Frame-Independent Angular Distributions as Density Matrix Invariants

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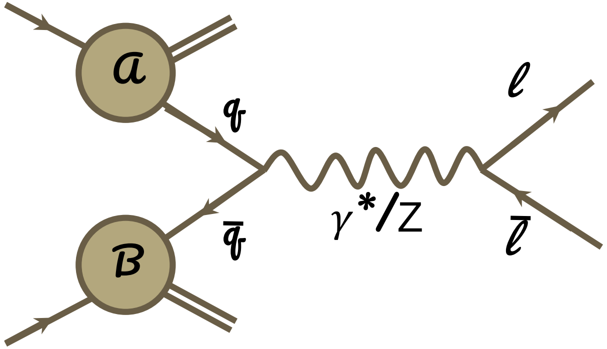
Outline

- Drell-Yan angular distributions and frame dependence; geometric model
- Hadronic tensor as density matrix (of virtual photon) and its invariants
- Counting and constraining the invariants
- Applications
- Conclusions

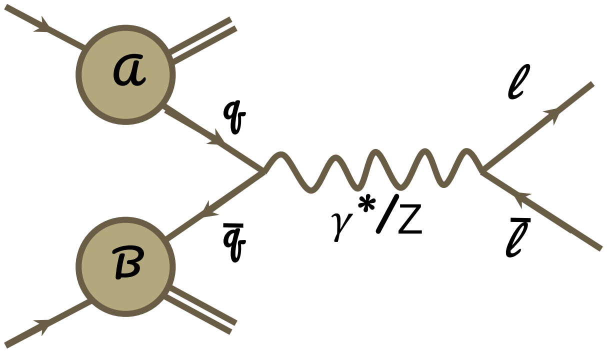
Angular distribution



general form of angular distribution (wrt particular frame)

$$
\begin{aligned}
\frac{1}{\sigma}\frac{d\sigma}{d\Omega} &= \frac{3}{4\pi}\frac{1}{3+\lambda_\theta}\left(1+\lambda_\theta\cos^2\theta+\lambda_{\theta\phi}\sin 2\theta\cos\phi\right.\\
&+ \lambda_\phi\sin^2\theta\cos 2\phi+\lambda_{\perp\phi}\sin^2\theta\sin 2\phi\\
&+ \lambda_{\perp\theta\phi}\sin 2\theta\sin\phi+2A_\theta\cos\theta\\
&+ \left.2A_\phi\sin\theta\cos\phi+2A_{\perp\phi}\sin\theta\sin\phi\right)
\end{aligned}
$$

Angular distribution



general form of angular distribution :

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&+ \lambda_{\perp\theta\phi}\sin 2\theta\sin\phi+2A_\theta\cos\theta\\
&+ \left.2A_\phi\sin\theta\cos\phi+2A_{\perp\phi}\sin\theta\sin\phi\right)
\end{aligned}
$$

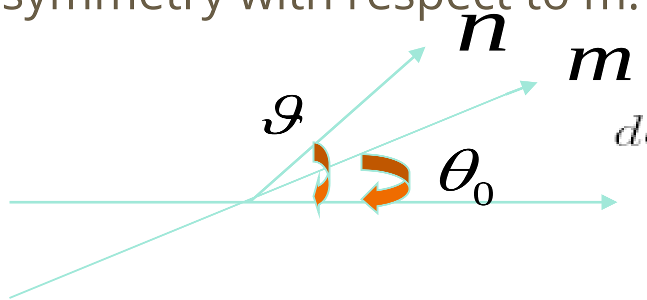
Invariants

➔ Facilitate comparison b/w experiments, theory and experiment

➔ Reveal systematic biases

Kinematic azimuthal asymmetry from polar one (OT'05)

Only polar asymmetry with respect to m!



$$d\sigma \propto 1 + \lambda_0 (\vec{n}\vec{m})^2 = 1 + \lambda_0 \cos^2\theta_{nm}^2$$

$$\cos\theta_{nm} = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos\phi$$

$$\lambda = \lambda_0 \frac{2 - 3\sin^2\theta_0}{2 + \lambda_0\sin^2\theta_0}$$

$$\nu = \lambda_0 \frac{2sin^2\theta_0}{2 + \lambda_0\sin^2\theta_0}$$

- azimuthal angle appears with new
- λ_0 - invariant!

Generalized Lam-Tung relation

- Relation between coefficients (high school math sufficient!)

$$\lambda_0 = \frac{\lambda + \frac{3}{2}\nu}{1 - \frac{1}{2}\nu}$$

- Reduced to standard LT relation for transverse polarization (λ_0 =1)
- LT - contains two very different inputs: kinematical asymmetry+transverse polarization
- Non-coplanarity – violation of LT (talk of W.-C. Chang)

General approach: the number of independent invariants

- ➔ 8 parameters describe distribution
- ➔ 3 Euler angles parameterize rotation

→ **8 - 3 = 5 independent invariants**

Hadronic tensor in terms of observables

$$\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto W^{\mu\nu} L_{\mu\nu}$$

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$$q_\mu W^{\mu\nu} = 0 \quad \longrightarrow \quad W^{ij} = \begin{pmatrix} d_1 & a_1 + i a_2 & b_1 + i b_2 \\ a_1 - i a_2 & d_2 & c_1 + i c_2 \\ b_1 - i b_2 & c_1 - i c_2 & d_3 \end{pmatrix}$$

Hadronic tensor in terms of observables

$$\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto W^{\mu\nu} L_{\mu\nu}$$

$$q_\mu W^{\mu\nu} = 0 \longrightarrow W^{ij} = \begin{pmatrix} d_1 & a_1 + ia_2 & b_1 + ib_2 \\ a_1 - ia_2 & d_2 & c_1 + ic_2 \\ b_1 - ib_2 & c_1 - ic_2 & d_3 \end{pmatrix}$$

$$L_{ij} \propto \delta_{ij} - n_i n_j + ig\epsilon_{ijk} n^k$$

Hadronic tensor in terms of observables

$$\begin{aligned}\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &\propto (d_1 + \frac{1}{2}d_2 + \frac{1}{2}d_3) + (-d_1 + \frac{1}{2}d_2 + \frac{1}{2}d_3)\cos^2\theta \\ &- a_1 \sin 2\theta \cos\phi + \frac{1}{2}(d_3 - d_2)\sin^2\theta\cos 2\phi \\ &- b_1 \sin 2\theta \sin\phi - c_1 \sin^2\theta \sin 2\phi \\ &- 2a_2 \sin\theta\sin\phi + 2b_2\sin\theta\cos\phi - 2c_2\cos\theta\end{aligned}$$

compare with the general expression

$$\begin{aligned}\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{3}{4\pi}\frac{1}{3+\lambda_\theta}\big(1 + \lambda_\theta\cos^2\theta + \lambda_{\theta\phi}\sin 2\theta\cos\phi \\ &+ \lambda_\phi \sin^2\theta\cos 2\phi + \lambda_{\perp\phi}\sin^2\theta\sin 2\phi \\ &+ \lambda_{\perp\theta\phi}\sin 2\theta\sin\phi + 2A_\theta\cos\theta \\ &+ 2A_\phi\sin\theta\cos\phi + 2A_{\perp\phi}\sin\theta\sin\phi\big)\end{aligned}$$

Hadronic tensor in terms of observables

$$\boxed{W^{ij} = \frac{2}{3+\lambda_\theta}\begin{pmatrix} \frac{1-\lambda_\theta}{2} & -\lambda_{\theta\phi} - iA_{\perp\phi} & -\lambda_{\perp\theta\phi} + iA_\phi \\ -\lambda_{\theta\phi} + iA_{\perp\phi} & \frac{1+\lambda_\theta-2\lambda_\phi}{2} & -\lambda_{\perp\phi} - iA_\theta \\ -\lambda_{\perp\theta\phi} - iA_\phi & -\lambda_{\perp\phi} + iA_\theta & \frac{1+\lambda_\theta+2\lambda_\phi}{2} \end{pmatrix}}$$

(normalization condition Tr W = 1)

Hadron tensor in terms of observables

$$\boxed{W^{ij} = \frac{2}{3+\lambda_\theta}\begin{pmatrix} \frac{1-\lambda_\theta}{2} & -\lambda_{\theta\phi} - iA_{\perp\phi} & -\lambda_{\perp\theta\phi} + iA_\phi \\ -\lambda_{\theta\phi} + iA_{\perp\phi} & \frac{1+\lambda_\theta-2\lambda_\phi}{2} & -\lambda_{\perp\phi} - iA_\theta \\ -\lambda_{\perp\theta\phi} - iA_\phi & -\lambda_{\perp\phi} + iA_\theta & \frac{1+\lambda_\theta+2\lambda_\phi}{2} \end{pmatrix}}$$

(normalization condition Tr W = 1)

$$W = \frac{1}{3}\cdot\mathbf{1} + W_s + iW_a$$

Spin 1: symmetric and antisymmetric part do not mix: rotational invariants can be found as eigenvalues of matrices:

$$W, W_s, W_a, W_sW_a, W_sW_aW_s, \ldots$$

Set of characteristic equations for W_x, W_z, $W_x + iW_z$ and $W_x W_z$

$$w^{(x)}\left(w^{(x)^2} + 4U_1\right) = 0$$

$$w^{(z)^3} - \frac{4}{3}U_2 w^{(z)} - \frac{8}{27}T = 0$$

$$w^3 - \left(4U_1 + \frac{4}{3}U_2\right)w - \frac{8}{27}(T + R) = 0$$

$$w^{(xz)}\left(w^{(xz)^2} + \frac{16}{9}M\right) = 0$$

Invariants

$$U_1 = \frac{A_\theta^2 + A_\phi^2 + A_{\perp\theta\phi}^2}{(3+\lambda_\theta)^2}, \qquad U_2 = \frac{\lambda_\theta^2 + 3\left(\lambda_\phi^2 + \lambda_{\theta\phi}^2 + \lambda_{\perp\phi}^2 + \lambda_{\perp\theta\phi}^2\right)}{(3+\lambda_\theta)^2}$$

$$T = \frac{(\lambda_\theta + 3\lambda_\phi)\left(2\lambda_\theta^2 - 6\lambda_\theta\lambda_\phi + 9\lambda_{\theta\phi}^2\right) + 9\left(\lambda_\theta\lambda_{\perp\theta\phi}^2 - 2\lambda_\theta\lambda_{\perp\phi}^2 + 6\lambda_{\theta\phi}\lambda_{\perp\theta\phi}\lambda_{\perp\phi} - 3\lambda_\phi\lambda_{\perp\theta\phi}^2\right)}{(3+\lambda_\theta)^3}$$

$$R = \frac{1}{(\lambda_\theta+3)^3}\left(54\left(A_\theta A_\phi\lambda_{\theta\phi} + A_\theta A_{\perp\phi}\lambda_{\perp\theta\phi} + A_{\perp\phi}A_\phi\lambda_{\perp\phi}\right) + 9\lambda_\theta\left(2A_\theta^2 - A_{\perp\phi}^2 - A_\phi^2\right)\right.$$
$$\left. +27\lambda_\phi\left(A_\phi^2 - A_{\perp\phi}^2\right)\right)$$

$$M = \frac{1}{(3+\lambda_\theta)^4}\left\{A_\theta^2\left(\lambda_\theta^2 - 9\lambda_\phi^2 - 9\lambda_{\perp\phi}^2\right) - A_\phi^2\left(2\lambda_\theta\left(\lambda_\theta + 3\lambda_\phi\right) + 9\lambda_{\perp\theta\phi}^2\right)\right.$$
$$+A_{\perp\phi}^2\left(6\lambda_\theta\lambda_\phi - 2\lambda_\theta^2 - 9\lambda_{\theta\phi}^2\right) + 6A_\theta A_{\perp\phi}\left(\lambda_{\perp\theta\phi}\left(\lambda_\theta - 3\lambda_\phi\right) + 3\lambda_{\theta\phi}\lambda_{\perp\phi}\right)$$
$$\left. +6A_\phi\left[A_\theta\left(\lambda_{\theta\phi}\left(\lambda_\theta + 3\lambda_\phi\right) + 3\lambda_{\perp\theta\phi}\lambda_{\perp\phi}\right) + A_{\perp\phi}\left(3\lambda_{\theta\phi}\lambda_{\perp\theta\phi} - 2\lambda_\theta\lambda_{\perp\phi}\right)\right]\right\}$$

Restrictions on Invariants

Positivity condition together with normalization lead to the following inequalities:

$$
\begin{gathered}
0 \leq w_{1,2,3} \leq 1 \\
0 \leq w_1 w_2 + w_1 w_3 + w_2 w_3 \leq \frac{1}{3} \\
0 \leq w_1 w_2 w_3 \leq \frac{1}{27}
\end{gathered}
$$

Constraints for Invariants

Positivity condition together with normalization lead to the following inequalities:

$$0 \le w_{1,2,3} \le 1$$
$$0 \le w_1 w_2 + w_1 w_3 + w_2 w_3 \le \tfrac{1}{3}$$
$$0 \le w_1 w_2 w_3 \le \tfrac{1}{27}$$

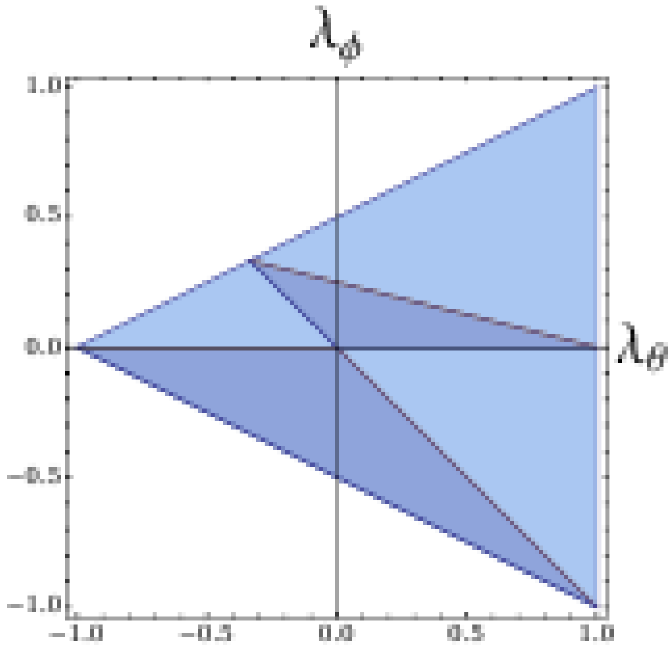
$$\boxed{U_1 + \frac{1}{3} U_2 \le \frac{1}{12} \quad \rightarrow \quad U_1 \le \frac{1}{12}, \; U_2 \le \frac{1}{4}}$$

$$\boxed{-\frac{1}{8} \le R + T \le \frac{3}{8}}$$

Positivity domain and geometric model for PHENIX J/Ψ data

$$\lambda_0 = \frac{\lambda_\theta + 3\lambda_\phi}{1 - \lambda_\phi}$$

$$\sin^2 \xi = \frac{2\lambda_\phi}{\lambda_\theta + 3\lambda_\phi}$$



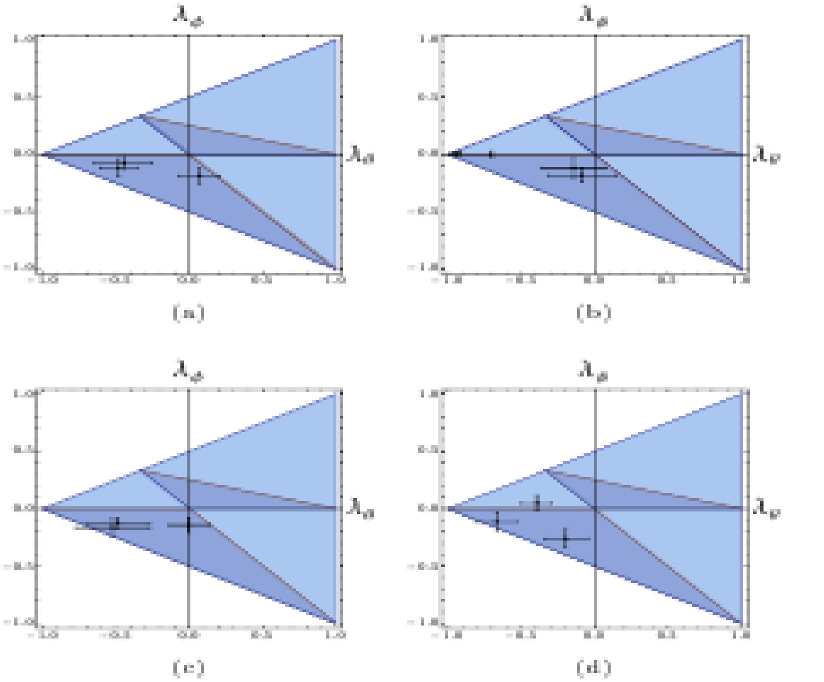


FIG. 2. Angular coefficients λ_θ and λ_ϕ measured b PHENIX [4]: (a) – HX frame, (b) – CS frame, (c) – GJI frame, (d) – GJF frame. Different points correspond to di ferent values of transverse momentum. Only statistical erro are shown.

How invariant are they?

TABLE V. The values of invariants U_2 and T calculated for angular coefficients measured by PHENIX collaboration [4] in $J/\psi \to \mu^+\mu^-$ decays for $1.2 < y < 2.2$ in four reference frames at different values of p_T: (2 − 3 GeV), (3 − 4 GeV) and (4 − 10 GeV). Only statistical errors are taken into account.

	U_2			T		
p_T [GeV/c]	2 − 3	3 − 4	4 − 10	2 − 3	3 − 4	4 − 10
HX	3.0 ± 2.5	2.8 ± 1.7	1.2 ± 0.8	4.8 ± 6.5	1.7 ± 3.7	1.3 ± 1.4
CS	$> (6.2 \pm 0.4)$	1.0 ± 0.8	0.5 ± 0.8	$> (15.2 \pm 1.5)$	-0.1 ± 1.6	-0.3 ± 0.8
GJB	5.0 ± 3.6	3.8 ± 3.6	1.0 ± 0.6	10.5 ± 12.1	0.7 ± 6.9	0.9 ± 0.8
GJF	8.7 ± 4.4	3.0 ± 1.7	4.3 ± 2.2	24.3 ± 20.1	3.3 ± 3.5	7.4 ± 7.1

Invariants discussed in literature

- ➔ Faccioli et al. (PRL 105 (2010) 061601)

$$\mathcal{F} = \frac{1+\lambda_\theta+\lambda_\phi}{3+\lambda_\theta}$$

- ➔ Faccioli et al. (PRD 81 (2010) 111502), Palestini (PRD 83 (2011) 031503)

(λ_0 (OT'05) =) $\tilde{\lambda} = \dfrac{\lambda_\theta+(3/2)\lambda_\phi}{1-(1/2)\lambda\phi} = \dfrac{3\mathcal{F}-1}{1-\mathcal{F}}$

Faccioli et al. (PRD 83 (2011) 056008)

$$G = \frac{1+\lambda_\theta-\lambda_\phi}{3+\lambda_\theta}$$

- ➔ Faccioli et al. (PRD 83 (2011) 056008)

$$\lambda_\theta$$

- ➔ Ma,Qiu,Zhang (arXiv: 1703.04752 (2017))

rotations along the y-axis in the dilepton rest frame

along the x-axis

along the z-axis

general method

Connection to invariants from arXiv: 1703.04752

$$\widetilde{U}_1 = \frac{3}{\pi} U_1$$

$$\widetilde{U}_2 = \frac{1}{5\pi} U_2$$

$$\widetilde{W}_3 = \frac{1}{70\pi^2} \left(T + 7R \right)$$

$$\widetilde{W}_4 = \frac{9}{20\pi} \widetilde{U}_1^{\,2} + \frac{15}{28\pi} \widetilde{U}_2^{\,2} + \frac{27}{14\pi} \widetilde{U}_1 \widetilde{U}_2 - \frac{9}{35\pi^3} \frac{1}{144} \left(45U_1 + 10R + 36M \right)$$

$$\widetilde{W}_5 = \frac{5}{2\pi} \left(\frac{3}{7} \widetilde{U}_1 + \frac{5}{11} \widetilde{U}_2 \right) \widetilde{W}_3 + \frac{3}{539\pi^4} \left(U_1 \left(-\frac{143}{4} + 429U_2 - 297T \right) + \frac{7}{3} U_2 R \right)$$

Invariants in terms of eigenvalues

$$w_1^{(a)}w_2^{(a)} + w_1^{(a)}w_3^{(a)} + w_2^{(a)}w_3^{(a)} = w_2^{(a)}w_3^{(a)} = 4U_1$$

$$w_1^{(s)}w_2^{(s)} + w_1^{(s)}w_3^{(s)} + w_2^{(s)}w_3^{(s)} = -\frac{4}{3}U_2$$
$$w_1^{(s)}w_2^{(s)}w_3^{(s)} = \frac{8}{27}T$$

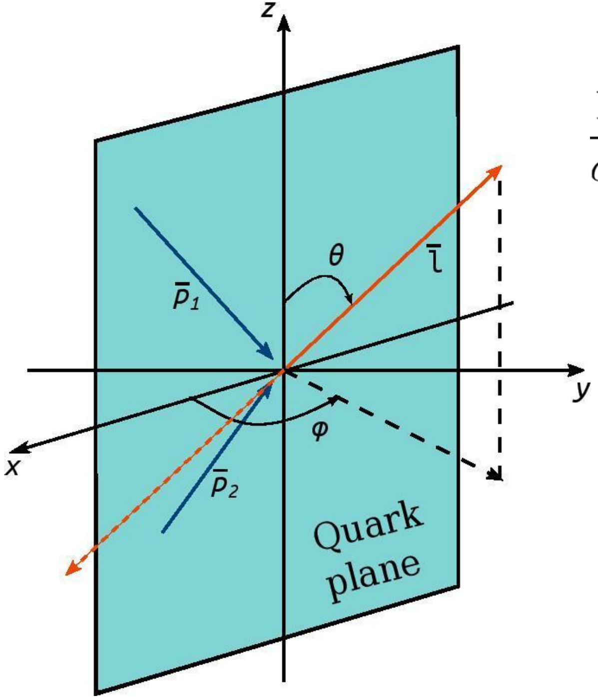
$$w_1w_2 + w_1w_3 + w_2w_3 = -4U_1 - \frac{4}{3}U_2$$
$$w_1w_2w_3 = \frac{8}{27}(T + R)$$

$$w_1^{(as)}w_2^{(as)} + w_1^{(as)}w_3^{(as)} + w_2^{(as)}w_3^{(as)} = w_2^{(as)}w_3^{(as)} = 4M$$

Special case

- ➔ Gottfried-Jackson frame - z || quark momenta
- ➔ Collins-Soper frame - z || bisection

related by rotation along the y-axis



$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega} = \frac{1}{3+\lambda_\theta}\left(1+\lambda_\theta\cos^2\theta+\lambda_{\theta\phi}\sin 2\theta\cos\phi+\lambda_\phi\sin^2\theta\cos 2\phi\right)$$

$$W^{ij} = \frac{1}{3+\lambda_\theta}\begin{pmatrix} 1-\lambda_\theta & -2\lambda_{\theta\phi} & 0 \\ -2\lambda_{\theta\phi} & 1+\lambda_\theta-2\lambda_\phi & 0 \\ 0 & 0 & 1+\lambda_\theta+2\lambda_\phi \end{pmatrix}$$

$$\boxed{w_1 = \frac{1+\lambda_\theta+2\lambda_\phi}{3+\lambda_\theta} \qquad w_{2,3} = \frac{1-\lambda_\phi \pm \sqrt{(\lambda_\theta-\lambda_\phi)^2+4\lambda_{\theta\phi}^2}}{3+\lambda_\theta}}$$

Faccioli invariant

Other special rotations

$$W^{ij} = \frac{1}{3+\lambda_\theta}\begin{pmatrix} 1-\lambda_\theta & -2\lambda_{\theta\phi} & -2\lambda_{\perp\theta\phi} \\ -2\lambda_{\theta\phi} & 1+\lambda_\theta-2\lambda_\phi & -2\lambda_{\perp\phi} \\ -2\lambda_{\perp\theta\phi} & -2\lambda_{\perp\phi} & 1+\lambda_\theta+2\lambda_\phi \end{pmatrix} \qquad \bar{e}_x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \bar{e}_z = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\downarrow$$

$$e_x^\mu W_{\mu\nu} e_x^\nu = \frac{1+\lambda_\theta-\lambda_\phi}{3+\lambda_\theta} = G$$

$$e_z^\mu W_{\mu\nu} e_z^\nu = \frac{1-\lambda_\theta}{3+\lambda_\theta} \Rightarrow \lambda_\theta = \text{invariant}$$

CONCLUSIONS/OUTLOOK

- Invariants may be expressed via eigenvalues of density matrix and its components
- Corresponds to general structure of quantum theory
- May/should be used for data analysis (errors are currently large)

- Incorporating to calculations
- Higher spin
- Crossed processes