Non-linear evolution in QCD at low-x beyond leading order

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OUTLINE

- LO BK equation and solutions
- NLO BK equation in projectile rapidity $Y$ and instability
- Time ordering and resummation in $Y$ and issues
- NLO BK equation in target rapidity $\eta$ and instability
- Resummed evolution in $\eta$ via shifts
- Fits, thoughts on saturation momentum $Q_s^2$
- Conclusion
Scatter dipole \( r = (x, y) \) and \( q^+ \) on hadron with \( Q_0^2 \) and \( q_0^+ = Q_0^2/2q_0^- \). Typically \( r^2Q_0^2 < 1 \). Emit soft gluon.

\[
\begin{align*}
\partial S_{xy}(Y) \frac{\partial S_{xy}(Y)}{\partial Y} &= \frac{\bar{\alpha}_s}{2\pi} \int \frac{d^2z (x - y)^2}{(x - z)^2(z - y)^2} \left[ S_{xz}(Y)S_{zy}(Y) - S_{xy}(Y) \right].
\end{align*}
\]

Resum powers of \( \bar{\alpha}_s Y = \bar{\alpha}_s \ln q^+/q_0^+ \) in presence of strong target field \( \rightsquigarrow \) BK equation. \( Y \equiv \) “projectile” rapidity.
Solution to BK Equation at LO

Saturation line $T(Y, r^2 \sim 1/Q_s^2) = 1/2$

Asymptotically $\lambda_s = d \ln Q_s^2 / dY \simeq 4.88\bar{\alpha}_s$

Scaling $T(Y, r^2) \sim (r^2 Q_s^2)^{\gamma_0}$ with $\gamma_0 \simeq 0.63$

Can calculate preasymptotics

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BK Evolution at NLO (I)

Emit soft + non-soft (in general)

\[ N_{c} \]

\[ N_{f} \]

+ many more
\[
\frac{\partial S_{xy}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{d^2z (x-y)^2}{(x-z)^2(z-y)^2} \left\{ 1 + \bar{\alpha}_s \left[ -\frac{1}{2} \ln \frac{(x-z)^2}{(x-y)^2} \ln \frac{(y-z)^2}{(x-y)^2} \right] \right. \\
\left. + \bar{b} \ln(x-y)^2 \mu^2 - \bar{b} \frac{(x-z)^2 - (y-z)^2}{(x-y)^2} \ln \frac{(x-z)^2}{(y-z)^2} \right\} [S_{xz}S_{zy} - S_{xy}] \\
+ \frac{\bar{\alpha}_s^2}{8\pi^2} \int d^2z d^2u \ K(x, y, z, u) [S_{xu}S_{uz}S_{zy} - S_{xu}S_{uy}] \\
+ \mathcal{O}(N_f) + \mathcal{O}(1/N_c^2)
\]
Solution to BK Equation at NLO: Unstable

Double logarithm main source of problem
Large Double Logarithm and Time Ordering

Strongly ordered large perturbative dipoles (DLA)

\[ r^2 \ll (x - z)^2 \sim (z - y)^2 \ll 1/Q_s^2 \]

Large dipoles interact stronger, real terms dominate

\[
\frac{dT(Y, r)}{dY} = \bar{\alpha}_s \int_{r^2}^{1/Q_s^2} \frac{r^2 d\bar{z}^2}{\bar{z}^4} \left( 1 - \frac{\bar{\alpha}_s}{2} \ln^2 \frac{\bar{z}^2}{r^2} \right) T(Y, \bar{z})
\]

NLO > LO, expansion in \( \bar{\alpha}_s \) fails

Lifetime of fluctuation \( \tau_{\text{fluct}} \sim k_{\text{fluct}}^+ r_{\text{fluct}}^2 \)

Problem for large daughters: \( k_{2}^+ < k_{1}^+ \Rightarrow \tau_2 < \tau_1 \)

LO does not care, NLO “tries” correcting \( \sim \) big correction
Resum Double Logs in $Y$ Evolution (I)

Local equation with resummed kernel

$$\frac{\partial S_{xy}(Y)}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{d^2z}{(x-z)^2(z-y)^2} K_{DLA} \left( \sqrt{\ln \frac{(x-z)^2}{(x-y)^2}} \ln \frac{(y-z)^2}{(x-y)^2} \right)$$

$$\times [S_{xz}(Y)S_{zy}(Y) - S_{xy}(Y)] .$$

with

$$K_{DLA}(\rho) = \frac{J_1 \left( 2\sqrt{\bar{\alpha}_s \rho^2} \right)}{\sqrt{\bar{\alpha}_s \rho^2}} = 1 - \frac{\bar{\alpha}_s \rho^2}{2} + \frac{(\bar{\alpha}_s \rho^2)^2}{12} + \cdots$$

Oscillations cut “unphysically“ large daughter dipoles

Add remaining $O(\bar{\alpha}_s^2)$ to fully match NLO BK

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"Delay" equation with LO dipole kernel

\[
\frac{\partial S_{xy}(Y)}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{d^2 z (x-y)^2}{(x-z)^2(z-y)^2} \Theta(Y - \rho_{\text{min}}) \\
\times \left[ S_{xz}(Y - \Delta_{xz}; r) S_{zy}(Y - \Delta_{zy}; r) - S_{xy}(Y) \right]
\]

with \( \rho_{\text{min}} = \ln(1/r_{\text{min}}^2 Q_0^2) \) and

\[
\Delta_{xz; r} \equiv \max \left\{ 0, \ln \frac{(x-z)^2}{r^2} \right\} \quad \text{(canonical shift)}
\]

Big log shift for large daughters, no shift otherwise

Again can fully match to NLO BK

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**Solution to Resummed Equation: Stable**

- \( T(L, Y) \) vs. \( L = \log(1/r^2) \)
- \( \alpha_s = 0.25 \)
- Various values of \( Y \):
  - \( Y = 0 \)
  - \( Y = 4 \)
  - \( Y = 8 \)
  - \( Y = 12 \)
  - \( Y = 16 \)

**Graphs:**

- **Left Graph:**
  - \( L = \log(1/r^2) \)
  - Various curves for different \( Y \) values

- **Right Graph:**
  - \( d \log[Q_s^2(Y)]/dY \) vs. \( Y \)
  - Different curves for \( \alpha_s = 0.25 \)
  - Comparison between LO and NLO+resum

**Text:**

*Everybody happy, go home? Not really ...*
Issues/Comments in $Y$ (Projectile) Evolution

- $\lambda_s$ very small (when including all NLO), $\gamma_s \sim 1$ too large

- Target rapidity determines kinematics

\[
\eta \equiv \ln \frac{1}{x_{Bj}} = \ln \frac{s}{Q^2} = \ln \frac{2q^+q^-}{2q^+q^-} = \ln \frac{q^-}{q^-}
\]

\[
= \ln \frac{Q_0^2}{2q_0^+} \frac{2q^+}{Q^2} \rightarrow Y - \ln \frac{1}{r^2 Q_0^2} \equiv Y - \rho
\]

Saturation is a target property, need $Q_s^2(\eta)$

- Not really an Initial Condition problem, but a Boundary one
  Extremely difficult to solve properly

- “Close eyes” and try various motivated Initial Conditions
  Need to express the results in terms of $\eta$ (easy)
Slope and Speed for $\eta$-Front

Front in $Y \Leftrightarrow$ Front in $\eta$, physical

Scaling in $Y$: $-\ln T = \gamma_s (\rho - \lambda_s Y)$

Change variable $Y = \eta + \rho$

Scaling in $\eta$: $-\ln \bar{T} = \bar{\gamma}_s (\rho - \bar{\lambda}_s \eta)$

$\bar{\gamma}_s = \gamma_s (1 - \lambda_s)$ and $\bar{\lambda}_s = \frac{\lambda_s}{1 - \lambda_s}$

$\lambda_s = \mathcal{O}(\bar{\alpha}_s)$, difference is NLO. In practice it is large.

Physical $\eta$-front less steep and faster

What we wanted, but not under control
Unphysical inflection point

Even larger than LO result

Canonical shift behaves properly (still, IC is not under control)
NLO BK in Target Rapidity $\eta$

- Evolve the projectile: dipole kernels in transverse space
- Variable change $\eta = Y - \rho$ in longitudinal space

\[
S(Y, r) = S(\eta + \ln(1/r^2 Q_0^2), r) \equiv \bar{S}(\eta, r)
\]
\[
S(Y, z) = S(\eta + \ln(1/z^2 Q_0^2) - \ln(r^2/z^2), z) = \bar{S}(\eta - \ln(r^2/z^2), z)
\]

Non-local, expand at NLO. Extra term for NLO BK in $\eta$

\[
\frac{\bar{\alpha}_s^2}{2\pi^2} \int \frac{d^2 z d^2 u (x-y)^2}{(x-u)^2 (u-z)^2 (z-y)^2} \ln \frac{(u-y)^2}{(x-y)^2} \bar{S}_{xu} [\bar{S}_{uz} \bar{S}_{zy} - \bar{S}_{uy}]
\]
Solution to NLO BK in Target Rapidity $\eta$

Milder but still problematic instability due to small dipoles

$$\frac{\partial \bar{T}(\eta, r)}{\partial \eta} \simeq \bar{\alpha}_s \int_0^r \frac{dz^2}{z^2} \left( 1 - \bar{\alpha}_s \ln^2 \frac{r^2}{z^2} \right) \bar{T}(\eta, z) \quad \text{for} \quad z \ll r$$

No double logs in solution (color transparency)
But solution develops anomalous dimension $\rightarrow$ big NLO correction
Resum Double Logs in $\eta$ Evolution

Lifetimes $\tau$, or $k^-$, automatically ordered
But ordering in $k^+$ is not guaranteed any more
Enforcing it resums double logs for small dipoles to all orders

\[
\frac{\partial \bar{S}_{xy}(\eta)}{\partial \eta} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{d^2z (x-y)^2}{(x-z)^2(z-y)^2} \Theta(\eta - \delta_{xyz}^>) \\
\times \left[ \bar{S}_{xz}(\eta - \delta_{xz;r}) \bar{S}_{zy}(\eta - \delta_{zy;r}) - \bar{S}_{xy}(\eta) \right]
\]

with \( \delta_{xz;r} \equiv \max \left\{ 0, \ln \frac{r^2}{(x-z)^2} \right\} \) (canonical shift)

Typically hard to soft, but BFKL/BK is diffusive, cut small dipoles
IC problem, directly the result in \( \eta \sim \ln(1/x_{Bj}) \)
Can fully match to NLO BK
Correction of order $\mathcal{O}(\bar{\alpha}_s)$

Physically acceptable results

Inclusion of all NLO terms would reduce uncertainty by a factor $\bar{\alpha}_s$. 
Preasymptotic dependence on $\eta$

Analytic expression valid down to $\eta = 6 \div 8$
Geometric scaling in $\eta$ Evolution

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Running Coupling in $\eta$ Evolution

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LO impact factor ($\gamma^* WF$) + Resummed in $\eta$ (DL’s, SL’s and RC)

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## HERA Data Fit (II)

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<th>RC</th>
<th>$\chi^2$/npts</th>
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Going to higher $Q^2$

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Thoughts on $Q_s^2(\eta)$

$Q_s^2(\eta)$ marginally perturbative, BUT:

- $x$ - dependent scale
- $b$ - averaged, in principle higher $Q_s^2(\eta)$ in “center”
- Saturation determines form even for $Q^2 > Q_s^2(\eta)$
- Extrapolate to nuclei $\sim A^{1/3}$ (maybe a factor of 3)
- $Q_s^2(\eta)$ in adjoint, factor of $\sim 3 > C_A/C_F = 9/4$
Conclusions

- Resummed evolution in target rapidity $\eta \sim \ln 1/x_{Bj}$
  $\sim$ physical front directly
- Stable results, front in $\eta$ faster and less steep
- Compared to LO: $\delta \lambda_s/\lambda_s \simeq O(\bar{\alpha}_s)$, roughly the same $\gamma_s$
- Can match to full NLO BK evolution
- Fit to DIS HERA data
- Higher than expected $Q_s^2$ in adjoint