

Soft corrections to inclusive DIS at four loops and beyond.

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with S. Moch and A. Vogt

Based on arXiv:1904.soon



Motivation.

- Form factor at **four loops** are available in the large color approximation and beyond with recent progress in loop calculations.
[Henn, Lee, Smirnov, Smirnov, Steinhauser]
[Manteuffel, Scabinger]
- An order by order analysis of the **structure** of corresponding unfactorized quantities in Dimensional Regularization.
- Inclusive DIS is phenomenologically and theoretically an important **benchmark process** in pQCD.
- Data on the cross-section expressed in terms of the structure function form a primary source of information on the PDFs.
- **Soft-virtual and resummation** gives extra sheds on the higher order corrections and in general its structure and **improvement over fixed order**.

Plan of the talk.

- Formalism for the soft-virtual(SV) cross-section
- Soft gluon resummation
- Numerical Study
- Summary

SV Cross-section.

- Finite Soft+Virtual cross-section:

$$\Delta_{I,P}^{sv}(z, q^2, \mu_R^2, \mu_F^2) = C \exp \left(\Psi_P^I(z, q^2, \mu_R^2, \mu_F^2, \varepsilon) \right) \Big|_{\varepsilon=0}$$

[Ravindran]

$$\begin{aligned} \Psi_P^I(z, q^2, \mu_R^2, \mu_F^2, \varepsilon) &= \left(\ln \left(Z^I(\hat{a}_s, \mu_R^2, \mu^2, \varepsilon) \right)^2 + \ln |\hat{F}^I(\hat{a}_s, Q^2, \mu^2, \varepsilon)|^2 \right) \delta(1-z) \\ &\quad + 2 \Phi_P^I(\hat{a}_s, q^2, \mu^2, z, \varepsilon) - 2 m C \ln \Gamma_H(\hat{a}_s, \mu^2, \mu_F^2, z, \varepsilon), \end{aligned}$$

Overall Op. Renorm

Form Factor

Soft distribution function

DGLAP Kernel

- For DIS: $m = 1/2$, $Z^I = 1$; z - Bjorken variable
- Total sum has to be **UV and IR finite!**

SV Cross-section.

- Finite Soft+Virtual cross-section:

$$\Delta_{I,P}^{sv}(z, q^2, \mu_R^2, \mu_F^2) = C \exp \left(\Psi_P^I(z, q^2, \mu_R^2, \mu_F^2, \varepsilon) \right) \Big|_{\varepsilon=0}$$

[Ravindran]

$$\begin{aligned} \Psi_P^I(z, q^2, \mu_R^2, \mu_F^2, \varepsilon) &= \left(\ln \left(Z^I(\hat{a}_s, \mu_R^2, \mu^2, \varepsilon) \right)^2 + \ln |\hat{F}^I(\hat{a}_s, Q^2, \mu^2, \varepsilon)|^2 \right) \delta(1-z) \\ &\quad + 2 \Phi_P^I(\hat{a}_s, q^2, \mu^2, z, \varepsilon) - 2 m C \ln \Gamma_H(\hat{a}_s, \mu^2, \mu_F^2, z, \varepsilon), \end{aligned}$$

Overall Op. Renorm

Form Factor

Soft distribution function

DGLAP Kernel

- For DIS: $m = 1/2$, $Z^I = 1$; z - Bjorken variable
- Total sum has to be **UV and IR finite!**

Form Factor: K-G.

- Form-factor satisfies K-G equation:

Gauge invariance, RG invariance

$$\frac{d \ln \hat{F}^I(\hat{a}_s, Q^2, \mu^2, \epsilon)}{d \ln Q^2} = \frac{1}{2} \left[K^I(\hat{a}_s, \epsilon, \mu_R^2, \mu^2) + G^I(\hat{a}_s, \epsilon, Q^2, \mu_R^2, \mu^2) \right]$$

[Sen, Mueller, Collins, Magnea]

Poles in ϵ
Q-independent

Finite in $\epsilon \rightarrow 0$
Q-dependent

- RG invariance of the Form-factor:

$$\mu_R^2 \frac{d}{d \mu_R^2} K^I\left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, \epsilon\right) = -\mu_R^2 \frac{d}{d \mu_R^2} G^I\left(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon\right) = -A^I(a_s(\mu_R^2))$$

cusp anomalous dimensions

- Casimir duality

$$\frac{A_q}{A_g} = \frac{C_F}{C_A}$$

Valid upto 3-loops

Form Factor: Solution.

- Solution of the Form-factor:

$$\ln \hat{F}^I(\hat{a}_s, Q^2, \mu^2, \epsilon) = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{Q^2}{\mu^2} \right)^{i \frac{\epsilon}{2}} S_{\epsilon}^i \hat{\mathcal{L}}_F^{I,(i)}(\epsilon)$$

Upto three-loop:

$$\begin{aligned}\hat{\mathcal{L}}_F^{I,(1)}(\epsilon) &= \frac{1}{\epsilon^2} [-2A_1^I] + \frac{1}{\epsilon} [G_1^I(\epsilon)] \\ \hat{\mathcal{L}}_F^{I,(2)}(\epsilon) &= \frac{1}{\epsilon^3} [\beta_0 A_1^I] + \frac{1}{\epsilon^2} \left[-\frac{1}{2} A_2^I - \beta_0 G_1^I(\epsilon) \right] + \frac{1}{\epsilon} \left[\frac{1}{2} G_2^I(\epsilon) \right] \\ \hat{\mathcal{L}}_F^{I,(3)}(\epsilon) &= \frac{1}{\epsilon^4} \left[-\frac{8}{9} \beta_0^2 A_1^I \right] + \frac{1}{\epsilon^3} \left[\frac{2}{9} \beta_1 A_1^I + \frac{8}{9} \beta_0 A_2^I + \frac{4}{3} \beta_0^2 G_1^I(\epsilon) \right] \\ &\quad + \frac{1}{\epsilon^2} \left[-\frac{2}{9} A_3^I - \frac{1}{3} \beta_1 G_1^I(\epsilon) - \frac{4}{3} \beta_0 G_2^I(\epsilon) \right] + \frac{1}{\epsilon} \left[\frac{1}{3} G_3^I(\epsilon) \right]\end{aligned}$$

- All terms can be found, only computation of finite piece is needed.

Form Factor: Solution.

- Solution of the Form-factor:

$$\ln \hat{F}^I(\hat{a}_s, Q^2, \mu^2, \epsilon) = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{Q^2}{\mu^2} \right)^{i \frac{\epsilon}{2}} S_{\epsilon}^i \hat{\mathcal{L}}_F^{I,(i)}(\epsilon)$$

$$G_1 = 2B_1^q + f_1^q + \epsilon f_{01}^q ,$$

$$G_2 = 2B_2^q + (f_2^q + \beta_0 f_{01}^q) + \epsilon f_{02}^q ,$$

$$G_3 = 2B_3^q + (f_3^q + \beta_1 f_{01}^q + \beta_0 f_{02}^q) + \epsilon f_{03}^q$$

[Moch, Vermaseren, Vogt]
[Dixon, Magnea, Sterman]
[Ravindran]

Coll. Ano. Dim.

Soft Ano. Dim.

Explicit
Computation

Origin from eikonal FF

[Dixon, Magnea, Sterman]

Form Factor @ 4loops.

- Solution at 4-loops:

$$\ln \hat{F}^I(\hat{a}_s, Q^2, \mu^2, \varepsilon) = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{Q^2}{\mu^2} \right)^{i \frac{\varepsilon}{2}} S_{\varepsilon}^i \hat{\mathcal{L}}_F^{I,(i)}(\varepsilon)$$

$$\begin{aligned}\hat{\mathcal{L}}_F^{I,(4)} &= \frac{1}{\varepsilon^5} \left(\beta_0^3 A_1^I \right) + \frac{1}{\varepsilon^4} \left(-\frac{2}{3} \beta_0 \beta_1 A_1^I - \frac{3}{2} \beta_0^2 A_2^I - 2 \beta_0^3 G_1^I(\varepsilon) \right) \\ &\quad + \frac{1}{\varepsilon^3} \left(\frac{1}{12} \beta_2 A_1^I + \frac{1}{4} \beta_1 A_2^I + \frac{3}{4} \beta_0 A_3^I + \frac{4}{3} \beta_0 \beta_1 G_1^I(\varepsilon) + 3 \beta_0^2 G_2^I(\varepsilon) \right) \\ &\quad + \frac{1}{\varepsilon^2} \left(-\frac{1}{8} A_4^I - \frac{1}{6} \beta_2 G_1^I(\varepsilon) - \frac{1}{2} \beta_1 G_2^I(\varepsilon) - \frac{3}{2} \beta_0 G_3^I(\varepsilon) \right) + \frac{1}{\varepsilon} \left(\frac{1}{4} G_4^I(\varepsilon) \right)\end{aligned}$$

[Henn, Lee, Smirnov, Smirnov, Steinhauser]
[Manteuffel, Scabinger]

ongoing ...

Form Factor @ 4loops.

- Solution at 4-loops:

$$\ln \hat{F}^I(\hat{a}_s, Q^2, \mu^2, \varepsilon) = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{Q^2}{\mu^2} \right)^{i \frac{\varepsilon}{2}} S_{\varepsilon}^i \hat{\mathcal{L}}_F^{I,(i)}(\varepsilon)$$

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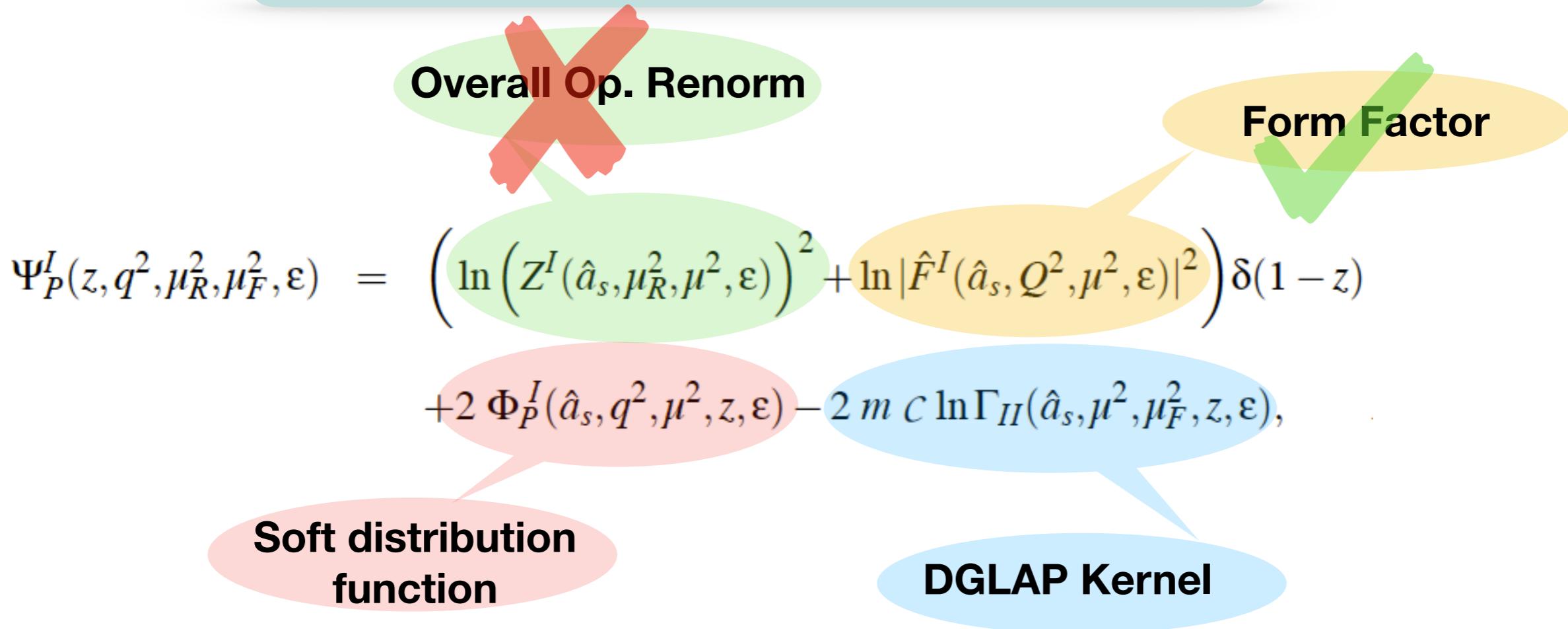
[Henn, Lee, Smirnov, Smirnov, Steinhauser]
[Manteuffel, Scabinger]

$$G_4 = 2B_4^q + (f_4^q + \beta_2 f_{01}^q + \beta_1 f_{02}^q + \beta_0 f_{03}^q) + \epsilon f_{04}^q$$

SV Cross-section.

- Finite Soft+Virtual cross-section:

$$\Delta_{I,P}^{sv}(z, q^2, \mu_R^2, \mu_F^2) = C \exp \left(\Psi_P^I(z, q^2, \mu_R^2, \mu_F^2, \varepsilon) \right) \Big|_{\varepsilon=0}$$



Collinear RGE.

- RGE for DGLAP kernel Γ :

$$\mu_F^2 \frac{d}{d\mu_F^2} \Gamma(z, \mu_F^2, \varepsilon) = \frac{1}{2} P(z, \mu_F^2) \otimes \Gamma(z, \mu_F^2, \varepsilon)$$

Altarelli-Parisi Splitting functions

- In SV cross-section, diagonal part contributes

$$P_{II}^{(i)}(z) = 2 \left[B_{i+1}^I \delta(1-z) + A_{i+1}^I \mathcal{D}_0 \right] + P_{reg,II}^{(i)}(z)$$

Coll. Ano. Dim.

Cusp. Ano. Dim.

Γ is known upto $\mathcal{O}(a_s^4)$

[Moch, Vermaseren, Vogt]

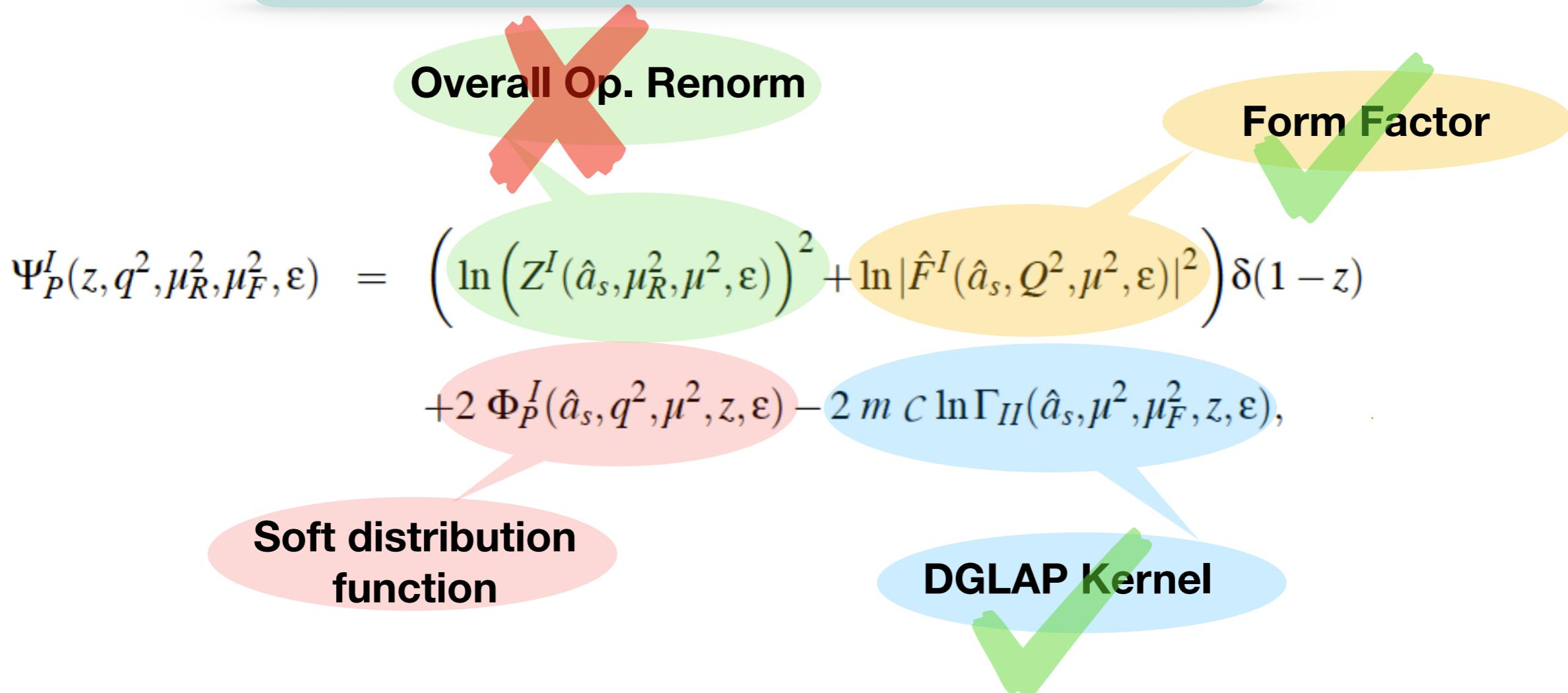
Approximate five-loop cusp ano. dim. is also estimated.

[Herzog, Moch, Ruijl, Ueda, Vermaseren, Vogt]

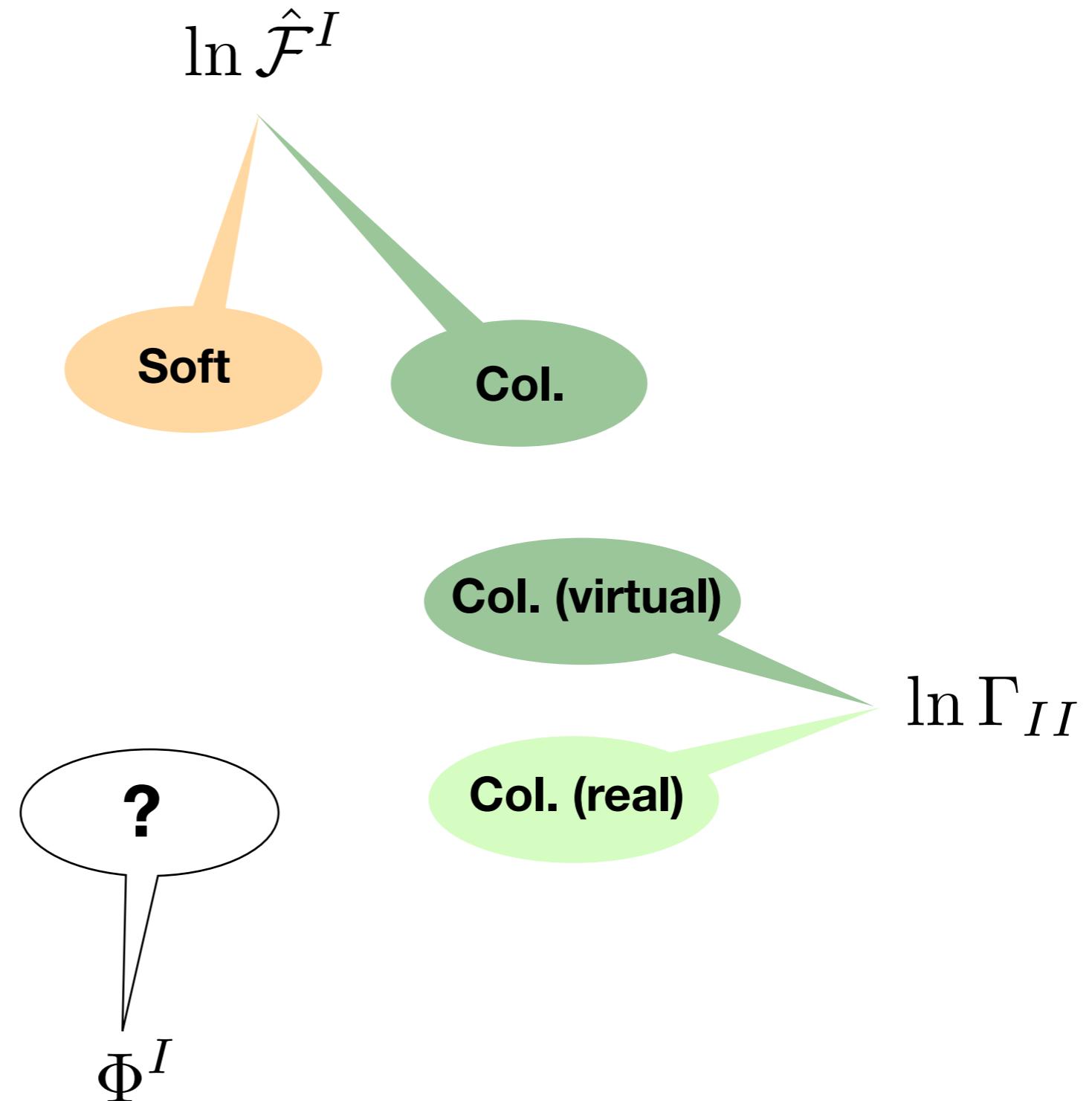
SV Cross-section.

- Finite Soft+Virtual cross-section:

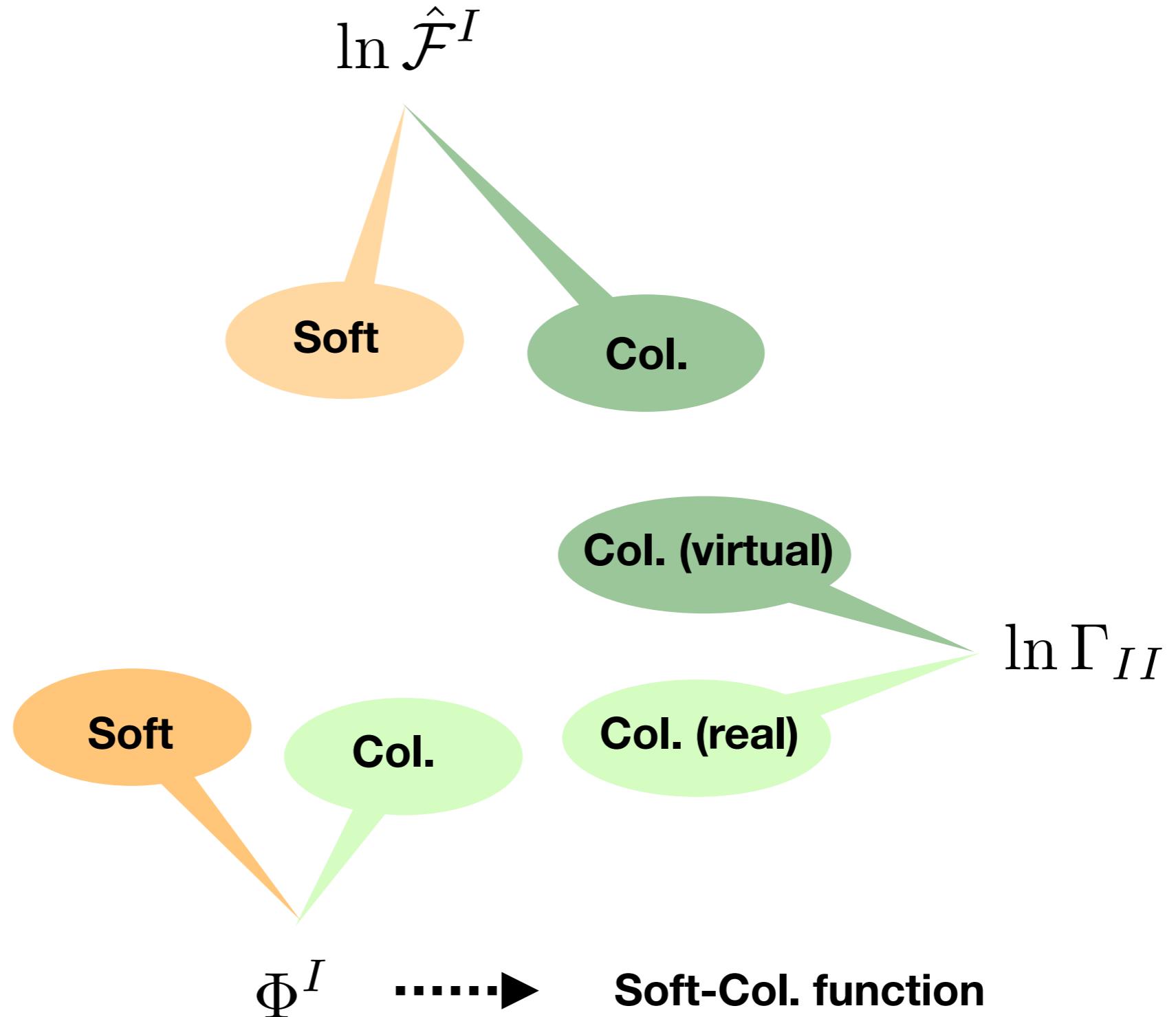
$$\Delta_{I,P}^{sv}(z, q^2, \mu_R^2, \mu_F^2) = C \exp \left(\Psi_P^I(z, q^2, \mu_R^2, \mu_F^2, \varepsilon) \right) \Big|_{\varepsilon=0}$$



Pole cancellation: DIS.



Pole cancellation: DIS.



Soft-Col. Function.

- Demand **finiteness** in $\lim_{\epsilon \rightarrow 0} \Psi_d^I(z_i, q^2, \epsilon)$
- **RGE** for the Soft function:

$$\mu_R^2 \frac{d}{d\mu_R^2} \Phi^I(\hat{a}_s, q^2, \mu^2, z, \epsilon) = 0$$

K-G type equation for Soft function

[Ravindran]

$$q^2 \frac{d}{dq^2} \Phi^I(\hat{a}_s, q^2, \mu^2, z, \epsilon) = \frac{1}{2} \left[\bar{K}^I \left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, z, \epsilon \right) + \bar{G}^I \left(\hat{a}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, z, \epsilon \right) \right]$$

Poles in ϵ

Finite in $\epsilon \rightarrow 0$

Soft-Col. Function.

- One possible ansatz:

$$\Phi^I(\hat{a}_s, q^2, \mu^2, z, \epsilon) = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{q^2(1-z)^2}{\mu^2} \right)^{i\frac{\epsilon}{2}} S_\epsilon^i \left(\frac{i\epsilon}{1-z} \right) \hat{\phi}^{I,(i)}(\epsilon)$$

[Ravindran, Smith, van Neerven]

- Solution: $\hat{\phi}^{I,(i)}(\epsilon) = \hat{\mathcal{L}}_F^{I,(i)}(\epsilon) \left(A^I \rightarrow -A^I, G^I(\epsilon) \rightarrow \bar{\mathcal{G}}^I(\epsilon) \right)$

$$\bar{\mathcal{G}}_{P,1}^I(\epsilon) = -\left(f_1^I + B_1^I \delta_{P,SJ}\right) + \sum_{k=1}^{\infty} \epsilon^k \bar{\mathcal{G}}_{P,1}^{I,(k)}$$

$$\bar{\mathcal{G}}_{P,2}^I(\epsilon) = -\left(f_2^I + B_2^I \delta_{P,SJ}\right) - 2\beta_0 \bar{\mathcal{G}}_{P,1}^{I,(1)} + \sum_{k=1}^{\infty} \epsilon^k \bar{\mathcal{G}}_{P,2}^{I,(k)}$$

$$\bar{\mathcal{G}}_{P,3}^I(\epsilon) = -\left(f_3^I + B_3^I \delta_{P,SJ}\right) - 2\beta_1 \bar{\mathcal{G}}_{P,1}^{I,(1)} - 2\beta_0 \left(\bar{\mathcal{G}}_{P,2}^{I,(1)} + 2\beta_0 \bar{\mathcal{G}}_{P,1}^{I,(2)} \right) + \sum_{k=1}^{\infty} \epsilon^k \bar{\mathcal{G}}_{P,3}^{I,(k)}$$

Verified to 3-loop for DY and Higgs

Catani, Cieri, de Florian, Ferrera, Frazzini;
Li, von Manteuffel, Scabinger, Zhu

Soft-Col. Function.

- 4-loop solution:

$$\begin{aligned}\bar{\mathcal{G}}_{P,4}^I(\varepsilon) = & - \left(f_4^I + B_4^I \delta_{P,SJ} \right) - 2\beta_2 \bar{\mathcal{G}}_{P,1}^{I,(1)} - 2\beta_1 \left(\bar{\mathcal{G}}_{P,2}^{I,(1)} + 4\beta_0 \bar{\mathcal{G}}_{P,1}^{I,(2)} \right) \\ & - 2\beta_0 \left(\bar{\mathcal{G}}_{P,3}^{I,(1)} + 2\beta_0 \bar{\mathcal{G}}_{P,2}^{I,(2)} + 4\beta_0^2 \bar{\mathcal{G}}_{P,1}^{I,(3)} \right) + \sum_{k=1}^{\infty} \varepsilon^k \bar{\mathcal{G}}_{P,4}^{I,(k)}\end{aligned}$$

Explicit FO computation

- All poles including single pole of Soft-collinear function can be predicted from that of Form Factor, Mass-fact Kernel.
- Terms with β_i are due to coupling const. renorm.
- Single pole is proportional to $A, - (f^I + B^I \delta_{P,SJ})$

Soft-Col. Function.

- 4 loop solution:

$$\begin{aligned}\bar{\mathcal{G}}_{P,4}^I(\varepsilon) = & - \left(f_4^I + B_4^I \delta_{P,SJ} \right) - 2\beta_2 \bar{\mathcal{G}}_{P,1}^{I,(1)} - 2\beta_1 \left(\bar{\mathcal{G}}_{P,2}^{I,(1)} + 4\beta_0 \bar{\mathcal{G}}_{P,1}^{I,(2)} \right) \\ & - 2\beta_0 \left(\bar{\mathcal{G}}_{P,3}^{I,(1)} + 2\beta_0 \bar{\mathcal{G}}_{P,2}^{I,(2)} + 4\beta_0^2 \bar{\mathcal{G}}_{P,1}^{I,(3)} \right) + \sum_{k=1}^{\infty} \varepsilon^k \bar{\mathcal{G}}_{P,4}^{I,(k)}\end{aligned}$$

Explicit FO
computation

- The new coefficients A_4^q, B_4^q are known.
- Maximally non-abelian up to 3-loop.
- Soft distributions are universal!

SV Cross-section.

- Finite Soft+Virtual cross-section:

$$\Delta_{I,P}^{sv}(z, q^2, \mu_R^2, \mu_F^2) = C \exp \left(\Psi_P^I(z, q^2, \mu_R^2, \mu_F^2, \varepsilon) \right) \Big|_{\varepsilon=0}$$

$$\begin{aligned} \Psi_P^I(z, q^2, \mu_R^2, \mu_F^2, \varepsilon) &= \left(\ln \left(Z^I(\hat{a}_s, \mu_R^2, \mu^2, \varepsilon) \right)^2 + \ln |\hat{F}^I(\hat{a}_s, Q^2, \mu^2, \varepsilon)|^2 \right) \delta(1-z) \\ &\quad + 2 \Phi_P^I(\hat{a}_s, q^2, \mu^2, z, \varepsilon) - 2 m_C \ln \Gamma_{II}(\hat{a}_s, \mu^2, \mu_F^2, z, \varepsilon), \end{aligned}$$

The diagram illustrates the components of the formula. A large red 'X' is placed over the term 'Overall Op. Renorm'. Three green checkmarks are placed over the terms 'Form Factor', 'Soft distribution function', and 'DGLAP Kernel'.

Soft-Virtual Structure.

- The Soft-virtual expression has the following structure:

$$\Delta_{sv}^{(n)} = \Delta_\delta \delta(1-z) + \sum_{k=0}^{2n-1} \Delta_{\mathcal{D}_k} \mathcal{D}_k$$

Virtual , Soft radiation



$$\delta(1-z)$$

Soft Radiation



$$\mathcal{D}_k = \left(\frac{\ln^k(1-z)}{(1-z)} \right)_+$$

Significant contributions from singular terms when $z \rightarrow 1$

Threshold Limit

Towards Threshold Resummation.

- Soft function solution:

$$\begin{aligned}\Phi_P^I(\hat{a}_s, q^2, \mu^2, z, \varepsilon) &= \left(\frac{m}{1-z} \left\{ \int_{\mu_R^2}^{q^2(1-z)^{2m}} \frac{d\lambda^2}{\lambda^2} A_I(a_s(\lambda^2)) + \overline{G}_P^I(a_s(q^2(1-z)^{2m}), \varepsilon) \right\} \right)_+ \\ &+ \delta(1-z) \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{q^2}{\mu^2} \right)^{i\frac{\varepsilon}{2}} S_{\varepsilon}^i \hat{\phi}_P^{I,(i)}(\varepsilon) \\ &+ \left(\frac{m}{1-z} \right)_+ \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{\mu_R^2}{\mu^2} \right)^{i\frac{\varepsilon}{2}} S_{\varepsilon}^i \overline{K}^{I,(i)}(\varepsilon)\end{aligned}$$

- Soft function decomposition:

$$\Phi^I \left(a_s(\mu_F^2), \frac{q^2}{\mu_F^2}, z, \varepsilon \right) = \Phi_{pole}^I \left(a_s(\mu_F^2), \frac{q^2}{\mu_F^2}, z, \frac{1}{\varepsilon} \right) + \Phi_{fin}^I \left(a_s(\mu_F^2), \frac{q^2}{\mu_F^2}, z, \varepsilon \right)$$

Towards Threshold Resummation.

- Finite piece directly relates to the threshold exponent

$$2 \int_0^1 dz z^{N-1} \Phi_{fin}^I \left(a_s(\mu_F^2), \frac{q^2}{\mu_F^2}, z, \varepsilon = 0 \right) = \int_0^1 dz \frac{z^{N-1} - 1}{1-z} \left[D^I \left(a_s \left(q^2(1-z)^2 \right) \right) \right. \\ \left. + 2 \int_{\mu_F^2}^{q^2(1-z)^2} \frac{d\lambda^2}{\lambda^2} A^I \left(a_s(\lambda^2) \right) \right]$$

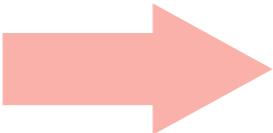
Resummed Exponent

$$+ H_S^I \left(a_s(\mu_F^2), \frac{q^2}{\mu_F^2} \right)$$

N-independent Finite piece

$$D^I \left(a_s \left(q^2(1-z)^2 \right) \right) = B_{DIS}^I \left(a_s \left(q^2(1-z) \right) \right) = \overline{G}_{SJ}^I \left(a_s \left(q^2(1-z) \right), \varepsilon \right) \Big|_{\varepsilon=0}$$

Soft-gluon resummation.

- Δ_I^{sv}  Distributions $\left(\frac{\ln^k(1-z)}{(1-z)} \right)_+$ and Delta $\delta(1-z)$
- Threshold limit: $z \rightarrow 1$
 - Large logarithms !!**
 - Fixed order becomes unreliable!!**
- Sum all threshold large logs to all orders in perturbation theory
 - Threshold resummation**

Soft-gluon resummation.

- Sudakov type exponentiation of soft-function.
- Resummation in conjugate space!
Amplitude factorises both in z-space and N-space.
Phase space factorises only in N-space.
- Logarithmic enhanced contribution in N-space also **contains subleading terms** when transformed into z-space.

Catani, de Florian, Grazzini, Nason;
Bonvini, Forte, Ridolfi

Soft-gluon resummation.

- DIS Wilson coefficient in Mellin space

$$C^N(Q^2)/C_{\text{LO}}^N(Q^2) = g_0(Q^2) \cdot \exp [G^N(Q^2)] + \mathcal{O}(N^{-1} \ln^n N)$$

- For DIS:

Moch, Vermaseren, Vogt;
Catani, de Florian, Grazzini, Nason

$$G_{\text{DIS}}^N = \ln \Delta_q + \ln J_q$$

Col. soft-gluon radiation factor from initial state

$$\ln \Delta_q(Q^2, \mu_f^2) = \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \int_{\mu_f^2}^{(1-z)^2 Q^2} \frac{dq^2}{q^2} A^q(\alpha_s(q^2))$$

Col. radiation factor from unobserved final state

$$\ln J_q(Q^2) = \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \left[\int_{(1-z)^2 Q^2}^{(1-z)Q^2} \frac{dq^2}{q^2} A^q(\alpha_s(q^2)) + B^{\text{DIS}}(\alpha_s([1-z]Q^2)) \right]$$

Soft-gluon resummation.

- DIS Wilson coefficient in Mellin space

$$C^N(Q^2)/C_{\text{LO}}^N(Q^2) = g_0(Q^2) \cdot \exp [G^N(Q^2)] + \mathcal{O}(N^{-1} \ln^n N)$$

- For DIS:

$$G_{\text{DIS}}^N = \ln \Delta_q + \ln J_q$$

$$G_N = \ln \bar{N} g^{(1)}(\lambda) + g^{(2)}(\lambda) + a_s g^{(3)}(\lambda) + a_s^2 g^{(4)}(\lambda) + a_s^3 g^{(5)}(\lambda)$$

LL

NLL

NNLL

$N^3 LL$

$N^4 LL$

new

Soft-gluon resummation

- Resummed coefficient:

$$G_N = \ln \bar{N} g^{(1)}(\lambda) + g^{(2)}(\lambda) + a_s g^{(3)}(\lambda) + a_s^2 g^{(4)}(\lambda) + a_s^3 g^{(5)}(\lambda)$$

$\mathcal{O}(a_s)$	$\ln^2(\bar{N})$	$\ln(\bar{N})$				
$\mathcal{O}(a_s^2)$	$\ln^3(\bar{N})$	$\ln^2(\bar{N})$	$\ln(\bar{N})$			
$\mathcal{O}(a_s^3)$	$\ln^4(\bar{N})$	$\ln^3(\bar{N})$	$\ln^2(\bar{N})$	$\ln(\bar{N})$		
$\mathcal{O}(a_s^4)$	$\ln^5(\bar{N})$	$\ln^4(\bar{N})$	$\ln^3(\bar{N})$	$\ln^2(\bar{N})$	$\ln(\bar{N})$	
$\mathcal{O}(a_s^5)$	$\ln^6(\bar{N})$	$\ln^5(\bar{N})$	$\ln^4(\bar{N})$	$\ln^3(\bar{N})$	$\ln^2(\bar{N})$	
⋮	⋮	⋮	⋮	⋮	⋮	⋮
	LL	NLL	NNLL	N ³ LL	N ⁴ LL	
Resummed Logs:	$a_s^m \ln^{m+1}(\bar{N})$	$a_s^m \ln^m(\bar{N})$	$a_s^{m+1} \ln^m(\bar{N})$	$a_s^{m+2} \ln^m(\bar{N})$	$a_s^{m+3} \ln^m(\bar{N})$	

Numerical Study.

- New quartic color factors.

$$d_{xy}^{(4)} \equiv d_x^{abcd} d_y^{abcd}$$

$$d_r^{abcd} = \frac{1}{6} \text{Tr} (T_r^a T_r^b T_r^c T_r^d + \text{five } bcd \text{ permutations})$$

- In a general SU(N):

$$\frac{d_{FA}^{(4)}}{n_c} = \frac{1}{48} (n_c^2 - 1)(n_c^2 + 6)$$

$$\frac{d_{FF}^{(4)}}{n_c} = \frac{1}{96} (n_c - n_c^{-1})(n_c^2 - 6 + 18n_c^{-2})$$

[Ritbergen, Schellekens, Vermaseren]

Numerical Study: A, B.

- Status of the coefficients A, B:

Color Factor	A_4^q	B_4^q
C_F^4	0	$197. \pm 3.$
$C_F^3 C_A$	0	$-687. \pm 10.$
$C_F^2 C_A^2$	0	$1219. \pm 12.$
$C_F C_A^3$	610.25 ± 0.1	295.6 ± 2.4
$d_{FA}^{(4)}/n_F$	-507.0 ± 2.0	$-996. \pm 45.$
<hr/>		
$n_f C_F^3$	-31.05543	81.4 ± 2.2
$n_f C_F^2 C_A$	38.79538	-455.7 ± 1.1
$n_f C_F C_A^2$	-440.6670	-274.4 ± 1.1
$n_f d_{FF}^{(4)}/n_F$	-123.8949	-143.5 ± 1.2
<hr/>		
$n_f^2 C_F^2$	-21.31439	-5.775288
$n_f^2 C_F C_A$	58.36737	51.03056
$n_f^3 C_F$	2.454258	2.261237

[Moch, Ruijl, Ueda, Vermaseren, Vogt]

Numerical Study: A, B.

- Status of the coefficients A, B:

quark	gluon	$A_{4,q}$	$A_{4,g}$
C_F^4	—	0	—
$C_F^3 C_A$	—	0	—
$C_F^2 C_A^2$	—	0	—
$C_F C_A^3$	C_A^4	610.25 ± 0.1	
$d_{FA}^{(4)}/N_F$	$d_{AA}^{(4)}/N_A$	-507.0 ± 2.0	-507.0 ± 5.0
<hr/>			
$n_f C_F^3$	$n_f C_F^2 C_A$	-31.00554	
$n_f C_F^2 C_A$	$n_f C_F C_A^2$	38.75 ± 0.2	
$n_f C_F C_A^2$	$n_f C_A^3$	-440.65 ± 0.2	
$n_f d_{FF}^{(4)}/N_F$	$n_f d_{FA}^{(4)}/N_A$	-123.90 ± 0.2	-124.0 ± 0.6
<hr/>			
$n_f^2 C_F^2$	$n_f^2 C_F C_A$	-21.31439	
$n_f^2 C_F C_A$	$n_f^2 C_A^2$	58.36737	
—	$n_f^2 d_{FF}^{(4)}/N_A$	—	0.0 ± 0.1
$n_f^3 C_F$	$n_f^3 C_A$	2.454258	2.454258

[Herzog, Moch, Ruijl, Ueda, Vermaseren, Vogt]

Numerical Study: B^{DIS} .

- For resummation we also need the coefficient B_q^{DIS} :

$$B_1^{DIS} = -f_1^q - B_1^q ,$$

$$B_2^{DIS} = -f_2^q - B_2^q - \beta_0 \left(f_{01}^q + c_{2,q}^{(1)} \Big|_{\text{const}_N} - \frac{1}{2} \zeta_2 A_1 \right) ,$$

$$B_3^{DIS} = -f_3^q - B_3^q - \beta_1 \left(f_{01}^q + c_{2,q}^{(1)} \Big|_{\text{const}_N} - \frac{1}{2} \zeta_2 A_1 \right) + \beta_0^2 \left(\zeta_2 f_1^q + \zeta_2 B_1^q + \frac{2}{3} \zeta_3 A_1 \right)$$

$$- \beta_0 \left(f_{02}^q + 2c_{2,q}^{(2)} \Big|_{\text{const}_N} - (c_{2,q}^{(1)})^2 \Big|_{\text{const}_N} - \zeta_2 A_2 \right) ,$$

Large-nc / Exact

Numerical Study: B^{DIS} .

- For resummation we also need the coefficient B_q^{DIS} .
- At fourth order:

$$\begin{aligned} B_4^{DIS} = & -f_4^q - B_4^q - \beta_2 \left(f_{01}^q + c_{2,q}^{(1)} \Big|_{\text{const}_N} - \frac{1}{2} \zeta_2 A_1 \right) \\ & + \beta_0^3 \left(3\zeta_2 f_{01}^q + 3\zeta_2 c_{2,q}^{(1)} \Big|_{\text{const}_N} + 2\zeta_3 f_1^q + 2\zeta_3 B_1^q - \frac{3}{8} \zeta_4 A_1 \right) \\ & + \beta_0 \beta_1 \left(\frac{5}{2} \zeta_2 f_1^q + \frac{5}{2} \zeta_2 B_1^q + \frac{5}{3} \zeta_3 A_1 \right) + \beta_0^2 (3\zeta_2 f_2^q + 3\zeta_2 B_2^q + 2\zeta_3 A_2) \\ & - \beta_1 \left(f_{02}^q + 2c_{2,q}^{(2)} \Big|_{\text{const}_N} - (c_{2,q}^{(1)})^2 \Big|_{\text{const}_N} - \zeta_2 A_2 \right) \\ & - \beta_0 \left(f_{03}^q + 3c_{2,q}^{(3)} \Big|_{\text{const}_N} - 3c_{2,q}^{(2)} c_{2,q}^{(1)} \Big|_{\text{const}_N} + (c_{2,q}^{(1)})^3 \Big|_{\text{const}_N} - \frac{3}{2} \zeta_2 A_3 \right) \end{aligned}$$

Numerical Study: B^{DIS} .

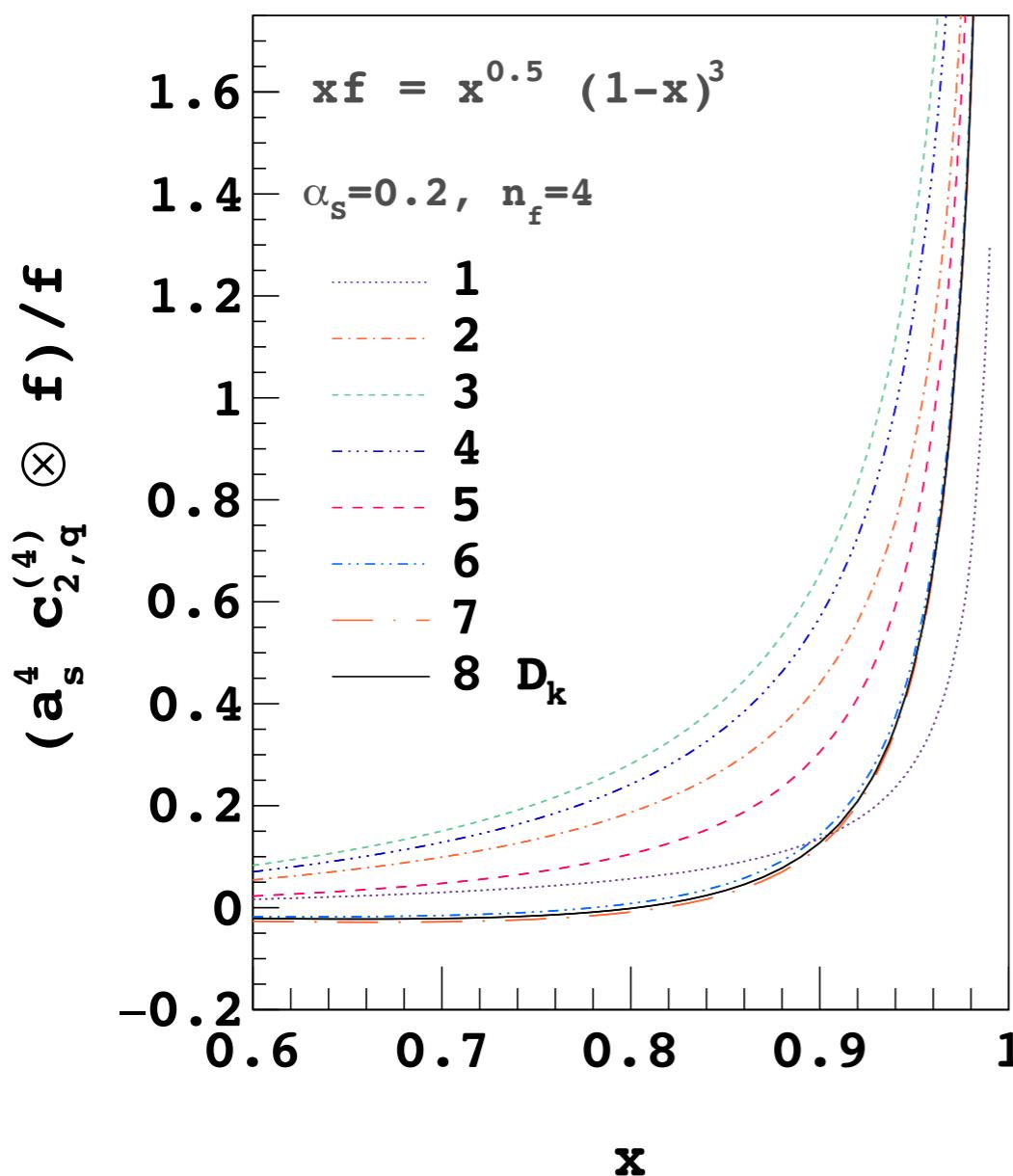
- For resummation we also need the coefficient B_q^{DIS} .
- At fourth order: Still unknown

$$f_{4, d_F^{abcd} d_A^{abcd}}^q, f_{4, n_f C_F^3}^q, f_{4, n_f C_F^2 C_A}^q$$

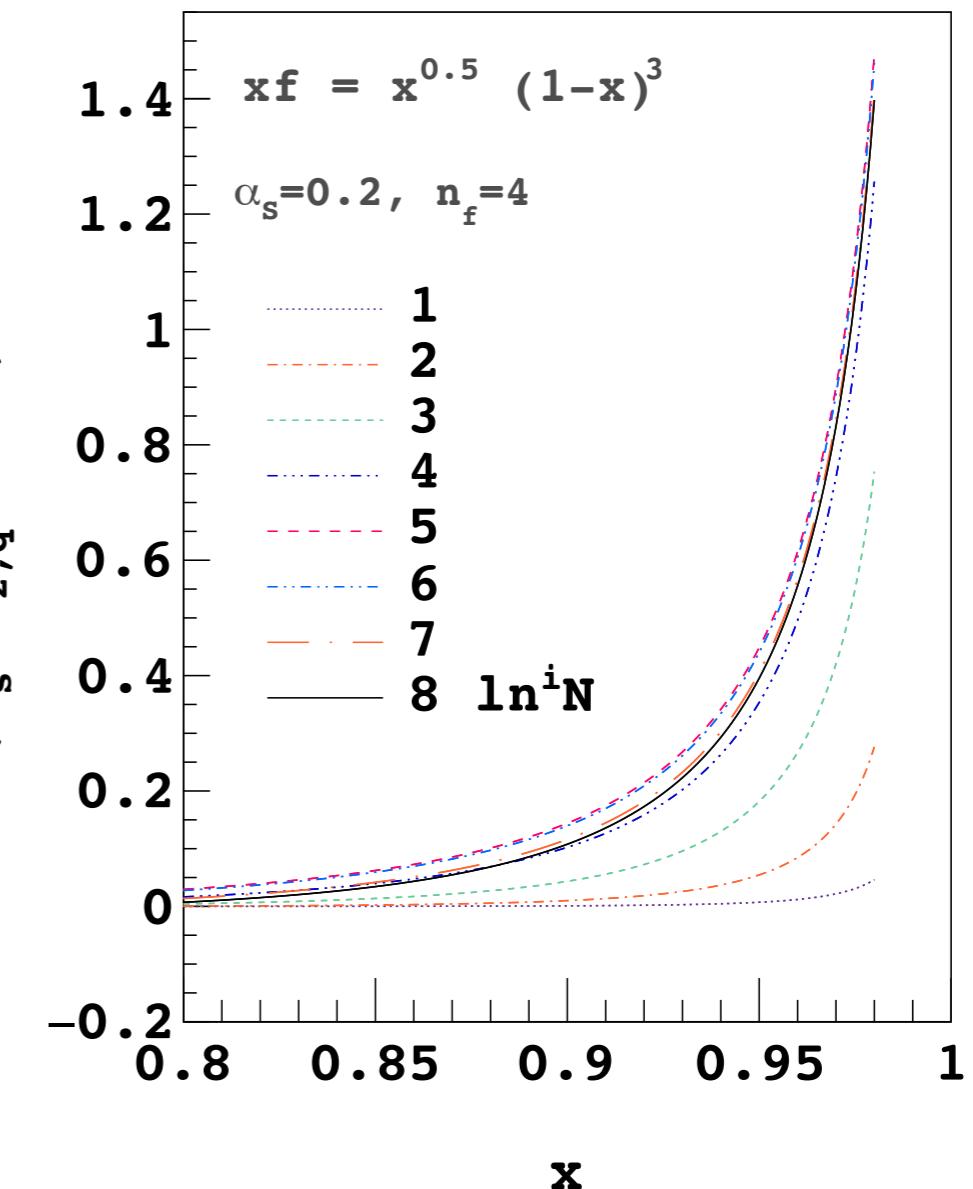
At large- n_c they drop out.

Numerical Study.

- Successive plus distributions and logarithms at fourth order:

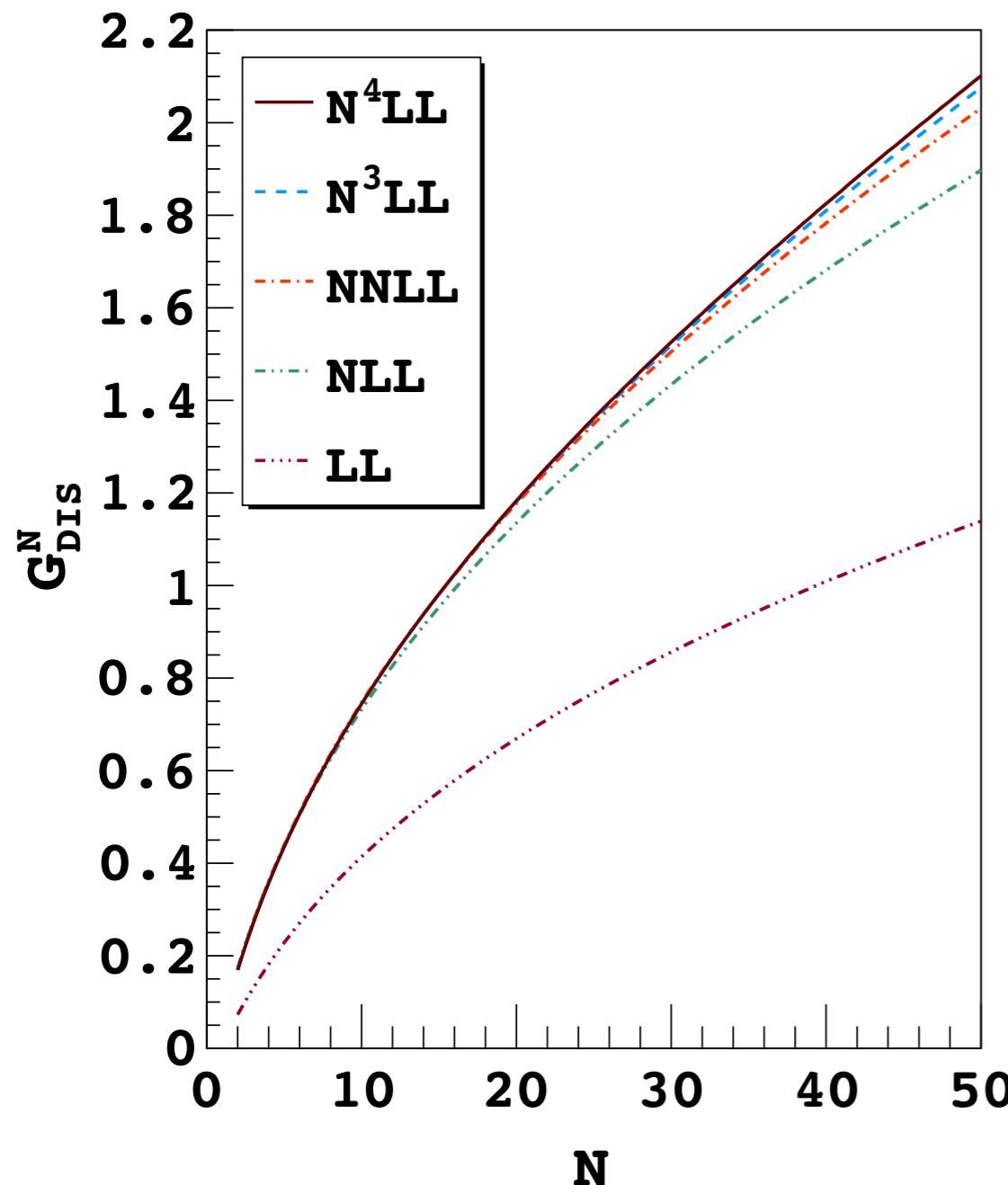


Slow convergence of successive distributions



Faster convergence of successive logarithms

Numerical Study.

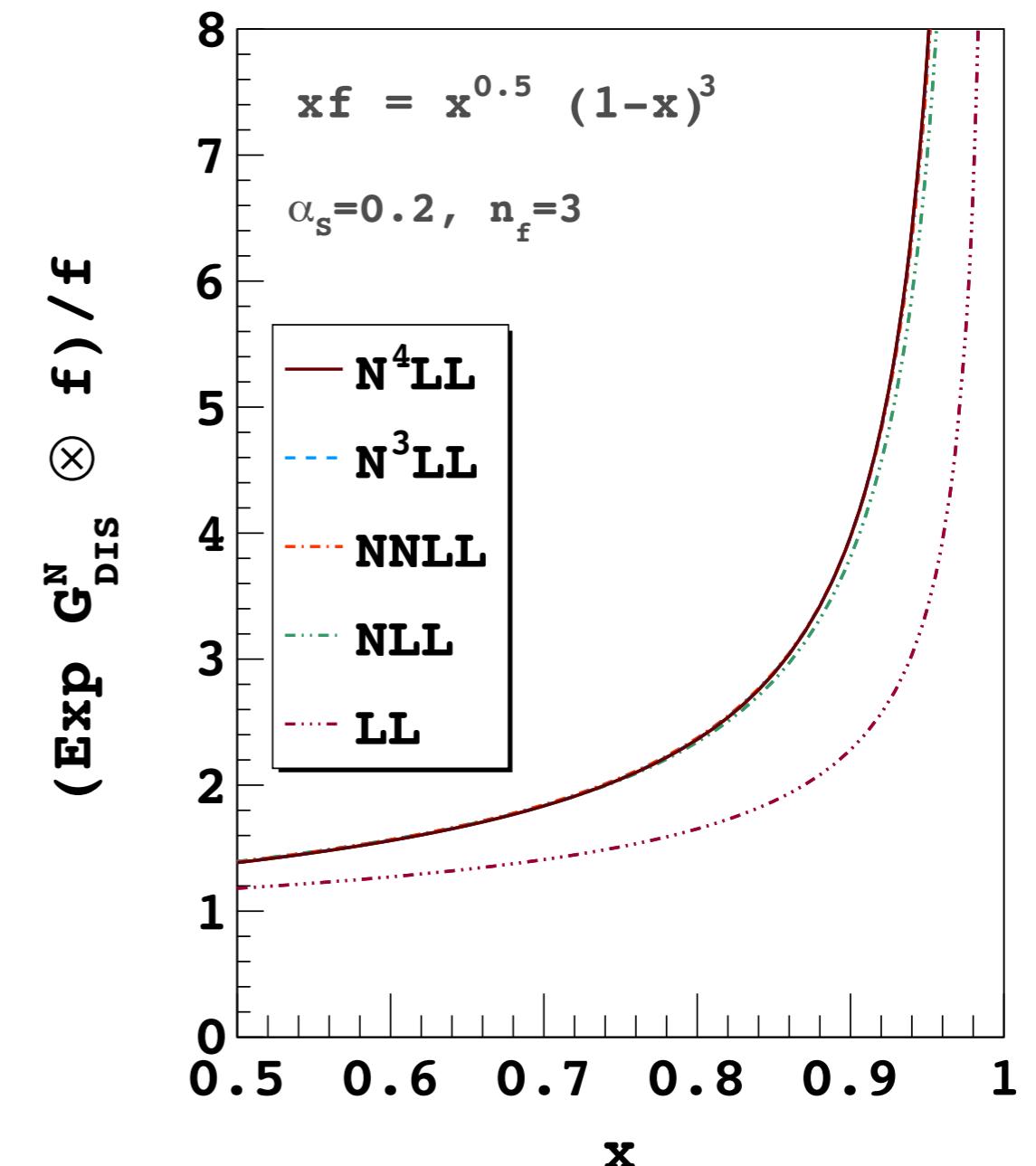


LL to NLL: 66%

NNLL to N3LL: 2.2%

NLL to NNLL: 7.1%

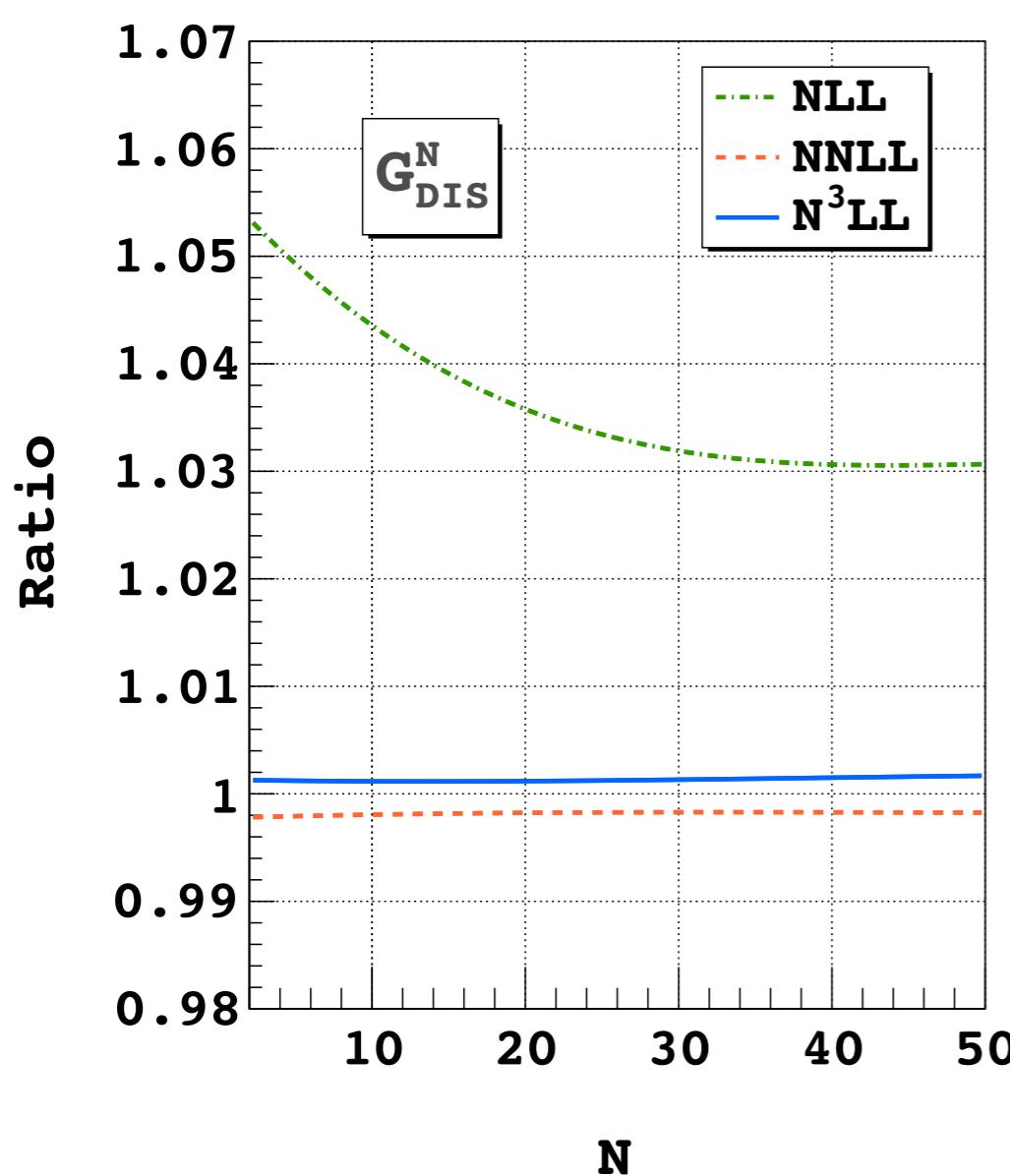
N3LL to N4LL: 1.2%



At N4LL: < 1%

Numerical Study.

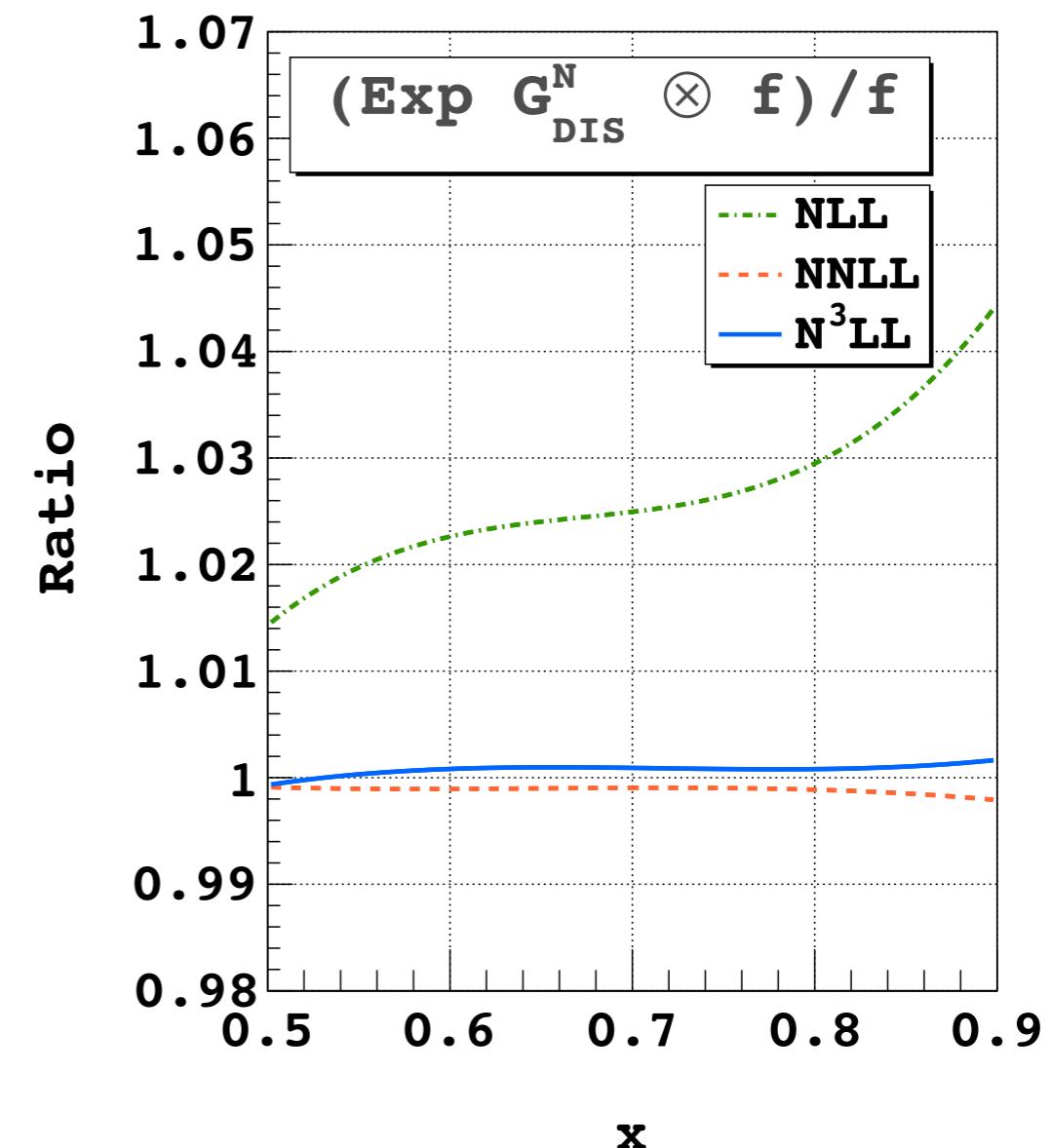
- Large-color approximation at lower orders:



At $x=0.95$

Large-nc to Exact: 5.42%, -0.28%, 0.23%

Corrections received from lower order: 92.9%, 2.5%, -2.1%



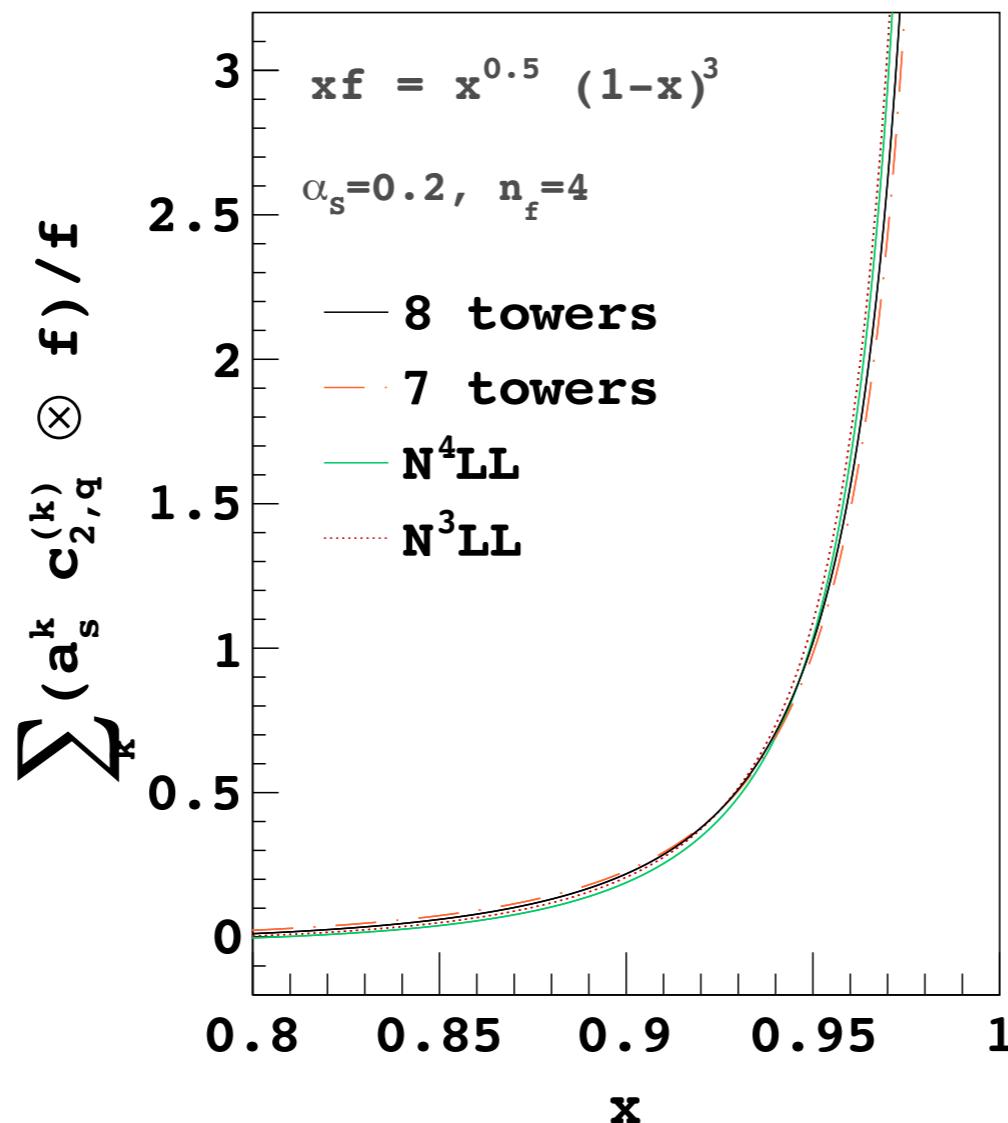
Fourth order contribution changes by -1.2%

Summary.

- We have studied a systematic way of resuming **threshold logs** for DIS and provided resummed exponent to $N^4 LL$ accuracy.
- We exploited the **factorisation**, **RG invariance** and **K+G equations** to four loops.
- We have compared the large-nc approximations at lower orders and found correction received from lower orders are reasonably large compared to large-nc errors.
- Work in progress to present approximate estimate for missing color pieces in f4 from DIS Mellin moments.

BK: Numerical Study.

Comparison between resummed coefficients and Tower Expansion:



Agree in lower-x region. Tower expansion deviates at large-x.