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Local Analytic Sector Subtraction at NNLO

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based on [Magnea et al., arXiv:1806.09570, arXiv:1809.05444]

Motivation

- The Standard Model (SM) is not the end of the story: new physics under investigation at high-energy colliders.
- No spectacular discoveries at the LHC: search for small effects.
  Very precise knowledge of the SM required.
- QCD predictions more stable when including higher-order corrections.

Needed theoretical predictions with the highest possible precision, in order to compare them with experimental data.

Strong efforts in calculations
at next-to-next-to-leading order in α_S (NNLO QCD).

Many schemes available for NNLO subtraction:

→ N-jettiness, qT-subtraction, Antenna, Sector Improved, Nested Soft-Collinear, Colorful, Projection to Born, Sector Decomposition, ε-prescription, Geometric, Unsubtraction

Anatomy of subtractions at NLO

Massless, partons in the final state only: NLO contribution

$$\frac{d\sigma_{\rm NLO}}{dX} \quad = \quad \int d\Phi_n \, V \, \delta_n + \int d\Phi_{n+1} \, R \, \delta_{n+1} \, . \tag{1}$$

where $R = \left|\mathcal{A}^{(0)}_{n+1}\right|^2$, $V = 2\,\mathbf{Re}\left[\mathcal{A}^{(0)*}_n \, \mathcal{A}^{(1)}_n\right]$, and $X =$ IRC safe obs., $X_i =$ computed with $i^{\rm th}$-body kinematics, $\delta_i = \delta(X - X_i)$.

In $d = 4 - 2\epsilon$, phase-space integration of R results in explicit infrared (IR) poles in ϵ, which cancel those of V, if X infrared safe, ensuring the cross section is finite (KLN).

Subtraction procedure: avoiding analytic integration of the full R amplitudes by adding and subtracting to Eq. (1) a counterterm

$$\left.\frac{d\sigma_{\rm NLO}}{dX}\right|_{\rm ct} = \int d\Phi_{n+1} \, \overline{K} \, \delta_n \, , \qquad I = \int d\Phi_{\rm rad} \, \overline{K} \, , \tag{2}$$

$d\Phi_{n+1} \, \overline{K}$ has the same singular limits of $d\Phi_{n+1} \, R$ and must be simple to be analytically integrated in d dim.

$$\frac{d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \Big(V + I\Big) \, \delta_n + \int d\Phi_{n+1} \, \Big(R \, \delta_{n+1} - \overline{K} \, \delta_{n+1}\Big) \tag{3}$$

First and the second terms separately finite in $d = 4$: efficient numerical integration.

FKS and CS schemes: advantages and bottlenecks

Two main subtraction schemes at NLO:

Frixione-Kunst-Signer procedure [Frixione, Kunst, Signer, hep-ph/9512328]

- Partition of the radiative phase-space with sector functions
- Different parameterization for each sector
- Analytical integration, after eliminating sector functions (sum rules): can be not trivial, due to non optimal parameterization

Catani-Seymour procedure [Catani, Seymour, hep-ph/9605323]

- A counterterm reproduces the IR singularities related to a parton pair in all of phase-space: complicated structure
- Sum of counterterms, each reparameterized with a specific kinematic mapping
- Analytical integration of each term: can be non trivial, as counterterms have a complicated structure

FKS and CS schemes: advantages and bottlenecks

How much can we simplify the NLO subtraction procedure? Our strategy:

Frixione-Kunst-Signer procedure [Frixione, Kunst, Signer, hep-ph/9512328]

- Partition of the radiative phase-space with sector functions
- Different parameterization for each sector
- Analytical integration, after eliminating sector functions (sum rules): can be not trivial, due to non optimal parameterization

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A minimal scheme at NLO (1)

Partition of the radiative phase-space Φ_{n+1} with sector functions $\mathcal{W}_{ij}$ ($i, j = 1, \ldots, n+1$ final-state partons)

- minimal singularity structure: $R\,\mathcal{W}_{ij}$ is singular only in one soft ($\mathbf{S}_i$) and one collinear ($\mathbf{C}_{ij}$) configuration
- normalization and sum rules:

$$\sum_{i,\,j\neq i} \mathcal{W}_{ij} = 1, \qquad \mathbf{S}_i \sum_{j\neq i} \mathcal{W}_{ij} = 1\,, \qquad \mathbf{C}_{ij} \sum_{ab\,\in\,\mathrm{perm}(ij)} \mathcal{W}_{ab} = 1\,,$$

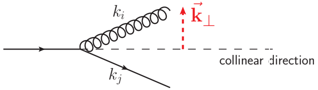
- Summing over sectors sharing a singularity, and taking the singular limit on the sum, the $\mathcal{W}$'s disappear: simplified analytic integration of the counterterm.
- Choice of sector functions very similar to FKS ($s_{qi} = 2\,q_{\mathrm{cm}} \cdot k_i\,, \quad s_{ij} = 2\,k_i \cdot k_j$):

$$\boxed{\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum\limits_{k,\,l\neq k} \sigma_{kl}}\,, \quad \text{with} \quad \sigma_{ij} = \frac{1}{e_i\, w_{ij}}\,,\ e_i = \frac{s_{qi}}{s}\,,\ w_{ij} = \frac{s\, s_{ij}}{s_{qi}\, s_{qj}}\,.}$$

A minimal scheme at NLO (2)

- Singular structure of R in sector ij in terms of dot products $\{s_{ab} = 2p_a \cdot p_b\}$:

$\mathbf{S}_i\, R$ = leading term in R ($k_i^\mu \to 0$),
$\mathbf{C}_{ij}\, R$ = leading term in R ($k_\perp^\mu \to 0$).



$\longrightarrow$ combinations of universal IR kernels and Born matrix-elements

$$\mathbf{S}_i\, R(\{k\}) = -\mathcal{N}_1 \sum_{l,\,m} \underbrace{\delta_{f_i g}\, \frac{s_{lm}}{s_{il}\, s_{im}}}_{\text{Eikonal kernel}} B_{lm}(\{k\}_{\not i})$$

$$\mathbf{C}_{ij}\, R(\{k\}) = \frac{\mathcal{N}_1}{s_{ij}} \underbrace{P_{ij}^{\mu\nu}(s_{ir}, s_{jr})}_{\text{Altarelli}-\text{Parisi split. f.}} B_{\mu\nu}\Big(\{k\}_{\not i \not j}, k\Big)$$

$$\mathbf{S}_i\, \mathbf{C}_{ij}\, R(\{k\}) = 2\,\mathcal{N}_1\, C_{f_j}\, \delta_{f_i g}\, \frac{s_{jr}}{s_{ij}\, s_{ir}}\, B\big(\{k\}_{\not i}\big)$$

- Candidate counterterm in sector ij: $K_{ij} = (\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i\, \mathbf{C}_{ij}) R\, \mathcal{W}_{ij}$

- Minimal structure as FKS, but no parametrisation yet: freedom to be exploited to simplify analytic integration.

A minimal scheme at NLO (3)

Need a momentum mapping $\{k_1, ..., k_{n+1}\} \to \{\bar{k}_1, ..., \bar{k}_n\}$ to factorize radiation $d\Phi_{\rm rad}$ from Born phase-space $d\Phi_n$, and integrate counterterm $\to$ Catani-Seymour mappings

- Mapping $\{k\} \to \{\bar{k}\}^{(abc)}$: freedom to choose a, b, c as we want.
  Optimal choice: adapt to invariants appearing in the kernels.

$$\overline{\mathbf{S}}_i \, R\,(\{k\}) = -\,\mathcal{N}_1 \sum_{l,\,m} \delta_{f_i g} \, \frac{s_{lm}}{s_{il}\, s_{im}} \, B_{lm}\Big(\{\bar{k}\}^{(ilm)}\Big)$$

$$\overline{\mathbf{C}}_{ij} \, R\,(\{k\}) = \frac{\mathcal{N}_1}{s_{ij}} \, P_{ij}^{\mu\nu}(s_{ir}, s_{jr}) \, B_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)}\Big)$$

$$\overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, R\,(\{k\}) = 2\,\mathcal{N}_1 \, C_{f_j} \, \delta_{f_i g} \, \frac{s_{jr}}{s_{ij}\, s_{ir}} \, B\Big(\{\bar{k}\}^{(ijr)}\Big)$$

- Definition for the local counterterm (barred limits on $\mathcal{W}$'s act as unbarred):

$$\overline{K}_{ij} \equiv \Big(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij}\Big) \, R \, \mathcal{W}_{ij} \,, \qquad \overline{K} = \sum_{i, j \neq i} \overline{K}_{ij} \,, \qquad (4)$$

A minimal scheme at NLO (4)

Getting rid of sector functions in $\overline{K}$:

$$\overline{K} = \sum_{i,j\neq i} \overline{K}_{ij} = \sum_i (\overline{\mathbf{S}}_i R) \underbrace{\left[\mathbf{S}_i \sum_{j\neq i} \mathcal{W}_{ij}\right]}_{=1\text{ , sum rules}} + \sum_{i,j>i} (\overline{\mathbf{C}}_{ij} R) \underbrace{\left[\mathbf{C}_{ij}(\mathcal{W}_{ij} + \mathcal{W}_{ji})\right]}_{=1\text{ , sum rules}} - \sum_{i,j\neq i} \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} R$$

$$= \sum_i \overline{\mathbf{S}}_i \, R + \sum_{i,\,j>i} \overline{\mathbf{C}}_{ij} \left(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j\right) R \,. \tag{5}$$

Recall : $\dfrac{d\sigma_{\text{NLO}}}{dX} = \displaystyle\int d\Phi_n \left(V + I\right) \delta_n + \int d\Phi_{n+1} \left(R\, \delta_{n+1} - \overline{K}\, \delta_{n+1}\right)$

Final result for the integrated counterterm (over $d\Phi_{\text{rad}}$):

$$I(\{\bar{k}\}) = -\mathcal{N}_1 \sum_{l,\, m\neq l} \frac{(4\pi)^{\epsilon-2}}{\bar{s}_{lm}^{\,\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} B_{lm}(\{\bar{k}\})$$

$$-\mathcal{N}_1 \sum_p \frac{(4\pi)^{\epsilon-2}}{\bar{s}_{pr}^{\,\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)} \left(\frac{C_A + 4\,T_R\,N_f}{2(3-2\epsilon)} \delta_{f_p g} + \frac{C_F}{2} \delta_{f_p\{q,\bar{q}\}}\right) B(\{\bar{k}\}). \tag{6}$$

exact in ϵ, virtual poles analytically reproduced, finite parts checked for a variety of differential distributions (against `MadGraph_aMC@NLO`).

A local analytic scheme at NNLO

NLO: bridge between FKS (minimality and phase space sectoring) and CS (Lorentz invariance and kinematics mappings) $\rightarrow$ Simplified analytic cterm integration

These nice properties can be exported to NNLO

Anatomy of subtractions at NNLO (1)

$$\frac{d\sigma_{\text{NNLO}}}{dX} = \int d\Phi_n\, VV\, \delta_n + \int d\Phi_{n+1}\, RV\, \delta_{n+1} + \int d\Phi_{n+2}\, RR\, \delta_{n+2}. \tag{7}$$

Add and subtract local counterterms:

$$\int d\Phi_{n+2}\, \overline{K}^{(1)}\, \delta_{n+1}\,, \qquad \int d\Phi_{n+2}\, (\overline{K}^{(2)} + \overline{K}^{(12)})\, \delta_n\,, \qquad \int d\Phi_{n+1}\, \overline{K}^{(\mathbf{RV})}\, \delta_n\,. \tag{8}$$

$\overline{K}^{(1)}$: same single-unresolved singularities as RR

$(\overline{K}^{(2)} + \overline{K}^{(12)})$: same double-unresolved singularities as RR, the first features double-unres. limits (pure), the second single-unres. limits of double-unres. ones (mixed)

$\overline{K}^{(\mathbf{RV})}$: same single-unresolved singularities as RV

Integrated counterterms (in d-dimensions):

$$I^{(\mathbf{i})} = \int d\Phi_{\text{rad},i}\, \overline{K}^{(\mathbf{i})}\,, \quad I^{(12)} = \int d\Phi_{\text{rad},1}\, \overline{K}^{(12)}\,, \quad I^{(\mathbf{RV})} = \int d\Phi_{\text{rad}}\, \overline{K}^{(\mathbf{RV})}\,,$$

Anatomy of subtractions at NNLO (2)

Subtracted NNLO

$$\begin{aligned}
\frac{d\sigma_{\mathrm{NNLO}}}{dX} \;=\; & \int d\Phi_n \underbrace{\left(VV + I^{(2)} + I^{(\mathbf{RV})}\right)}_{\text{finite in d=4 and in } \Phi_n} \delta_n \\
& + \int d\Phi_{n+1} \underbrace{\left[\underbrace{\left(RV + I^{(1)}\right)}_{\text{finite in d=4, singular in } \Phi_{n+1}} \delta_{n+1} - \underbrace{\left(\overline{K}^{(\mathbf{RV})} - I^{(12)}\right)}_{\text{finite in d=4, singular in } \Phi_{n+1}} \delta_n \right]}_{\text{finite in d=4 and in } \Phi_{n+1}} \\
& + \int d\Phi_{n+2} \underbrace{\left[RR\,\delta_{n+2} - \overline{K}^{(1)}\,\delta_{n+1} - \left(\overline{K}^{(2)} + \overline{K}^{(12)}\right)\delta_n \right]}_{\text{finite in d=4 and in } \Phi_{n+2}} . \qquad (9)
\end{aligned}$$

Sector functions at NNLO (1)

- Partition of Φ_{n+2} with sector functions $\mathcal{W}_{ijkl}$, $(\sum_{ijkl} \mathcal{W}_{ijkl} = 1)$. Our choice:

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sum\limits_{a,\, b\neq a} \sum\limits_{\substack{c\neq a \\ d\neq a,c}} \sigma_{abcd}}\,, \qquad \sigma_{ijkl} = \frac{1}{e_i^{\alpha}\, w_{ij}^{\beta}} \frac{1}{(e_k + \delta_{kj}\, e_i)\, w_{kl}}\,, \qquad \alpha > \beta > 1\,.$$

- $RR\,\mathcal{W}_{abcd}$ is singular only in few configurations ($\mathbf{S}_{ab} = a\,b$ uniformly soft, $\mathbf{C}_{ijk} = j\,k$ uniformly collinear to i, $\mathbf{SC}_{ijk} = i$ soft and j, k collinear, $\mathbf{CS}_{ijk} = i, j$ collinear and k soft)

$$\begin{aligned}
\mathcal{W}_{ijjk} \quad &: \quad \mathbf{S}_i\,, \ \ \mathbf{C}_{ij}\,, \ \ \mathbf{S}_{ij}\,, \ \ \mathbf{C}_{ijk}\,, \ \ \mathbf{SC}_{ijk}\,; \\
\mathcal{W}_{ijkj} \quad &: \quad \mathbf{S}_i\,, \ \ \mathbf{C}_{ij}\,, \ \ \mathbf{S}_{ik}\,, \ \ \mathbf{C}_{ijk}\,, \ \ \mathbf{SC}_{ijk}\,, \ \ \mathbf{CS}_{ijk}\,; \\
\mathcal{W}_{ijkl} \quad &: \quad \mathbf{S}_i\,, \ \ \mathbf{C}_{ij}\,, \ \ \mathbf{S}_{ik}\,, \ \ \mathbf{C}_{ijkl}\,, \ \ \mathbf{SC}_{ikl}\,, \ \ \mathbf{CS}_{ijk}\,.
\end{aligned}$$

- Sum rules in double-unresolved limits: summing over sectors sharing the same singularity, and taking that singular limit on the sum, $\mathcal{W}$ functions disappear.

Sector functions at NNLO (2)

- In the single-unresolved limits, NNLO sector functions factorise NLO sector functions. For example

$$\mathbf{C}_{ij}\,\mathcal{W}_{ijkl} \;=\; \mathcal{W}_{kl}\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,, \qquad\qquad \mathbf{S}_{i}\,\mathcal{W}_{ijkl} \;=\; \mathcal{W}_{kl}\,\mathbf{S}_{i}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,,$$

  where

$$\mathcal{W}_{ij}^{(\alpha\beta)} \;=\; \frac{\sigma_{ij}^{(\alpha\beta)}}{\sum\limits_{a,\,b\neq a}\sigma_{ab}^{(\alpha\beta)}}\,, \qquad\qquad \sigma_{ab}^{(\alpha\beta)} \;=\; \frac{1}{(e_a)^{\alpha}(w_{ab})^{\beta}}\,.$$

  with the same properties of NLO sector functions.

- Allows $(RV + I^{(\mathbf{1})})$ and $(K^{(\mathbf{RV})} - I^{(\mathbf{12})})$ to be finite in $d = 4$ NLO sector by NLO sector.

Counterterms at NNLO (1): sector $\mathcal{W}_{ijkj}$

- Candidate (not remapped) counterterms built collecting singular limits of $RR\,\mathcal{W}$, written in terms of dot products.
- Ex. sector $\mathcal{W}_{ijkj}$ (nonzero limits: $\mathbf{S}_i$, $\mathbf{C}_{ij}$, $\mathbf{S}_{ik}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{ijk}$, $\mathbf{CS}_{ijk}$):

$$
\begin{aligned}
K^{(1)}_{ijkj} &= \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\, RR\,\mathcal{W}_{ijkj}, \\
K^{(2)}_{ijkj} &= \Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk}) \\
&\qquad + \mathbf{CS}_{ijk}(1-\mathbf{SC}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\Big]\, RR\,\mathcal{W}_{ijkj}, \\
K^{(12)}_{ijkj} &= -\Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk}) \\
&\qquad + \mathbf{CS}_{ijk}(1-\mathbf{SC}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\Big]\, RR\,\mathcal{W}_{ijkj},
\end{aligned}
$$

- $\mathbf{S}_{ij}\,RR$, $\mathbf{C}_{ikj}\,RR$, $\mathbf{SC}_{ijk}\,RR$: universal kernels [Catani, Grazzini, hep-ph/9908523].
- Simplifications possible $\rightarrow$ $\mathbf{SC}_{ijk}$, $\mathbf{CS}_{ijk}$ cancel in the sum $K^{(2)} + K^{(12)}$.
- Limits on RR and on $\mathcal{W}$ functions commute.

Counterterms at NNLO (2): remapping and integration

Different kernels / different terms in the same kernel are parametrized with different NNLO mappings, to simplify integration.

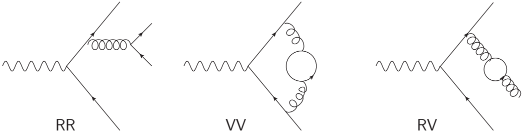
$\overline{K}^{(\mathbf{1})}$ – to be integrated over $d\Phi_{n+1}$, thanks to the factorization properties of $\mathcal{W}_{abcd}$ and the sum rules of $\mathcal{W}_{ab}^{(\alpha\beta)}$ (same integral as at NLO, $I^{(1)}$)
– $I^{(\mathbf{1})}$ has the same $1/\epsilon$ structure as RV, NLO sector by NLO sector.

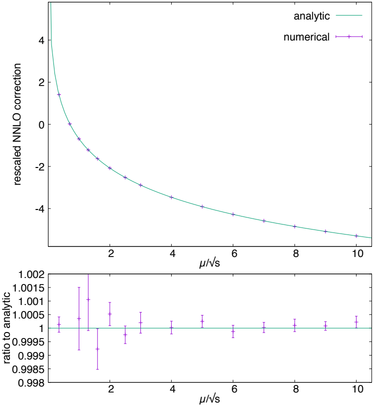
$\overline{K}^{(\mathbf{12})}$ – to be integrated over $d\Phi_{n+1}$, thanks to the factorization properties of $\mathcal{W}_{abcd}$ and the sum rules of $\mathcal{W}_{ab}^{(\alpha\beta)}$
– $I^{(\mathbf{12})}$ has the same $1/\epsilon$ structure as $\overline{K}^{(\mathbf{RV})}$, NLO sector by NLO sector.

$\overline{K}^{(\mathbf{2})}$ – to be integrated over $d\Phi_{n+2}$, $\mathcal{W}$ functions disappear from $\overline{K}^{(\mathbf{2})}$ as well as from $I^{(\mathbf{2})}$ thanks to sum rules
– analytic integration of NNLO kernels $\times$ Born matrix-elements, without $\mathcal{W}$ functions.

Proof-of-concept

$T_R\, C_F$ contributions to
$e^+e^- \to jj$ @ NNLO





Inclusive cross-section (NNLO correction) obtained via numerical implementation of the subtraction scheme, compared with the analytic result,

$$\frac{\sigma_{\text{NNLO}}}{\sigma_{\text{LO}}\left(\frac{\alpha_s}{2\pi}\right)^2 T_R\, C_F} = \left(-\frac{11}{2} + 4\,\zeta_3 - \log\frac{\mu^2}{s}\right)$$

with the renormalization-scale dependence.

Very good agreement ($\lesssim 0.1\%$ differences).

Conclusions and outlook

A subtraction procedure at NNLO: local, analytic, general, efficient.

Present procedure is designed for final-state QCD radiation and massless partons.

Status and wishlist:

- implementation in a differential numerical code is in progress,
- analytic integration of part of the double-unresolved and real-virtual counterterms is ongoing,
- we are planning to exted such a procedure to initial state radiation (conceptually straightforward), and to massive partons (more involved)

Backup slides

Mapping from NLO to Born kinematics

- Catani-Seymour massless final-state mapping $\{k\} \to \{\bar{k}\}^{(abc)}$:

$$\bar{k}_i^{(abc)} = k_i, \quad \text{if } i \neq a, b, c, \quad \bar{k}_b^{(abc)} = k_a + k_b - \frac{s_{ab}}{s_{ac} + s_{bc}}\, k_c\,, \quad \bar{k}_c^{(abc)} = \frac{s_{abc}}{s_{ac} + s_{bc}}\, k_c$$

with $s_{abc} = s_{ab} + s_{ac} + s_{bc}$, and $\bar{k}_b^{(abc)} + \bar{k}_c^{(abc)} = k_a + k_b + k_c$.

- Catani-Seymour variables $y, z \in [0,1]$ for mapping $\{k\} \to \{\bar{k}\}^{(abc)}$:

$$s_{ab} = y\, s_{abc}\,, \qquad s_{ac} = z(1-y)\, s_{abc}\,, \qquad s_{bc} = (1-z)(1-y)\, s_{abc}$$

- Phase-space factorization:

$$d\Phi_{n+1} = d\Phi_n^{(abc)}\, d\Phi_{\rm rad}^{(abc)}\,, \qquad d\Phi_{\rm rad}^{(abc)} \equiv d\Phi_{\rm rad}\left(\bar{s}_{bc}^{(abc)}; y, z, \phi\right)$$

$$\int d\Phi_{\rm rad}\,(s; y, z, \phi) \equiv N(\epsilon)\, s^{1-\epsilon} \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz \left[y(1-y)^2\, z(1-z)\right]^{-\epsilon} (1-y)\,,$$

$$N(\epsilon) \equiv \frac{(4\pi)^{\epsilon-2}}{\sqrt{\pi}\, \Gamma(1/2-\epsilon)}\,, \qquad \bar{s}_{bc}^{(abc)} \equiv 2\, \bar{k}_b^{(abc)} \cdot \bar{k}_c^{(abc)} = s_{abc}\,.$$

- ϕ = azimuth between $\vec{k}_a$ and an reference three-momentum ($\neq \vec{k}_b, \vec{k}_c$).

Sector functions at NNLO: sum rules

- Sum rules in double-unresolved limits: summing over sectors sharing the same singularity, and taking that singular limit on the sum, $\mathcal{W}$ functions disappear.

$$\mathbf{S}_{ik}\left(\sum_{b\neq i}\sum_{d\neq i,k}\mathcal{W}_{ibkd}+\sum_{b\neq k}\sum_{d\neq k,i}\mathcal{W}_{kbid}\right)=1\,,$$

$$\mathbf{C}_{ijk}\sum_{abc\,\in\,\mathrm{perm}(ijk)}\left(\mathcal{W}_{abbc}+\mathcal{W}_{abcb}\right)=1\,,\qquad \mathbf{C}_{ijkl}\sum_{\substack{ab\,\in\,\mathrm{perm}(ij)\\ cd\,\in\,\mathrm{perm}(kl)}}\left(\mathcal{W}_{abcd}+\mathcal{W}_{cdab}\right)=1\,,$$

$$\mathbf{SC}_{ikl}\sum_{b\neq i}\left(\mathcal{W}_{ibkl}+\mathcal{W}_{iblk}\right)=1\,,\qquad \mathbf{CS}_{ijk}\left(\sum_{d\neq i,k}\mathcal{W}_{ijkd}+\sum_{d\neq j,k}\mathcal{W}_{jikd}\right)=1\,.$$

Counterterms simplification at NNLO: sector $\mathcal{W}_{ijkj}$

- Simplifications possible, thanks to idempotency relations

$$(1 - \mathbf{S}_i)\, \mathbf{SC}_{icd}\, RR\, \mathcal{W}_{ibcd} = 0\,, \qquad (1 - \mathbf{C}_{ij})\, \mathbf{CS}_{ijk}\, RR\, \mathcal{W}_{ijkd} = 0\,.$$

- Limits **SC**, **CS**: disappear from $K^{(\mathbf{2})} + K^{(\mathbf{12})}$:

$$K^{(\mathbf{2})}_{ijkj} + K^{(\mathbf{12})}_{ijkj} \quad = \quad (1 - \mathbf{S}_i)(1 - \mathbf{C}_{ij})\Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}(1 - \mathbf{S}_{ik})\Big]\, RR\, \mathcal{W}_{ijkj}\,,$$

very simple structure!

- Still, since integrated $I^{(\mathbf{12})}$ and $I^{(\mathbf{2})}$ enter separately, they receive contributions from **SC** and **CS** (which however cancel in the sum).

Counterterms at NNLO: $\overline{K}^{(2)}$

Remapped pure double-unresolved counterterm, to be integrated over $d\Phi_{n+2}$:

$$
\begin{aligned}
\overline{K}^{(2)} = \sum_{i} \Bigg\{ & \sum_{j>i} \overline{\mathbf{S}}_{ij} + \sum_{j>i} \sum_{k>j} \overline{\mathbf{C}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\right) \\
& + \sum_{j>i} \sum_{\substack{k>i \\ k\neq j}} \sum_{\substack{l>k \\ l\neq j}} \overline{\mathbf{C}}_{ijkl} \left(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jl}\right) \\
& + \sum_{j\neq i} \sum_{\substack{k\neq i \\ k>j}} \overline{\mathbf{SC}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik}\right) \left(1 - \overline{\mathbf{C}}_{ijk} - \sum_{l\neq i,j,k} \overline{\mathbf{C}}_{iljk}\right) \\
& + \sum_{j>i} \sum_{k\neq i,j} \overline{\mathbf{CS}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\right) \left(1 - \overline{\mathbf{C}}_{ijk} - \sum_{l\neq i,j,k} \overline{\mathbf{C}}_{ijkl}\right) \Bigg\} RR \,,
\end{aligned}
$$