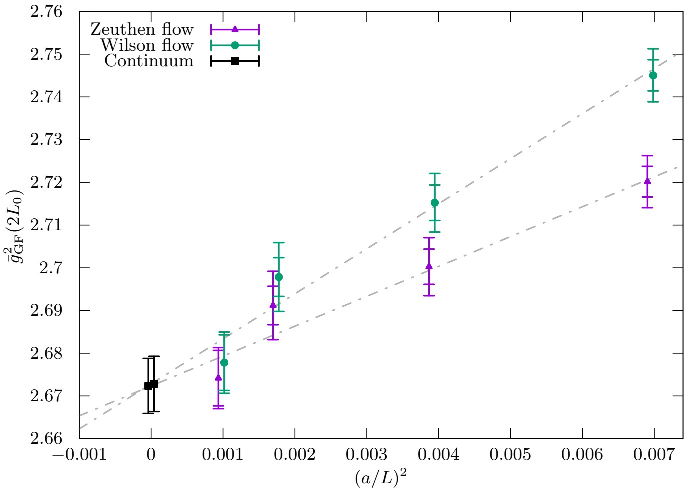
The strategy in a picture

(Fritzsch, Lattice '14)



Step I: matching the couplings

(MDB, Fritzsch, Korzec, Ramos, Sint, Sommer '17a)



$$\bar{g}^2_{\mathrm{SF}}(\mu_0) = 2.012 \quad \Rightarrow \quad \bar{g}^2_{\mathrm{GF}}(\mu_0/2) = 2.6723(64)$$

Step II: running to low-energy

(MDB, Fritzsch, Korzec, Ramos, Sint, Sommer '17a)

How do we do it?

Goal ✓

$$\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3} = \frac{\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3}}{\mu_0} \times \frac{\mu_0}{\mu_\mathrm{had}} \times \mu_\mathrm{had}$$

Step scaling function (SSF)

$$\sigma(u) = \lim_{a/L\to 0} \Sigma(u, a/L) \qquad \Sigma(u, a/L) = \bar{g}^2_\mathrm{GF}(\mu/2)\big|^{u=\bar{g}^2_\mathrm{GF}(\mu)}_{\overline{m}(\mu)=0} \qquad \left[\mu = L^{-1}\right]$$

β-function

$$\ln\left(\frac{\mu_2}{\mu_1}\right) = \int_{\bar{g}(\mu_1)}^{\bar{g}(\mu_2)} \frac{\mathrm{d}g}{\beta(g)} \quad \Rightarrow \quad \log 2 = \int_{\sqrt{\sigma(u)}}^{\sqrt{u}} \frac{\mathrm{d}g}{\beta(g)} \qquad \left[\beta(\bar{g}) = \mu\frac{\mathrm{d}\bar{g}(\mu)}{\mathrm{d}\mu}\right]$$

Ratios of scales

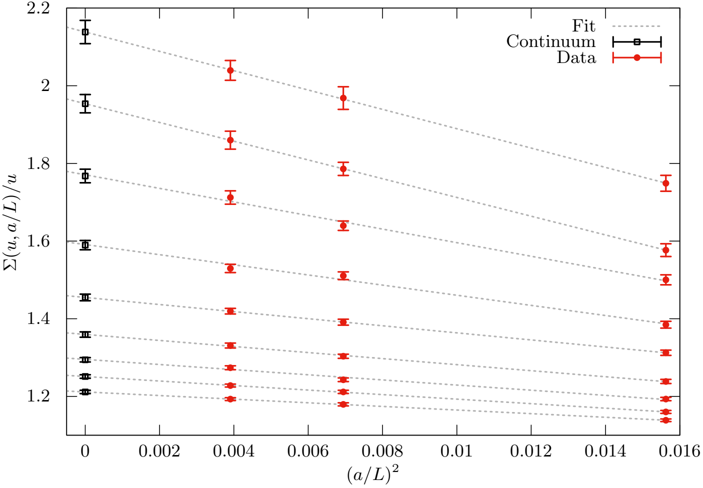
$$\bar{g}^2_\mathrm{GF}(\mu_\mathrm{had}) \equiv 11.31 \quad \Rightarrow \quad \frac{\mu_0}{\mu_\mathrm{had}} = 2\times\exp\left\{\int_{\bar{g}_\mathrm{GF}(\mu_\mathrm{had})}^{\bar{g}_\mathrm{GF}(\mu_0/2)} \frac{\mathrm{d}g}{\beta_\mathrm{GF}(g)}\right\} = 21.86(42)$$

!!Spoiler alert!! Turns out that: $\mu_\mathrm{had} \approx 200\,\mathrm{MeV}$

Step II: running to low-energy

(MDB, Fritzsch, Korzec, Ramos, Sint, Sommer '17a)

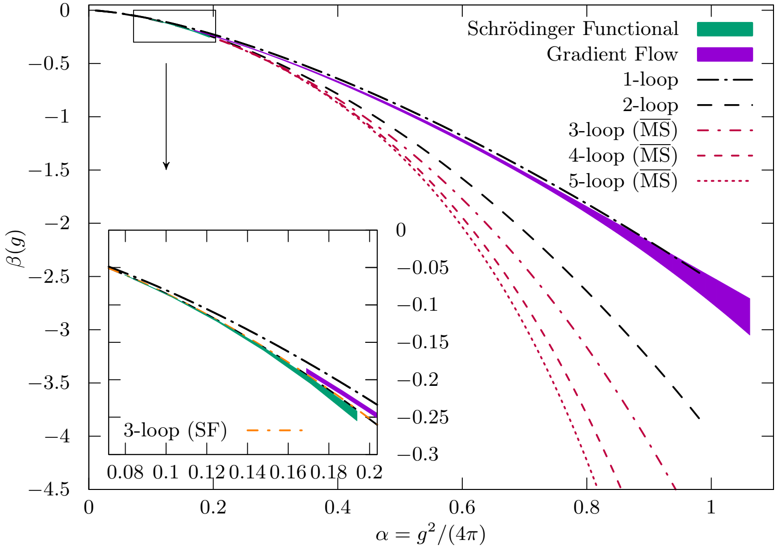
Taking the continuum limit



- Sizeable discretization effects $\rightarrow$ **cautious** extrapolations are needed
- Continuum results are nonetheless very **precise**

Step II: jogging to low-energy

The non-perturbative β-function(s)



Step III: matching to hadronic physics

Setting the scale

What's next?

$$\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3} = \frac{\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3}}{\mu_0} \times \frac{\mu_0}{\mu_\mathrm{had}} \times \mu_\mathrm{had}$$

A reference scale

(Lüscher '10)

$$\mu_\mathrm{ref} = (8t_0)^{-\frac{1}{2}} \quad \text{where} \quad t_0^2 \langle \mathrm{tr}\{G_{\mu\nu}(t_0,x)G_{\mu\nu}(t_0,x)\}\rangle = 0.3$$

- simply and accurately measured in simulations
- gluonic quantity w/ very mild m_π-dependence (Bär, Golterman '14)
- not directly measurable in experiments

Scale setting

(Bruno, Korzec, Schaefer '16)

$$\left.\begin{array}{l} \text{CLS + FLAG + PDG} \\ m_\pi, m_K, f_\pi, f_K \end{array}\right\} \Rightarrow \lim_{\substack{m_{\pi,K}\to m_{\pi,K}^\mathrm{phys} \\ a\to 0}} \frac{(a\mu_\mathrm{ref})}{(af_{\pi K})} = 3.22(3) \quad \left[f_{\pi K} = \tfrac{2}{3}(f_K + \tfrac{1}{2} f_\pi) \right]$$

- **CLS effort:** • $N_\mathrm{f} = 2+1$ • $m_\pi \approx 200 - 420\,\mathrm{MeV}$ • $a \approx 0.04 - 0.09\,\mathrm{fm}$
- Hadronic inputs corrected for QED and $m_u \neq m_d$ effects (FLAG, PDG)

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Result

$\mu_\mathrm{ref} = 475(5)\,\mathrm{MeV} \sim 1\% \rightarrow$ very **precise** relative scale!

Step III: matching to hadronic physics

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Result

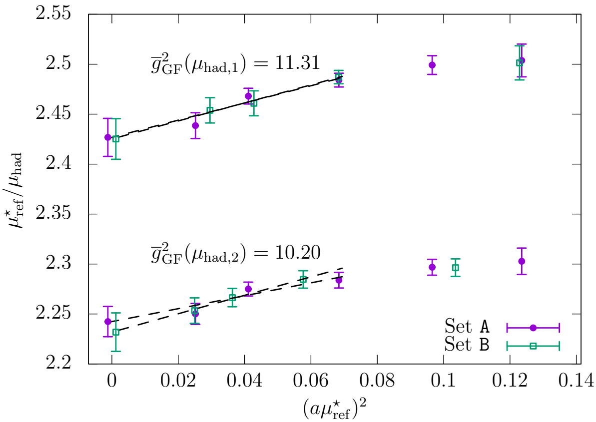
$\mu^*_{\rm ref} = 478(6)\,\mathrm{MeV} \sim 1.2\%$ @ $m_{u,d,s} = m_{\rm av,phys}$ where $m_\pi = m_K \approx 400\,\mathrm{MeV}$

Step III: matching to hadronic physics

The Λ-parameter of $N_\mathrm{f} = 3$ QCD (Bruno, MDB, Fritzsch, Korzec, Ramos, Schaefer, Simma, Sint, Sommer '17b)

Connecting small and infinite volume physics

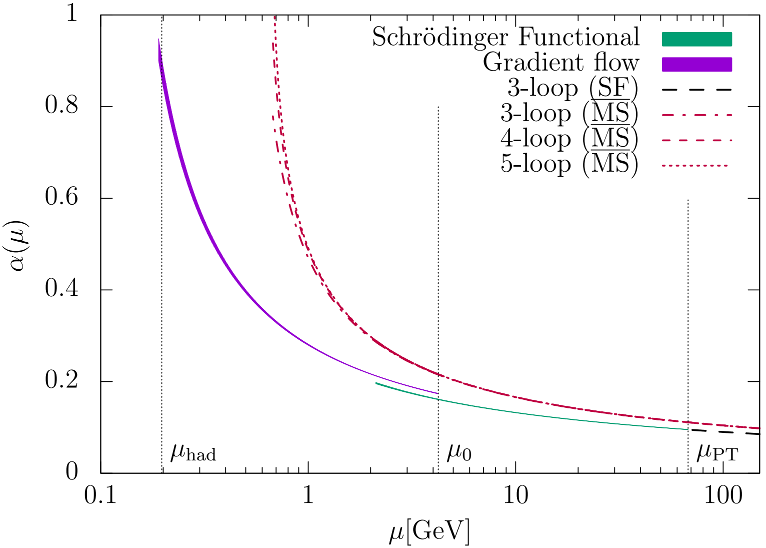
$$\frac{\mu^*_\mathrm{ref}}{\mu_\mathrm{had}} = \lim_{a\to 0} \frac{(a\mu^*_\mathrm{ref})}{(a\mu_\mathrm{had})}$$



Λ-parameter

$$\Lambda^{N_\mathrm{f}=3}_{\overline{\mathrm{MS}}} = \underset{\checkmark}{\frac{\Lambda^{N_\mathrm{f}=3}_{\overline{\mathrm{MS}}}}{\mu_0}} \times \underset{\checkmark}{\frac{\mu_0}{\mu_\mathrm{had}}} \times \underset{\checkmark}{\frac{\mu_\mathrm{had}}{\mu^*_\mathrm{ref}}} \times \underset{\checkmark}{\frac{\mu^*_\mathrm{ref}}{f_{\pi K}}} \times f^{(\mathrm{PDG})}_{\pi K} = 341(12)\ \mathrm{MeV} \ \sim 3.5\%$$

Non-perturbative running couplings in $N_\mathrm{f} = 3$ QCD



Heavy-quark effects & decoupling

Perturbative, or non-perturbative, that is the question

Q: How do we go from $\Lambda_{\overline{\rm MS}}^{N_{\rm f}=3} \to \Lambda_{\overline{\rm MS}}^{N_{\rm f}=5}$?

Heavy-quark effects

- **Type 1.** Charm (and bottom) contributions to (dimensionless) low-energy quantities:

$$\mathcal{Q}_{N_{\rm f}=4} = \mathcal{Q}_{N_{\rm f}=3} + \mathrm{O}(\Lambda^2/M_c^2) \quad \text{e.g.} \quad \mathcal{Q} = \frac{\mu_{\rm ref}}{f_{\pi K}}, \; \frac{\mu_{\rm ref}}{m_{\pi,K}}, \; \ldots$$

  Non-perturbative studies show: $\mathrm{O}(\Lambda^2/M_c^2) \lesssim 0.5\% \Rightarrow$ **not important**! (Knechtli et. al. '17)

- **Type 2.** Charm and bottom contributions to the running of α_s:

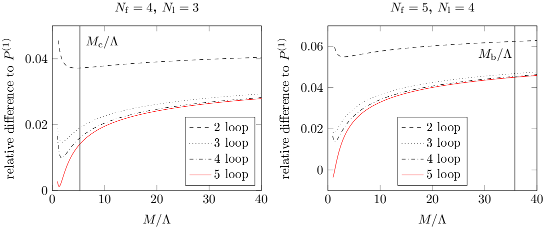
$$\alpha_{\overline{\rm MS}}^{(N_{\rm l})}(\mu) = \xi^2(\alpha_{\overline{\rm MS}}^{(N_{\rm f})}, \overline{m}/\mu)\, \alpha_{\overline{\rm MS}}^{(N_{\rm f})}(\mu) \;\Rightarrow\; \Lambda_{\overline{\rm MS}}^{N_{\rm l}}/\Lambda_{\overline{\rm MS}}^{N_{\rm f}} = P(M/\Lambda_{\overline{\rm MS}}^{N_{\rm f}})$$

Key observations (Athenodorou et. al. '18)

- P is very **well-behaved** in PT
- Non-perturbative $\mathrm{O}(\Lambda^2/M_c^2)$ corrections to P are $\lesssim \mathbf{0.5\%}$

Conclusion

$\Lambda_{\overline{\rm MS}}^{N_{\rm f}=3} \overset{\rm PT}{\to} \Lambda_{\overline{\rm MS}}^{N_{\rm f}=5}$ is accurate enough!



Conclusions & Outlook

The strong coupling α_s

Final result

- Using as inputs: the non-perturbative $\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3}$, and $\overline{m}^c_{\overline{\mathrm{MS}}}$, $\overline{m}^b_{\overline{\mathrm{MS}}}$ from the PDG, perturbative decoupling predicts:

$$\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3} \to \Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=4} = 298(12)(3)_{\mathrm{PT}}\,\mathrm{MeV} \to \Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=5} = 215(10)(3)_{\mathrm{PT}}\,\mathrm{MeV}$$

- Having $\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=5}$ we quote:

$$\alpha_{\overline{\mathrm{MS}}}(M_Z) = 0.11852(84)$$

ALPHA Collaboration (Bruno, MDB, Fritzsch, Korzec, Ramos, Schaefer, Simma, Sint, Sommer '17b)

which compares well with:

$$\alpha_{\overline{\mathrm{MS}}}^{\text{FLAG '16}}(M_Z) = 0.1182(12) \quad \text{and} \quad \alpha_{\overline{\mathrm{MS}}}^{\text{PDG '18}}(M_Z) = 0.1181(11)$$

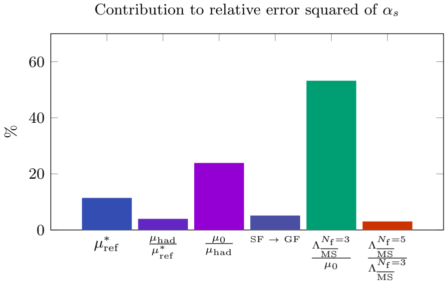
Conclusions

- Lattice QCD is a **natural** and **powerful** tool for determining α_s
- Using **finite-volume couplings** and a **step-scaling** strategy, we can keep uncertainties **under control** at all stages of the computation
- Our $\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3}$ result uses PT at scales of $\mathrm{O}(100\,\mathrm{GeV})$ where we proved it accurate
- The $N_\mathrm{f} = 3$ and $N_\mathrm{f} = 5$ theories are matched using **perturbative decoupling**. Non-perturbative corrections are **not important**, as long as: $\delta\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3}/\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3} \gtrsim 1\%$

Conclusions & Outlook

Error budget and future perspectives

$$\mu^*_{\rm ref} \times \frac{\mu_{\rm had}}{\mu^*_{\rm ref}} \times \frac{\mu_0}{\mu_{\rm had}} \times \frac{\Lambda^{N_{\rm f}=3}_{\overline{\rm MS}}}{\mu_0} \times \frac{\Lambda^{N_{\rm f}=5}_{\overline{\rm MS}}}{\Lambda^{N_{\rm f}=3}_{\overline{\rm MS}}} = \Lambda^{N_{\rm f}=5}_{\overline{\rm MS}} \to \alpha_s$$



Outlook

- The dominant source of error is the non-perturbative running from 4 to $70\,\mathrm{GeV}$
  This error is **statistical** ⇒ there is room for **improvement**!
- In the near future we can reach: $\delta\Lambda^{N_{\rm f}=3}_{\overline{\rm MS}}/\Lambda^{N_{\rm f}=3}_{\overline{\rm MS}} \approx 2\% \Rightarrow \delta\alpha_s/\alpha_s \approx 0.5\%$
- Significantly below this precision we need to reconsider several things:
  QED and iso-spin effects, non-perturbative decoupling, ...


Thanks

BACKUP

Perturbative decoupling

Ignoring $O((\overline{m}^h)^{-2})$ corrections, matching means:

$$\bar{g}^{(N_\mathrm{f})}(\mu) = \bar{g}^{(N_\mathrm{f}+1)}(\mu) \times \xi\left(g^{(N_\mathrm{f}+1)}(\mu), \frac{\overline{m}^h}{\mu}\right) \Rightarrow \frac{\Lambda^{N_\mathrm{f}}}{\Lambda^{N_\mathrm{f}+1}} = \frac{\varphi^{(N_\mathrm{f})}(\bar{g}^{(N_\mathrm{f}+1)} \times \xi)}{\varphi^{(N_\mathrm{f}+1)}(\bar{g}^{(N_\mathrm{f}+1)})}$$

where $[b_{0,1} \equiv b_{0,1}(N_\mathrm{f})]$

$$\varphi^{(N_\mathrm{f})}(\bar{g}) = (b_0\bar{g}^2)^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}^2}} \times \exp\left\{-\int_0^{\bar{g}} \mathrm{d}x \left[\frac{1}{\beta^{(N_\mathrm{f})}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

- ξ is known to 4-loops and β to 5-loops (Chetyrkin, Kühn, C. Sturm '06; Baikov, Chetyrkin, Kühn, '17)
- We take $\mu = m_*^h$ where $m_*^h = \overline{m}^h_{\overline{\mathrm{MS}}}(m_*^h)$, $h = c, b$
- PT looks surprisingly well-behaved already at $\mu = m_*^c$

n (ξ-loops)	$\alpha_{\overline{\mathrm{MS}}}^{(N_\mathrm{f}=5)}$	$\alpha_n - \alpha_{n-1}$
1	0.11699	
2	0.11827	0.00128
3	0.11846	0.00019
4	0.11852	0.00006

Within PT, a conservative error to attribute to α_s is: $|\alpha_2 - \alpha_4| \approx 0.0003$

The gradient flow β-function at high-energy

The pure SU(3) Yang-Mills case

(MDB, Ramos, in preparation)