

Sub-eikonal spin corrections and g_1 structure function at low- x

Giovanni Antonio Chirilli

University of Regensburg

Torino - Italy 8 - 12 April, 2019

- Motivation
- Brief review of Operator Product Expansion at high-energy
- Operator Product Expansion at high-energy with sub-eikonal corrections
 - Quark and Gluon propagator with sub-eikonal corrections
- Evolution equation for flavor non-Singlet F_2 structure function at small- x .
- Evolution equation for flavor non-Singlet and Singlet g_1 structure function at small- x .
- Conclusions and Outlook

Based on

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G. A. C. in preparation

- Unpolarized DIS at low- x_B : dynamics is driven by gluon structure functions
 - gluon structure function grows as $(1/x_B)^\lambda$ with $\lambda > 1$.
- Polarized DIS at low- x_B : polarized gluon structure function grows as $(1/x_B)^\lambda$ with λ close to 0.
 - The polarized quark and gluon structure functions are equally relevant.
- At Electron Ion Collider low- x_B spin TMDs and g_1 structure function are relevant
 - Highly polarized ($\sim 70\%$) electron and nucleon beams
- Understand how the proton's spin arises from the intrinsic and orbital angular momenta of the constituent quarks and gluons.

- Compare with results obtained in the Leading Log approximation by [Bartels-Ermolaev-Ryskin-\(1995-1996\)](#) and recent work in Saturation formalism obtained by [Kovchegov-Pytoniak-Sievert \(20016-2017\)](#)
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 - **non-Singlet case** Agrees with BER only in the large N_c limit
 - **Singlet case** Agrees with the Large N_c limit of BER for the Ladder Diagrams. They find disagreement for the non ladder diagrams.
- R. Boussarie, Y. Hatta, F. Yuan (2019)
 - Have generalized BER result to include quark and gluon Orbital Angular Momentum

Spin sum rule

$$\frac{1}{2}\Delta\Sigma + \Delta G + L_q + L_g = \frac{1}{2}$$

- **BER: quark and gluon helicity**
- **BHY: Orbital Angular Momentum**

Propagation in the shock wave: Wilson line (Spectator frame)



Boost of the fields

$$x_{\bullet} = \sqrt{\frac{s}{2}}x^{-} \quad x_{*} = \sqrt{\frac{s}{2}}x^{+} \quad x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}}$$

$$A_{\bullet}(x_{\bullet}, x_{*}, x_{\perp}) \rightarrow \lambda A_{\bullet}(\lambda^{-1}x_{\bullet}, \lambda x_{*}, x_{\perp})$$

$$A_{*}(x_{\bullet}, x_{*}, x_{\perp}) \rightarrow \lambda^{-1}A_{*}(\lambda^{-1}x_{\bullet}, \lambda x_{*}, x_{\perp})$$

$$A_{\perp}(x_{\bullet}, x_{*}, x_{\perp}) \rightarrow A_{\perp}(\lambda^{-1}x_{\bullet}, \lambda x_{*}, x_{\perp})$$

λ is the boost parameter.

$$\langle x | \frac{i}{\hat{P} + i\epsilon} | y \rangle \rightarrow \langle x | \frac{i}{\hat{p} + \alpha \frac{2}{s} \not{p}_2 \hat{A}_{\bullet} + i\epsilon} | y \rangle$$

$$[\hat{\alpha}, \hat{A}_{\mu}^{cl}] = 0 \quad \text{with} \quad \alpha = \sqrt{\frac{2}{s}}p^{+} \quad \text{and} \quad \not{p}_2 \propto \gamma^{+}$$

Infinite boost: particle does not have time to deviate from straight line



Eikonal interactions give a Wilson lines

$$U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]$$

$$[x, y] = P e^{ig \int_0^1 du (x-y)^\mu A_\mu(ux+(1-u)y)} \quad p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu$$

Propagation in the shock wave: Wilson line (Spectator frame)



Quark propagator with eikonal interactions

$$\begin{aligned}
 \langle x | \frac{i}{\hat{p} + i\epsilon} | y \rangle &= \left[\int_0^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_* - y_*) - \int_{-\infty}^0 \frac{d\alpha}{2\alpha} \theta(y_* - x_*) \right] e^{-i\alpha(x_\bullet - y_\bullet)} \\
 &\times \frac{1}{\alpha s} \langle x_\perp | e^{-i\frac{\hat{p}_\perp^2}{\alpha s} x_*} \hat{p} \not{p}_2 [x_*, y_*] \hat{p} e^{i\frac{\hat{p}_\perp^2}{\alpha s} y_*} | y_\perp \rangle
 \end{aligned}$$

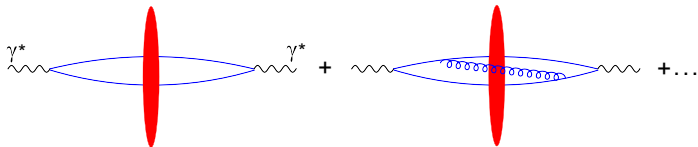
Propagation in the shock wave: Wilson line (Spectator frame)



Quark propagator with eikonal interactions

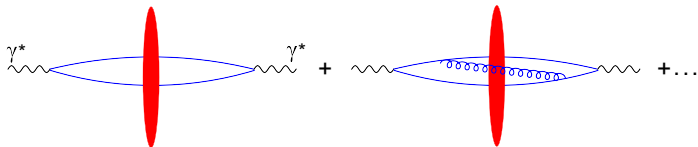
$$\begin{aligned}
 \langle x | \frac{i}{\hat{\not{p}} + i\epsilon} | y \rangle &= \left[\int_0^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_* - y_*) - \int_{-\infty}^0 \frac{d\alpha}{2\alpha} \theta(y_* - x_*) \right] e^{-i\alpha(x_\bullet - y_\bullet)} \\
 &\times \frac{1}{\alpha s} \langle x_\perp | e^{-i\frac{\hat{p}_\perp^2}{\alpha s} x_*} \hat{\not{p}} \not{p}_2 [\infty \not{p}_1 + z_\perp, -\infty \not{p}_1 + z_\perp] \hat{\not{p}} e^{i\frac{\hat{p}_\perp^2}{\alpha s} y_*} | y_\perp \rangle
 \end{aligned}$$

Diagrammatic representation Operator Product Expansion at High-energy



- The target is highly boosted

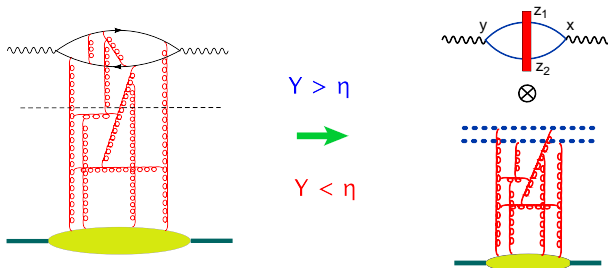
Operator Product Expansion at high energy



$$\begin{aligned}
 \langle P | T \{ \hat{j}_\mu(x) \hat{j}_\nu(y) \} | P \rangle &\simeq \int d^2 z_1 d^2 z_2 I_{\mu\nu}^{LO}(z_1, z_2; x, y) \langle P | \text{tr} \{ U_{z_1} U_{z_2}^\dagger \} | P \rangle \\
 &+ \frac{\alpha_s}{\pi} \int d^2 z_1 d^2 z_2 d^2 z_3 I_{\mu\nu}^{NLO}(z_1, z_2, z_3; x, y) \\
 &\quad \times \langle P | \left[\text{tr} \{ U_{z_1} U_{z_3}^\dagger \} \text{tr} \{ U_{z_3} U_{z_2}^\dagger \} - N_c \text{tr} \{ U_{z_1} U_{z_2}^\dagger \} \right] | P \rangle
 \end{aligned}$$

High-Energy Operator Product Expansion

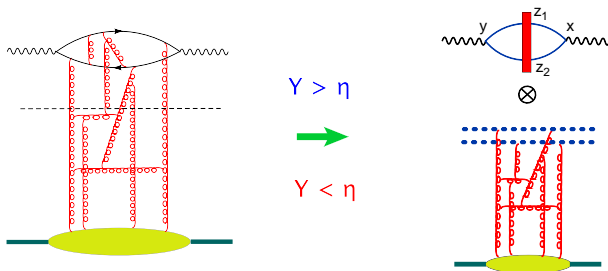
DIS amplitude is factorized in rapidity: η



$|P\rangle$ is the target state.

$$\langle P|T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|P\rangle = \int d^2z_1 d^2z_2 I_{\mu\nu}^{\text{LO}}(x, y; z_1, z_2) \langle P|\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}|P\rangle + \dots$$

High-Energy Operator Product Expansion

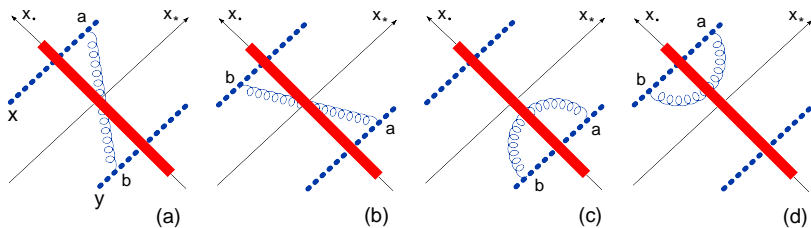


$$\langle P | T \{ \hat{j}_\mu(x) \hat{j}_\nu(y) \} | P \rangle = \int d^2 z_1 d^2 z_2 I_{\mu\nu}^{\text{LO}}(x, y; z_1, z_2) \langle P | \text{tr} \{ \hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta} \} | P \rangle + \dots$$

- If we use a model to evaluate $\langle P | \text{tr} \{ \hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta} \} | P \rangle$ we can calculate the DIS cross-section.
- If we want to include energy dependence to the DIS cross section, we need to find the evolution of $\langle P | \text{tr} \{ \hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta} \} | P \rangle$ with respect to the rapidity parameter η .

$$\frac{d}{d\eta} \text{tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\text{LO}} \text{tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \dots \Rightarrow$$

$$\frac{d}{d\eta} \langle \text{tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{\text{LO}} \text{tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}}$$



$$x_\bullet = \sqrt{\frac{s}{2}} x^-$$

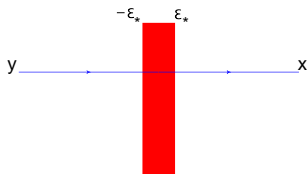
$$x_* = \sqrt{\frac{s}{2}} x^+$$

$$\hat{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

$$\frac{d}{d\eta}\hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2z}{(x-z)^2(y-z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\}$$

- LLA for DIS in pQCD \Rightarrow BFKL
 - (LLA: $\alpha_s \ll 1, \alpha_s \eta \sim 1$): describes proliferation of gluons.
- LLA for DIS in semi-classical-QCD \Rightarrow BK eqn
 - background field method: describes recombination process.
- Note: if $x_\perp \rightarrow z_\perp$ or $y_\perp \rightarrow z_\perp$ divergences cancel out.

Shock-wave with finite width



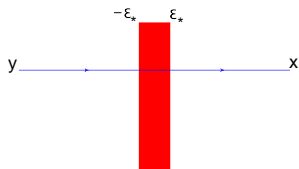
$$x_* = \sqrt{\frac{s}{2}}x^+ \quad x_\bullet = \sqrt{\frac{s}{2}}x^-$$

$$\begin{aligned} A_\bullet(x_\bullet, x_*, x_\perp) &\rightarrow \lambda A_\bullet(\lambda^{-1}x_\bullet, \lambda x_*, x_\perp) \\ A_*(x_\bullet, x_*, x_\perp) &\rightarrow \lambda^{-1} A_*(\lambda^{-1}x_\bullet, \lambda x_*, x_\perp) \\ A_\perp(x_\bullet, x_*, x_\perp) &\rightarrow A_\perp(\lambda^{-1}x_\bullet, \lambda x_*, x_\perp) \end{aligned}$$

λ is the boost parameter

- $p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu$
- **small** α gluons are **classical** fields **large** α gluons are **quantum** fields.
- Longitudinal size **classical fields**: $\epsilon_* = \frac{\alpha s}{l_\perp^2}$ with l_\perp trans. mom. of classical fields
- Distance traveled by **quantum fields**: $z_* = \frac{\alpha s}{k_\perp^2}$ with k_\perp trans. mom. of classical fields
- We are in the case $l_\perp \sim k_\perp$

Shock-wave with finite width



$$x_* = \sqrt{\frac{s}{2}}x^+ \quad x_\bullet = \sqrt{\frac{s}{2}}x^-$$

$$\begin{aligned} A_\bullet(x_\bullet, x_*, x_\perp) &\rightarrow \lambda A_\bullet(\lambda^{-1}x_\bullet, \lambda x_*, x_\perp) \\ A_*(x_\bullet, x_*, x_\perp) &\rightarrow \lambda^{-1}A_*(\lambda^{-1}x_\bullet, \lambda x_*, x_\perp) \\ A_\perp(x_\bullet, x_*, x_\perp) &\rightarrow A_\perp(\lambda^{-1}x_\bullet, \lambda x_*, x_\perp) \end{aligned}$$

λ is the boost parameter

sub-eikonal terms go like $\frac{1}{\lambda}$

$$\langle x | \frac{i}{\not{p} + i\epsilon} | y \rangle \rightarrow \langle x | \not{p} \frac{i}{p^2 + 2\alpha A_\bullet + ig\frac{2}{s}\not{p}_2\gamma^i F_{\bullet i} + \frac{1}{2}F_{ij}\sigma^{ij} + \dots + i\epsilon} | y \rangle$$

■ Note: $[\hat{\alpha}, \hat{A}_\mu^{cl}] = 0$ with $\alpha = \sqrt{\frac{2}{s}}p^+$ and $\not{p}_2 \propto \gamma^+$

$$e^{i\frac{\hat{p}_2^2}{\alpha s}z_*} \hat{A}_\bullet(z_*) e^{-i\frac{\hat{p}_2^2}{\alpha s}z_*} \simeq A_\bullet(z_*) - \frac{z_*}{\alpha s} \{p^i, F_{\bullet i}(z_*)\} - \frac{z_*^2}{2\alpha^2 s^2} \{p^j, \{p^i, D_j F_{\bullet i}(z_*)\}\} + \dots$$

Quark propagator with sub-eikonal corrections

$$x_* = \sqrt{\frac{s}{2}} x^+ \quad x_\bullet = \sqrt{\frac{s}{2}} x^- \quad x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}}$$

$$\begin{aligned} \langle x | \frac{i}{\hat{p} + i\epsilon} | y \rangle &= \left[\int_0^{+\infty} \frac{d\alpha}{2s\alpha^2} \theta(x_* - y_*) - \int_{-\infty}^0 \frac{d\alpha}{2s\alpha^2} \theta(y_* - x_*) \right] e^{-i\alpha(x_\bullet - y_\bullet)} \langle x_\perp | e^{-i\frac{\hat{p}_\perp^2}{\alpha s} x_*} \\ &\times \left\{ \hat{p} \not{p}_2 [x_*, y_*] \hat{p} + \hat{p} \not{p}_2 \hat{\mathcal{O}}_1(x_*, y_*; p_\perp) \hat{p} + \frac{1}{2} \hat{p} \not{p}_2 \hat{\mathcal{O}}_2(x_*, y_*) - \frac{1}{2} \hat{\mathcal{O}}_2(x_*, y_*) \not{p}_2 \hat{p} \right\} e^{i\frac{\hat{p}_\perp^2}{\alpha s} y_*} | y_\perp \rangle \\ &+ O(\lambda^{-2}) \end{aligned}$$

- Leading-eikonal term
- Sub-eikonal terms

Operators $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$ *measure* the deviation from the straight line.

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Quark propagator with sub-eikonal corrections

$$x_* = \sqrt{\frac{s}{2}} x^+ \quad x_\bullet = \sqrt{\frac{s}{2}} x^- \quad x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}}$$

$$\hat{\mathcal{O}}_1(x_*, y_*; p_\perp) = \frac{ig}{2\alpha} \int_{y_*}^{x_*} d\frac{2}{s} \omega_* \left([x_*, \omega_*] \frac{1}{2} \sigma^{ij} F_{ij}[\omega_*, y_*] + \{\hat{p}^i, [x_*, \omega_*] \frac{2}{s} \omega_* F_{i\bullet}(\omega_*) [\omega_*, y_*]\} \right. \\ \left. + g \int_{\omega_*}^{x_*} d\frac{2}{s} \omega'_* \frac{2}{s} (\omega_* - \omega'_*) [x_*, \omega'_*] F^i_{\bullet}[\omega'_*, \omega_*] F_{i\bullet}[\omega_*, y_*] \right)$$

$$\hat{\mathcal{O}}_2(x_*, y_*; p_\perp) = \frac{ig}{2\alpha} \int_{y_*}^{x_*} d\frac{2}{s} \omega_* \left[[x_*, \omega_*] \frac{i}{4} \{ (\not{p}_\perp F_{ij}), \gamma^i \gamma^j \} [\omega_*, y_*] + \{\hat{p}^k, [x_*, \omega_*] i F_{kj} \gamma^j [\omega_*, y_*]\} \right. \\ \left. + \int_{\omega_*}^{x_*} d\frac{2}{s} \omega'_* \left([x_*, \omega'_*] g F^k_{\bullet}[\omega'_*, \omega_*] i F_{kj} \gamma^j [\omega_*, y_*] - [x_*, \omega'_*] i F_{kj} \gamma^j [\omega'_*, \omega_*] g F^k_{\bullet}[\omega_*, y_*] \right) \right. \\ \left. + \int_{\omega_*}^{x_*} d\frac{2}{s} \omega'_* \left([x_*, \omega'_*] i \frac{2}{s} F_{\bullet*}[\omega'_*, \omega_*] \gamma^k g F_{k\bullet}[\omega_*, y_*] \right. \right. \\ \left. \left. - [x_*, \omega'_*] \gamma^k g F_{k\bullet}[\omega'_*, \omega_*] i \frac{2}{s} F_{\bullet*}[\omega_*, y_*] \right) + (\hat{\alpha} \not{p}_1 - \hat{p}_\perp) [x_*, \omega_*] i \frac{2}{s} F_{\bullet*}[\omega_*, y_*] \right]$$

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Quark propagator with sub-eikonal corrections

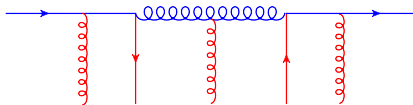
$$x_* = \sqrt{\frac{s}{2}} x^+ \quad x_\bullet = \sqrt{\frac{s}{2}} x^- \quad x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}}$$

$$\hat{\mathcal{O}}_1(x_*, y_*; p_\perp) = \frac{ig}{2\alpha} \int_{y_*}^{x_*} d\frac{2}{s} \omega_* \left([x_*, \omega_*] \frac{1}{2} \sigma^{ij} F_{ij}[\omega_*, y_*] + \{\hat{p}^i, [x_*, \omega_*] \frac{2}{s} \omega_* F_{i\bullet}(\omega_*) [\omega_*, y_*]\} \right. \\ \left. + g \int_{\omega_*}^{x_*} d\frac{2}{s} \omega'_* \frac{2}{s} (\omega_* - \omega'_*) [x_*, \omega'_*] F^i{}_\bullet[\omega'_*, \omega_*] F_{i\bullet}[\omega_*, y_*] \right)$$

$$\hat{\mathcal{O}}_2(x_*, y_*; p_\perp) = \frac{ig}{2\alpha} \int_{y_*}^{x_*} d\frac{2}{s} \omega_* \left[[x_*, \omega_*] \frac{i}{4} \{ (i\hat{p}_\perp F_{ij}), \gamma^i \gamma^j \} [\omega_*, y_*] + \{\hat{p}^k, [x_*, \omega_*] iF_{kj} \gamma^j [\omega_*, y_*]\} \right. \\ \left. + g \int_{\omega_*}^{x_*} d\frac{2}{s} \omega'_* \left([x_*, \omega'_*] F^k{}_\bullet[\omega'_*, \omega_*] iF_{kj} \gamma^j [\omega_*, y_*] - [x_*, \omega'_*] iF_{kj} \gamma^j [\omega'_*, \omega_*] F^k{}_\bullet[\omega_*, y_*] \right) \right. \\ \left. + g \int_{\omega_*}^{x_*} d\frac{2}{s} \omega'_* \left([x_*, \omega'_*] i\frac{2}{s} F_{\bullet*}[\omega'_*, \omega_*] \gamma^k F_{k\bullet}[\omega_*, y_*] \right. \right. \\ \left. \left. - [x_*, \omega'_*] \gamma^k g F_{k\bullet}[\omega'_*, \omega_*] i\frac{2}{s} F_{\bullet*}[\omega_*, y_*] \right) + (\hat{\alpha} \hat{p}_1 - \hat{p}_\perp) [x_*, \omega_*] i\frac{2}{s} F_{\bullet*}[\omega_*, y_*] \right]$$

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Quark propagator in the background of quark fields



$$\begin{aligned}
 & \langle \mathbf{T} \{ \psi(x) \bar{\psi}(y) \} \rangle_{\psi, \bar{\psi}} \\
 &= g^2 \left[\int_0^{+\infty} \frac{d\alpha}{2s^2 \alpha^3} \theta(x_* - y_*) - \int_{-\infty}^0 \frac{d\alpha}{2s^2 \alpha^3} \theta(y_* - x_*) \right] e^{-i\alpha(x_\bullet - y_\bullet)} \int_{y_*}^{x_*} dz_* \int_{y_*}^{z_*} dz'_* \\
 & \times \langle x_\perp | e^{-i \frac{\hat{p}_\perp^2}{\alpha s} x_*} (\alpha \not{p}_\perp + \not{p}_\perp) [x_*, z_*] \gamma_\perp^\mu t^a \psi(z_*) [z_*, z'_*]^{ab} \bar{\psi}(z'_*) t^b \gamma_\mu^\perp [z'_*, y_*] (\alpha \not{p}_\perp + \not{p}_\perp) e^{i \frac{\hat{p}_\perp^2}{\alpha s} y_*} | y_\perp \rangle
 \end{aligned}$$

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Gluon propagator in the background of gluon fields

$$i\langle A_\mu^a(x)A_\nu^b(y)\rangle \equiv iG_{\mu\nu}^{ab}(x,y) = \langle x|\frac{1}{\square^{\mu\nu} - P^\mu P^\nu + \frac{1}{w}p_2^\mu p_2^\nu}|y\rangle^{ab}$$

with $\square^{\mu\nu} = P^2 g^{\mu\nu} + 2i g F^{\mu\nu}$

$$\begin{aligned} & \langle A_\mu^a(x)A_\nu^b(y)\rangle_A \\ &= \left[-\int_0^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_* - y_*) + \int_{-\infty}^0 \frac{d\alpha}{2\alpha} \theta(y_* - x_*) \right] e^{-i\alpha(x_\bullet - y_\bullet)} \\ & \times \langle x_\perp | e^{-i\frac{\hat{p}_\perp^2}{\alpha s} x_*} \left(\delta_\mu^\xi - \frac{p_{2\mu}}{p_*} p^\xi \right) \mathcal{O}_\alpha(x_*, y_*) \left(g_{\xi\nu} - p_\xi \frac{p_{2\nu}}{p_*} \right) e^{i\frac{\hat{p}_\perp^2}{\alpha s} y_*} |y_\perp\rangle^{ab} + i \langle x | \frac{p_{2\mu} p_{2\nu}}{p_*^2} |y\rangle^{ab} \\ & + \left[-\int_0^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_* - y_*) + \int_{-\infty}^0 \frac{d\alpha}{2\alpha} \theta(y_* - x_*) \right] e^{-i\alpha(x_\bullet - y_\bullet)} \langle x_\perp | e^{-i\frac{\hat{p}_\perp^2}{\alpha s} x_*} \\ & \times \left[\mathfrak{G}_{1\mu\nu}^{ab}(x_*, y_*; p_\perp) + \mathfrak{G}_{2\mu\nu}^{ab}(x_*, y_*; p_\perp) + \mathfrak{G}_{3\mu\nu}^{ab}(x_*, y_*; p_\perp) + \mathfrak{G}_{4\mu\nu}^{ab}(x_*, y_*; p_\perp) \right] \\ & \times e^{i\frac{\hat{p}_\perp^2}{\alpha s} y_*} |y_\perp\rangle + O(\lambda^{-2}), \end{aligned}$$

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$$\mathcal{O}_\alpha(x_*, y_*) \equiv [x_*, y_*] + \frac{ig}{2\alpha} \int_{y_*}^{x_*} d\frac{2}{s}\omega_* \left(\left\{ p^i, [x_*, \omega_*] \frac{2}{s}\omega_* F_{i\bullet}(\omega_*) [\omega_*, y_*] \right\} \right. \\ \left. + g \int_{\omega_*}^{x_*} d\frac{2}{s}\omega'_* \frac{2}{s}(\omega_* - \omega'_*) [x_*, \omega'_*] F_{i\bullet}^i[\omega'_*, \omega_*] F_{i\bullet}[\omega_*, y_*] \right).$$

$$\mathfrak{G}_{1\mu\nu}^{ab}(x_*, y_*; p_\perp) = -\frac{g p_{2\mu} p_{2\nu}}{s^2 \alpha^3} \int_{y_*}^{x_*} d\frac{2}{s}\omega_* \left[4p^i [x_*, \omega_*] F_{ij}[\omega_*, y_*] p^j \right. \\ \left. + ig \int_{\omega'_*}^{x_*} d\frac{2}{s}\omega'_* \frac{2}{s}(\omega'_* - \omega_*) [x_*, \omega'_*] iD^i F_{i\bullet}[\omega'_*, \omega_*] iD^j F_{j\bullet}[\omega_*, y_*] \right]^{ab},$$

$$\mathfrak{G}_{2\mu\nu}^{ab}(x_*, y_*; p_\perp) = -\frac{g}{\alpha} \delta_\mu^i \delta_\nu^j \int_{y_*}^{x_*} d\frac{2}{s}\omega_* ([x_*, \omega_*] F_{ij}[\omega_*, y_*])^{ab},$$

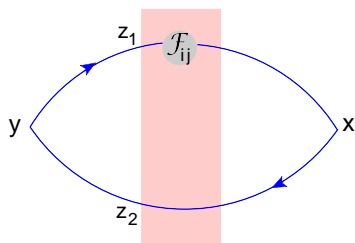
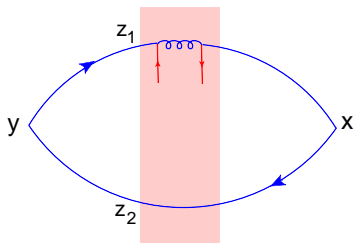
$$\mathfrak{G}_{3\mu\nu}^{ab}(x_*, y_*; p_\perp) = \frac{g}{\alpha^2 s} (\delta_\mu^j p_{2\nu} + \delta_\nu^j p_{2\mu}) \int_{y_*}^{x_*} d\frac{2}{s}\omega_* ([x_*, \omega_*] iD^i F_{ij}[\omega_*, y_*])^{ab},$$

$$\mathfrak{G}_{4\mu\nu}^{ab}(x_*, y_*; p_\perp) = -\frac{2g^2}{\alpha^2 s} \int_{y_*}^{x_*} d\frac{2}{s}\omega_* \int_{\omega_*}^{x_*} d\frac{2}{s}\omega'_* \left(\delta_\nu^j p_{2\mu} [x_* \omega'_*] F_{i\bullet}^i[\omega'_*, \omega_*] F_{ij}[\omega_*, y_*] \right. \\ \left. + \delta_\mu^j p_{2\nu} [x_* \omega'_*] F_{ij}[\omega'_*, \omega_*] F_{i\bullet}^i[\omega_*, y_*] \right)^{ab}.$$

$$\begin{aligned}
 & \langle A_\mu^a(x) A_\nu^b(y) \rangle_{\psi, \bar{\psi}} \\
 &= \left[- \int_0^{+\infty} \bar{d}\alpha \theta(x_* - y_*) + \int_{-\infty}^0 \bar{d}\alpha \theta(y_* - x_*) \right] e^{-i\alpha(x \bullet - y \bullet)} \\
 & \times \frac{g^2}{2\alpha^2 s^2} \int_{y_*}^{x_*} dz_{1*} \int_{y_*}^{z_{1*}} dz_{2*} \left[\langle x_\perp | e^{-i\frac{\hat{p}_\perp^2}{\alpha s} x_*} \left(g_{\perp\mu}^\xi - \frac{p_{2\mu}}{p_*} p_\perp^\xi \right) \right. \\
 & \times \bar{\psi}(z_{1*}) \gamma_\xi^\perp \not{p}_1 [z_{1*}, x_*] t^a [x_*, y_*] t^b [y_*, z_{2*}] \gamma_\perp^\sigma \psi(z_{2*}) \left(g_{\sigma\nu}^\perp - p_\sigma^\perp \frac{p_{2\nu}}{p_*} \right) e^{i\frac{\hat{p}_\perp^2}{\alpha s} y_*} |y_\perp \rangle \\
 & + \langle y_\perp | e^{-i\frac{\hat{p}_\perp^2}{\alpha s} y_*} \left(g_{\perp\nu}^\xi - \frac{p_{2\nu}}{p_*} p_\perp^\xi \right) \bar{\psi}(z_{2*}) \gamma_\xi^\perp \not{p}_1 [z_{2*}, y_*] t^b [y_*, x_*] t^a [x_*, z_{1*}] \gamma_\perp^\sigma \psi(z_{1*}) \\
 & \left. \times \left(g_{\sigma\mu}^\perp - p_\sigma^\perp \frac{p_{2\mu}}{p_*} \right) e^{i\frac{\hat{p}_\perp^2}{\alpha s} x_*} |x_\perp \rangle \right] + O(\lambda^{-2})
 \end{aligned}$$

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Quark propagator with sub-eikonal corrections



Let $|P\rangle$ be proton or nuclear target

$$\langle P|J^\mu(x)J^\nu(y)|P\rangle \rightarrow \langle J^\mu(x)J^\nu(y)\rangle_{A\psi\bar{\psi}} = \text{Tr}\left\{\gamma^\mu\langle x|\frac{1}{\not{P}+i\epsilon}|y\rangle\gamma^\nu\langle y|\frac{1}{\not{P}+i\epsilon}|y\rangle\right\}$$

Quark propagator with sub-eikonal corrections: \Rightarrow New evolution equations

$$\begin{aligned}
 \langle P | T \{ \hat{j}_\mu(x) \hat{j}_\nu(y) \} | P \rangle &\simeq \int d^2 z_1 d^2 z_2 \left[I_{\mu\nu}^{eikonal}(z_{1\perp}, z_{2\perp}; x, y) \langle P | \text{Tr} \{ U_{z_1} U_{z_2}^\dagger \} | P \rangle \right. \\
 &+ I_{1\mu\nu}^{sub\ eik}(z_1, z_2; x, y) \langle P | \left(Q_1(z_{1\perp}) \text{Tr} \{ U_{z_1} U_{z_2}^\dagger \} - \frac{1}{N_c} \text{Tr} \{ \tilde{Q}_1(z_{1\perp}) U_{z_2}^\dagger \} \right) + h.c | P \rangle \\
 &+ I_{5\mu\nu}^{sub\ eik}(z_{1\perp}, z_2; x, y) \langle P | \left(Q_5(z_{1\perp}) \text{Tr} \{ U_{z_1} U_{z_2}^\dagger \} - \frac{1}{N_c} \text{Tr} \{ \tilde{Q}_5(z_{1\perp}) U_{z_2}^\dagger \} \right) + h.c | P \rangle \\
 &\left. + 2 C_F I_{\mathcal{F}\mu\nu}^{sub\ eik}(z_{1\perp}, z_2; x, y) \langle P | \text{Tr} \{ \mathcal{F}_{z_1} U_{z_2}^\dagger \} + h.c | P \rangle \right]
 \end{aligned}$$

$$\not{x}_1 = \sqrt{\frac{s}{2}}\gamma^- \quad x_* = \sqrt{\frac{s}{2}}x^+$$

$$Q_1(x_\perp) = g^2 \int_{-\infty}^{+\infty} dz_* \int_{-\infty}^{z'_*} dz'_* \bar{\psi} i \not{x}_1(z_*, x_\perp) [z_*, z'_*]_x \psi(z'_*)$$

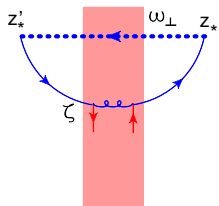
$$Q_5(x_\perp) = g^2 \int_{-\infty}^{+\infty} dz_* \int_{-\infty}^{z'_*} dz'_* \bar{\psi} \gamma^5 \not{x}_1(z_*, x_\perp) [z_*, z'_*]_x \psi(z'_*)$$

$$\tilde{Q}_1(x_\perp) = g^2 \int_{-\infty}^{+\infty} dz_* \int_{-\infty}^{z'_*} dz'_* [\infty p_1, z_*]_x \text{tr} \{ \psi(z_*, x_\perp) \bar{\psi}(z'_*, x_\perp) i \not{x}_1 \} [z'_*, -p_1 \infty]$$

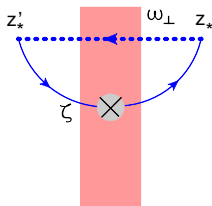
$$\tilde{Q}_5(x_\perp) = g^2 \int_{-\infty}^{+\infty} dz_* \int_{-\infty}^{z'_*} dz'_* [\infty p_1, z_*]_x \text{tr} \{ \psi(z_*, x_\perp) \bar{\psi}(z'_*, x_\perp) \gamma^5 \not{x}_1 \} [z'_*, -p_1 \infty]$$

$$\mathcal{F}(x_\perp) = \frac{ig}{2C_F} \frac{s}{2} \int_{-\infty}^{+\infty} dz_* [\infty, z_*]_x \epsilon^{ij} F_{ij}(z_*, x_\perp) [z_*, -\infty]_x$$

Sample of diagrams for the quark operator Q_1 or Q_5



a)



b)

Evolution equation flavor non-singlet case

Unpolarized case

$$\langle Q_{1\omega} \rangle = \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \frac{1}{(\omega - \zeta)_\perp^2} \left(\frac{1}{2} Q_{1\zeta} \text{Tr}\{U_\zeta U_\omega^\dagger\} - \frac{1}{2N_c} \text{Tr}\{U_\omega^\dagger \tilde{Q}_{1\zeta}\} \right)$$

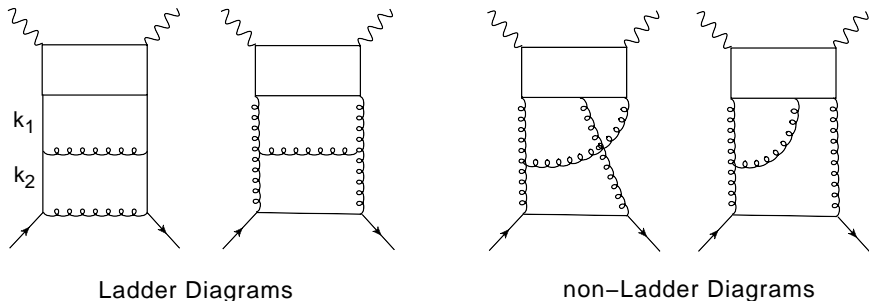
$$\begin{aligned} \langle \text{Tr}\{U_z^\dagger \tilde{Q}_{1\omega}\} \rangle &= \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \left[\frac{1}{(\omega - \zeta)_\perp^2} \left(\frac{1}{2} \text{Tr}\{U_z^\dagger U_\zeta\} Q_{1\zeta} - \frac{1}{2N_c} \text{Tr}\{U_z^\dagger \tilde{Q}_{1\zeta}\} \right) \right. \\ &\quad \left. + \frac{(\omega - z)_\perp^2}{(\omega - \zeta)_\perp^2 (\zeta - z)_\perp^2} \left(\text{Tr}\{U_\zeta U_z^\dagger\} \text{Tr}\{U_\zeta^\dagger \tilde{Q}_{1\omega}\} - \left(\frac{1}{N_c} + C_F \right) \text{Tr}\{U_z^\dagger \tilde{Q}_{1\omega}\} \right) \right] \end{aligned}$$

Polarized case

$$\langle Q_{5\omega} \rangle = \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \frac{1}{(\omega - \zeta)_\perp^2} \left(\frac{1}{2} \text{Tr}\{U_\omega^\dagger U_\zeta\} Q_{5\zeta} - \frac{1}{2N_c} \text{Tr}\{U_\omega^\dagger \tilde{Q}_{5\zeta}\} \right)$$

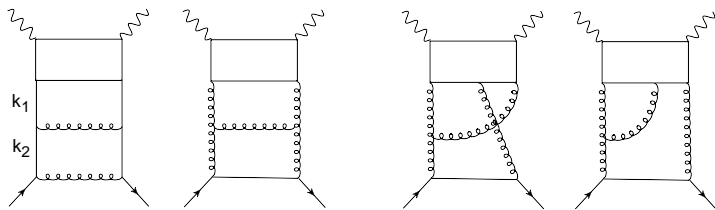
$$\begin{aligned} \langle \text{Tr}\{U_z^\dagger \tilde{Q}_{5\omega}\} \rangle &= \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \left[\frac{1}{(\omega - \zeta)_\perp^2} \left(\frac{1}{2} \text{Tr}\{U_z^\dagger U_\zeta\} Q_{5\zeta} - \frac{1}{2N_c} \text{Tr}\{U_z^\dagger \tilde{Q}_{5\zeta}\} \right) \right. \\ &\quad \left. + \frac{(\omega - z)_\perp^2}{(\omega - \zeta)_\perp^2 (\zeta - z)_\perp^2} \left(\text{Tr}\{U_\zeta U_z^\dagger\} \text{Tr}\{U_\zeta^\dagger \tilde{Q}_{5\omega}\} - \left(\frac{1}{N_c} + C_F \right) \text{Tr}\{U_z^\dagger \tilde{Q}_{5\omega}\} \right) \right] \end{aligned}$$

Infrared Evolution Equation



- Frolov, Gorshkov, Gribov, Lipatov (1966): QED
- Kirschner, Lipatov (1983)
- Bartels, Ermolaev, Ryskin-(1995-1996)

Infrared Evolution Equation



Ladder Diagrams

non-Ladder Diagrams

$$k^\mu = \alpha p_1^\mu + \beta p_2^\mu + k_\perp \quad \alpha = \sqrt{\frac{2}{s}}k^+ \quad \text{and} \quad \beta = \sqrt{\frac{2}{s}}k^-$$

$$1 \gg \alpha_1 \gg \alpha_2 \quad \text{and} \quad 1 \gg \beta_2 \gg \beta_1$$

$$\frac{k_{1\perp}^2}{\alpha_1 s} \ll \frac{k_{2\perp}^2}{\alpha_2 s} \quad \Rightarrow \quad \alpha_1 x_{1\perp}^2 \gg \alpha_2 x_{2\perp}^2$$

$$\alpha_s \int_0^{\alpha_1} \frac{d\alpha}{\alpha} \int \frac{dx_{2\perp}^2}{x_{2\perp}^2} \rightarrow \alpha_s \ln^2 \frac{\alpha_1}{\alpha_0}$$

Unpolarized case

$$\langle Q_{1\omega} \rangle = \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \frac{1}{(\omega - \zeta)_\perp^2} \left(\frac{1}{2} Q_{1\zeta} \text{Tr}\{U_\zeta U_\omega^\dagger\} - \frac{1}{2N_c} \text{Tr}\{U_\omega^\dagger \tilde{Q}_{1\zeta}\} \right)$$

$$\langle \text{Tr}\{U_z^\dagger \tilde{Q}_{1\omega}\} \rangle = \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \left[\frac{1}{(\omega - \zeta)_\perp^2} \left(\frac{1}{2} \text{Tr}\{U_z^\dagger U_\zeta\} Q_{1\zeta} - \frac{1}{2N_c} \text{Tr}\{U_z^\dagger \tilde{Q}_{1\zeta}\} \right) \right]$$

Polarized case

$$\langle Q_{5\omega} \rangle = \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \frac{1}{(\omega - \zeta)_\perp^2} \left(\frac{1}{2} \text{Tr}\{U_\omega^\dagger U_\zeta\} Q_{5\zeta} - \frac{1}{2N_c} \text{Tr}\{U_\omega^\dagger \tilde{Q}_{5\zeta}\} \right)$$

$$\langle \text{Tr}\{U_z^\dagger \tilde{Q}_{5\omega}\} \rangle = \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \left[\frac{1}{(\omega - \zeta)_\perp^2} \left(\frac{1}{2} \text{Tr}\{U_z^\dagger U_\zeta\} Q_{5\zeta} - \frac{1}{2N_c} \text{Tr}\{U_z^\dagger \tilde{Q}_{5\zeta}\} \right) \right]$$

Evolution equation flavor non-singlet case in double log approx

Unpolarized case

$$\langle Q_{1\omega} \rangle = \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \frac{1}{(\omega - \zeta)_\perp^2} \left(\frac{1}{2} Q_{1\zeta} \text{Tr}\{U_\zeta U_\omega^\dagger\} - \frac{1}{2N_c} \text{Tr}\{U_\omega^\dagger \tilde{Q}_{1\zeta}\} \right)$$

$$\langle \text{Tr}\{U_z^\dagger \tilde{Q}_{1\omega}\} \rangle = \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \left[\frac{1}{(\omega - \zeta)_\perp^2} \left(\frac{1}{2} \text{Tr}\{U_z^\dagger U_\zeta\} Q_{1\zeta} - \frac{1}{2N_c} \text{Tr}\{U_z^\dagger \tilde{Q}_{1\zeta}\} \right) \right]$$

Polarized case

$$\langle Q_{5\omega} \rangle = \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \frac{1}{(\omega - \zeta)_\perp^2} \left(\frac{1}{2} \text{Tr}\{U_\omega^\dagger U_\zeta\} Q_{5\zeta} - \frac{1}{2N_c} \text{Tr}\{U_\omega^\dagger \tilde{Q}_{5\zeta}\} \right)$$

$$\langle \text{Tr}\{U_z^\dagger \tilde{Q}_{5\omega}\} \rangle = \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \left[\frac{1}{(\omega - \zeta)_\perp^2} \left(\frac{1}{2} \text{Tr}\{U_z^\dagger U_\zeta\} Q_{5\zeta} - \frac{1}{2N_c} \text{Tr}\{U_z^\dagger \tilde{Q}_{5\zeta}\} \right) \right]$$

After linearization we recover **both, Unpolarized and Polarized BER** result in non-singlet flavor case: $\frac{N_c}{2} - \frac{1}{2N_c} = C_F$

Evolution equation flavor non-singlet case in double log approx

$$1 - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\} = \mathcal{U}_{xy} \text{ and } \Psi_{xy} = \mathcal{Q}_x - \text{Tr}\{\tilde{\mathcal{Q}}_x U_y^\dagger\}$$

$$\begin{aligned} \frac{d}{d\eta} \mathcal{Q}_{5\omega} \mathcal{U}_{\omega z} = & \frac{\alpha_s}{2\pi^2} \int d^2\zeta \left[\frac{1}{(\omega - \zeta)_\perp^2} \left(C_F \mathcal{Q}_{5\zeta} \mathcal{U}_{\omega z} + 2C_F \mathcal{F}_{\zeta\omega} - \frac{N_c}{2} \mathcal{Q}_{5\zeta\omega} \mathcal{U}_{\omega z} + \frac{1}{2N_c} \Psi_{\zeta\omega} \mathcal{U}_{\omega z} \right) \right. \\ & \left. + \frac{N_c (\omega - z)_\perp^2}{(\omega - \zeta)_\perp^2 (\zeta - z)_\perp^2} \left(\mathcal{Q}_{5\omega} \mathcal{U}_{\omega\zeta} - \mathcal{Q}_{5\omega} \mathcal{U}_{\omega z} + \mathcal{Q}_{5\omega} \mathcal{U}_{\zeta z} - \mathcal{Q}_{5\omega} \mathcal{U}_{\omega\zeta} \mathcal{U}_{1\zeta z} \right) \right] \end{aligned}$$

- Double Log resummation $\alpha_s \ln^2 s$
- UV and IR finite: Leading Log resummation: $\alpha_s \ln s$

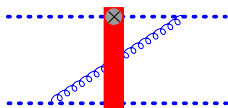
Evolution equation flavor Singlet case for g_1 structure function

$$\langle Q_{5\omega} \rangle = \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \frac{1}{(\omega - \zeta)_\perp^2} \left(\frac{1}{2} \text{Tr}\{U_\omega^\dagger U_\zeta\} Q_{5\zeta} - \frac{1}{2N_c} \text{Tr}\{U_\omega^\dagger \tilde{Q}_{5\zeta}\} + 2C_F \text{Tr}\{U_\omega^\dagger \mathcal{F}(\zeta_\perp)\} \right)$$

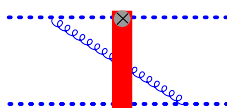
$$\begin{aligned} \langle \text{Tr}\{U_z^\dagger \tilde{Q}_{5\omega}\} \rangle = & \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \left[\frac{1}{(\omega - \zeta)_\perp^2} \left(\frac{1}{2} \text{Tr}\{U_z^\dagger U_\zeta\} Q_{5\zeta} - \frac{1}{2N_c} \text{Tr}\{U_z^\dagger \tilde{Q}_{5\zeta}\} \right. \right. \\ & \left. \left. + 2C_F \text{Tr}\{U_z^\dagger \mathcal{F}(\zeta)\} \right) + \frac{(\omega - z)_\perp^2}{(\omega - \zeta)_\perp^2 (\zeta - z)_\perp^2} \left(\text{Tr}\{U_\zeta U_z^\dagger\} \text{Tr}\{U_\zeta^\dagger \tilde{Q}_{5\omega}\} - \left(\frac{1}{N_c} + C_F \right) \text{Tr}\{U_z^\dagger \tilde{Q}_{5\omega}\} \right) \right] \end{aligned}$$

Evolution equation case: Gluon operator

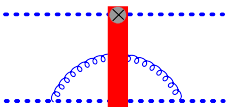
BK-type diagrams



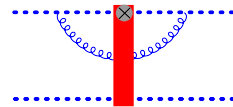
a)



b)



c)



d)

$$\langle \text{Tr}\{\mathcal{F}_\omega U_z^\dagger\} \rangle = \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \frac{(\omega - z)_\perp^2}{(\omega - \zeta)_\perp^2 (\zeta - z)_\perp^2} \\ \times \left[\text{Tr}\{U_\zeta U_z^\dagger\} \text{Tr}\{U_\zeta^\dagger \mathcal{F}_\omega\} - \left(\frac{1}{N_c} + C_F\right) \text{Tr}\{U_z^\dagger \mathcal{F}_\omega\} \right]$$

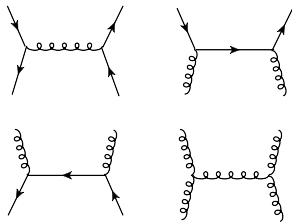
Unpolarized case

$$\begin{aligned}
 \langle \text{Tr} \{ i \frac{s}{2} \mathcal{F}_\omega U_z^\dagger \} \rangle &= \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \\
 &\times \left\{ \frac{N_f}{2} \left[\frac{(\vec{\omega} - \vec{\zeta}) \times (\vec{\zeta} - \vec{z})}{(\omega - \zeta)_\perp^2 (z - \zeta)_\perp^2} \left(\text{Tr} \{ U_z^\dagger \tilde{Q}_{1\zeta} \} \text{Tr} \{ U_\zeta^\dagger U_\omega \} - \frac{1}{N_c} \text{Tr} \{ U_\omega U_z^\dagger U_\zeta \tilde{Q}_{1\zeta}^\dagger \} \right. \right. \right. \\
 &- \frac{1}{N_c} \text{Tr} \{ U_z^\dagger U_\omega \tilde{Q}_{1\zeta}^\dagger U_\zeta \} + \frac{1}{N_c^2} \text{Tr} \{ U_z^\dagger U_\omega \} \mathcal{Q}_{1\zeta}^\dagger - \text{Tr} \{ U_\omega \tilde{Q}_{1\zeta}^\dagger \} \text{Tr} \{ U_z^\dagger U_\zeta \} + \frac{1}{N_c} \text{Tr} \{ U_\omega U_z^\dagger \tilde{Q}_{1\zeta} U_\zeta^\dagger \} \\
 &\left. \left. \left. + \frac{1}{N_c} \text{Tr} \{ U_z^\dagger U_\omega U_\zeta^\dagger \tilde{Q}_{1\zeta} \} - \frac{1}{N_c^2} \text{Tr} \{ U_z^\dagger U_\omega \} \mathcal{Q}_{1\zeta} \right) \right] \\
 &+ 2C_F \left[\frac{(\omega - \zeta, \zeta - z)}{(z - \zeta)_\perp^2 (\zeta - \omega)_\perp^2} + \frac{1}{(\omega - \zeta)_\perp^2} + 4\pi^2 \int d^2q_1 \frac{e^{i(q_1, z - \zeta)} - e^{i(q_1, \omega - \zeta)}}{q_{1\perp}^2} \delta^{(2)}(\zeta - \omega) \right] \\
 &\times \left(\text{Tr} \{ U_\omega U_\zeta^\dagger \} \text{Tr} \{ U_z^\dagger \mathcal{F}(\zeta_\perp) \} + \text{Tr} \{ U_z^\dagger U_\zeta \} \text{Tr} \{ U_\omega \mathcal{F}^\dagger(\zeta_\perp) \} \right) \Big\}
 \end{aligned}$$

Polarized case

$$\begin{aligned}
 \langle \text{Tr}\{\mathcal{F}_\omega U_z^\dagger\} \rangle &= \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \left\{ \frac{N_f}{2} \left[- \left(\frac{(\omega - \zeta, \zeta - z)}{(\omega - \zeta)_\perp^2 (z - \zeta)_\perp^2} + \frac{1}{(\omega - \zeta)_\perp^2} \right) \right. \right. \\
 &\times \left(\text{Tr}\{U_z^\dagger \tilde{Q}_{5\zeta}\} \text{Tr}\{U_\zeta^\dagger U_\omega\} - \frac{1}{N_c} \text{Tr}\{U_\omega U_z^\dagger U_\zeta \tilde{Q}_{5\zeta}^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_z^\dagger U_\omega \tilde{Q}_{5\zeta}^\dagger U_\zeta\} + \frac{1}{N_c^2} \text{Tr}\{U_z^\dagger U_\omega\} Q_{5\zeta}^\dagger \right. \\
 &\left. \left. + \text{Tr}\{U_\omega \tilde{Q}_{5\zeta}^\dagger\} \text{Tr}\{U_z^\dagger U_\zeta\} - \frac{1}{N_c} \text{Tr}\{U_\omega U_z^\dagger \tilde{Q}_{5\zeta} U_\zeta^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_z^\dagger U_\omega U_\zeta^\dagger \tilde{Q}_{5\zeta}\} + \frac{1}{N_c^2} \text{Tr}\{U_z^\dagger U_\omega\} Q_{5\zeta} \right) \right. \\
 &\left. + 2C_F \left[\frac{(\omega - \zeta, \zeta - z)}{(z - \zeta)_\perp^2 (\zeta - \omega)_\perp^2} + \frac{1}{(\omega - \zeta)_\perp^2} + 4\pi^2 \int d^2 q_1 \frac{e^{i(q_1, z - \zeta)} - e^{i(q_1, \omega - \zeta)}}{q_{1\perp}^2} \delta^{(2)}(\zeta - \omega) \right] \right. \\
 &\left. \times \left(\text{Tr}\{U_\omega U_\zeta^\dagger\} \text{Tr}\{U_z^\dagger \mathcal{F}(\zeta_\perp)\} + \text{Tr}\{U_z^\dagger U_\zeta\} \text{Tr}\{U_\omega \mathcal{F}^\dagger(\zeta_\perp)\} \right) \right\}
 \end{aligned}$$

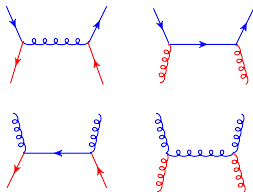
BER formalism



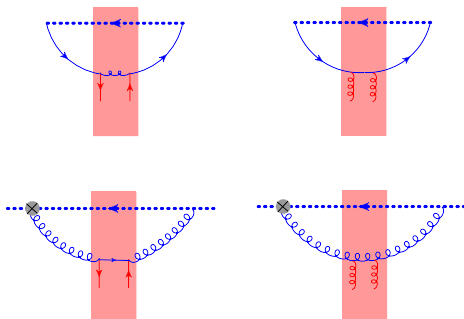
$$\begin{pmatrix} P_{q \leftarrow q} & P_{g \leftarrow q} \\ P_{q \leftarrow g} & P_{g \leftarrow g} \end{pmatrix} \longleftrightarrow \begin{pmatrix} C_F & 2C_F \\ -N_f & 4N_c \end{pmatrix}$$

Ladder Diagrams in Singlet case

BER formalism



Background-shock-wave formalism



$$\begin{pmatrix} C_F & 2C_F \\ -N_f & 4N_c \end{pmatrix}$$

$$\langle Q_{5\omega} \mathcal{U}_{\omega z} \rangle = \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \frac{1}{(\omega - \zeta)_\perp^2} \left(C_F Q_{5\zeta} \mathcal{U}_{\omega z} + 2C_F \mathcal{F}_{\zeta\omega} \right)$$

$$\langle \mathcal{F}_{\omega z} \rangle = \frac{\alpha_s}{2\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2\zeta \frac{1}{(\omega - \zeta)_\perp^2} \left[-N_f Q_{5\zeta} + 4N_c (\mathcal{F}_{\zeta z} + \mathcal{F}_{\zeta\omega}) \right]$$

- After linearization we find agreement with BER with the Ladder type of diagrams
- Refine linearization procedure to avoid UV divergences

- High-energy Quark and Gluon propagator with sub-eikonal corrections has been presented.
- BK-type of evolution equation for Unpolarized flavor-non-singlet F_2 structure function has been presented.
- BK-type of evolution equations for Polarized flavor-non-singlet q_1 structure function has been presented and agrees with BER result.
- BK-type of evolution equations for Polarized flavor-singlet q_1 structure function has been presented and agree with BER result for the ladder type of diagrams.

- Refine the linearization procedure.
- Solve the coupled evolution equations in the Double Log Approximation for the flavor-singlet case and compare with the non-ladder BER diagrams.
- Orbital Angular Momentum and spin-TMDs at small- x .