Is there a lower bound for shear viscosity to entropy density ratio?

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A. Jakovac, arXiv:0901.2802, arXiv:0911.3248

Outlines

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1. Introduction
   - Transport in plasma
   - Lower bound for η/s
2. eta/s in field theory
   - Generic formulae
   - Transport coefficient
   - Entropy density
3. Quasiparticle systems
   - Small width case
   - Wave function renormalization
   - High temperature effects
   - System with zero mass excitations
4. Conclusions

Outlines

Inhomogeneous distribution of conserved charge density N

$\Rightarrow$ way to equilibrium

- if elementary excitations are weakly interacting
  - independent smoothing of charge excess at each point
  - homogenization before equilibration
  - ballistic regime (gases)
- strongly interacting systems
  - induced currents depend on the local environment
    $\Rightarrow$ linear response theory $J = -D\nabla N \quad \Rightarrow \quad \dot{N} = D \triangle N$
  - equilibration before homogenization
  - diffusive regime (fluids)

Diffusion constant

The order of magnitude of the diffusion constant D:

(τ lifetime, ℓ mean free path, v velocity, σ cross section)

- $\dot{N} = D \triangle N \quad \Rightarrow \quad \frac{\delta N}{\tau} = D \frac{\delta N}{\ell^2} \quad \Rightarrow \quad D \sim \ell v \sim v^2 \tau \sim \frac{v}{n\sigma}$
- D is large in weakly, small in strongly interacting systems
- description sensible only in strongly interacting (fluid) systems

Determination of D

- in QFT from linear response $C(x) = \langle [J_i(x), J_i(0)] \rangle$

$$\Rightarrow \quad D = \lim_{\omega \to 0} \frac{C(\mathbf{k} = 0, \omega)}{\omega}$$

- Boltzmann equations

Viscosity

- transport coefficient in momentum transport $\Rightarrow$ viscosity
- $\varrho\dot{v} \sim \eta \triangle v$ (Navier-Stokes) $\Rightarrow$ $\eta \sim \varrho v^2 \tau \sim \epsilon\tau$ (ϵ energy density)
- damping rate of small perturbations: $\Gamma = \frac{4k^2}{3T}\frac{\eta}{s}$.

Typical values of η/s:

- water at room temperature ~ 30
- superfluid ^{4}He at λ-point ~ 0.8
- smallest at the phase transition point

$\Downarrow$

what is the smallest value for η/s?

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- argumentation: $\eta \sim \epsilon\tau,\ s \sim n \quad \Rightarrow \quad \frac{\eta}{s} \sim E\tau \gtrsim \hbar$

P. Danielewicz, M. Gyulassy, PRD 31, 53 (1985); P. Kovtun, D.T. Son, A.O. Starinets PRL 94, 111601 (2005).

- calculation: for $\mathcal{N}=4$ SYM theory at $N_c \gg 1$, $\lambda = g^2 N_c \gg 1$ from graviton absorbtion in the dual 5D AdS gravity:

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

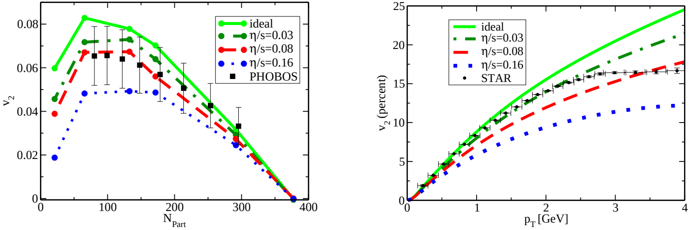
(P. Kovtun, D.T. Son, A.O. Starinets JHEP 0310, (2003) 064.)

- for weaker coupling we expect larger ratio: indeed, first λ, N_c corrections are positive (R.C. Myers, M.F. Paulos, A. Sinha, arXiv:0806.2156)
- universal for a wide class of theories (A. Buchel, R.C. Myers, M.F. Paulos, A. Sinha, Phys.Lett.B669:364-370,2008.; M. Haack, A. Yarom, arXiv:0811.1794)
- so far we did not find counterexamples experimentally

$\Rightarrow$ commonly accepted lower bound for η/s

RHIC data

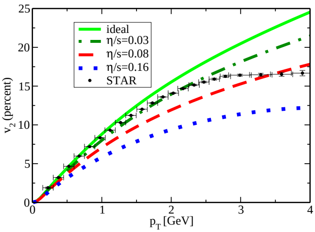
Non-central heavy ion collisions have initial anisotropy.
Time evolution of anisotropy: the larger the viscosity, the more extent the initial anisotropy is washed out



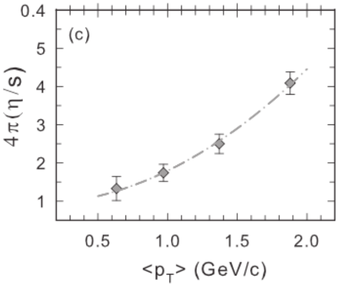
(P. Romatschke, U. Romatschke, Phys.Rev.Lett.99:172301,2007.)

upper bound: $\frac{\eta}{s} \lesssim 0.16 \quad \Rightarrow$ is there a lower bound?

RHIC data: is there a lower bound?



$\Rightarrow$



P. Romatschke, U. Romatschke, PRL.99:172301,2007.

R.A. Lacey, A. Taranenko, R. Wei, arXiv:0905.4368

Statistically $\dfrac{\eta}{s} < \dfrac{1}{4\pi}$ is not excluded (favored?)

$4\pi\eta/s = 1 \quad \Rightarrow \quad$ infinite coupling, N_c!

RHIC seriously challenges the lower bound!

failed?

Theoretical caveats

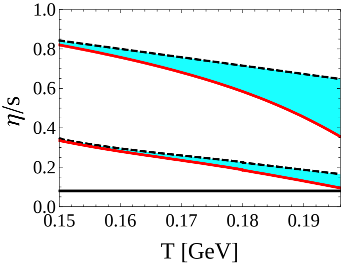
- $\mathcal{N} = 4$ SYM theory is not QCD
- higher curvature and dilaton corrections to 5D AdS actions
  $\Rightarrow \quad \frac{\eta}{s} < \frac{1}{4\pi}$ is conceivable   dual theory? unitarity?
- Counterexample: N non-interacting species $\Rightarrow$ η is not changed, $s \sim \ln N$ (mixing entropy) $\Rightarrow \frac{\eta}{s} \sim \frac{1}{\ln N}$
  (A. Cherman, T. D. Cohen, and P. M. Hohler, JHEP 02, 026 (2008), 0708.4201.)
  metastable system? what is the case with limited N?

Theoretical caveats

- $E\tau \gtrsim \hbar$? In fact $\Delta E\tau \gtrsim \hbar$! First is true only if $E > \Delta E$
  Condition for (small width) quasiparticle system

$\Rightarrow$ the argumentation is applicable only in quasiparticle systems

- in QFT there is always a continuum – effect on η/s?



pure hadron gas vs. hadron gas with continuum $\Rightarrow$ considerable difference

(J. Noronha-Hostler, J. Noronha and C. Greiner, PRL 103, 172302 (2009), 0811.1571)

We need exact statements about $\frac{\eta}{s}$ from first principles!

Result: there is a non-universal lower bound at finite entropy density; for smal s:

$$\left.\frac{\eta}{s}\right|_{min} \sim \frac{s}{N_Q L T^4},$$

where

- N_Q: number of relevant quantum channels (species)
- L: interaction range

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Continuous spectrum: energy levels are not discrete

$\Rightarrow$ represented by a spectral function (density of states, DoS):

$$\sum_n |n\rangle\langle n| = V\sum_{\mathcal{Q}}\int\frac{d^4p}{(2\pi)^4}\,\varrho_{\mathcal{Q}}(p)\,|p,\mathcal{Q}\rangle\langle p,\mathcal{Q}| \equiv \int_Q |p,\mathcal{Q}\rangle\langle p,\mathcal{Q}|\,,$$

where $\mathcal{Q}$ denotes conserved quantities (quantum channel), $p=(p_0,\mathbf{p})$ is the total energy-momentum of the state.

- QM-based description
  use volume normalization to calculate densities
- effects of interaction
  - continuous DoS
  - temperature dependent DoS

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$C_J(x) = \langle [J_i(x), J_i(0)] \rangle \quad \Rightarrow \quad \eta_J = \lim_{\omega\to 0} C_J(\omega)/\omega$ generic transport coefficient.

- insert complete basis of energy-momentum eigenstates

$$C_J(x) \; = \frac{1}{Z}\sum_{n,m}\left[\langle n\,|e^{-\beta H}J(x)|\,m\rangle\,\langle m\,|J(0)|\,n\rangle - \{x\leftrightarrow 0\}\right]$$

- translation:
  $J(x) = e^{iPx}A(0)e^{-iPx} \quad \Rightarrow \quad \langle n\,|J(x)|\,m\rangle = e^{i(P_n-P_m)x}\,\langle n\,|J(0)|\,m\rangle$
- Fourier transformation, $p = (p_0, \mathbf{p})$

$$C_J(p) \; = \frac{1}{Z}\sum_{n,m}\left(e^{-\beta E_n} - e^{-\beta E_m}\right)(2\pi)^4\delta(p+P_n-P_m)|\,\langle m\,|J(0)|\,n\rangle\,|^2$$

- introduce spectral densities

$$\eta_J = \beta \frac{V^2}{\mathcal{Z}} \sum_{\mathcal{Q}} \int \frac{d^4k}{(2\pi)^4}\, \varrho_{\mathcal{Q}}^2(k)\, e^{-\beta k_0} |\,\langle k,\mathcal{Q}|J_i|k,\mathcal{Q}\rangle\,|^2.$$

- current matrix element: $\langle k,\mathcal{Q}|J_i|k,\mathcal{Q}\rangle = \mathcal{J}_{\mathcal{Q}}(k)\dfrac{k_i}{Vk_0}$
  - $\sim k_i/k_0 \sim v_i$ since J_i is a current
  - in free case $\mathcal{J}_{\mathcal{Q}}(k)$ is the charge carried by the current: for electric current $\mathcal{J} = e$ charge, for viscosity $\mathcal{J} \sim k_j$ momentum
  - in nonperturbative case $\mathcal{J}_{\mathcal{Q}}(k)$ can be momentum dependent.
- angular averaging

Finally:

$$\eta_J = \beta \frac{1}{3\mathcal{Z}} \sum_{\mathcal{Q}} \int \frac{d^4k}{(2\pi)^4}\, \frac{\mathbf{k}^2}{k_0^2}\, e^{-\beta k_0} \left(\mathcal{J}_{\mathcal{Q}}(k)\varrho_{\mathcal{Q}}(k)\right)^2.$$

Outlines

free energy density $\mathcal{Z} = e^{-\beta F} = \mathrm{Tr}\, e^{-\beta H} = V \sum_{\mathcal{K}} \int \frac{d^4 k}{(2\pi)^4} \varrho_{\mathcal{K}}(k) e^{-\beta k_0}$

Volume dependence of the free energy:

- for small sizes it is arbitrary
- as $V \to \infty$ we recover linear volume dependence
- crossover at scale L
  - volume elements larger than L interact weakly (surface interaction)
  - coarse graining scale $\Rightarrow$ free energy density cen be defined at scales larger than L
  - L is also an effective IR cutoff for the interactions.

$\Rightarrow$ we choose a volume $V = L^3$ to define free energy density:

$$f = -\frac{T}{L^3} \ln \left(1 + L^3 \sum_{\mathcal{K}} \int \frac{d^4 k}{(2\pi)^4} \varrho_{\mathcal{K}}(k)\, e^{-\beta k_0} \right).$$

The eta/s ratio

with $s = -\dfrac{\partial f}{\partial T}$ and with angular averaging

$$\frac{\eta}{s} = \frac{\dfrac{\beta}{3\mathcal{Z}} \sum_{\mathcal{K}} \int \dfrac{d^4k}{(2\pi)^4} \, \dfrac{\mathbf{k}^2}{k_0^2} \, e^{-\beta k_0} \left(\mathcal{J}_{\mathcal{K}}(k) \varrho_{\mathcal{K}}(k)\right)^2}{-\dfrac{\partial}{\partial T} \dfrac{T}{L^3} \ln\left(1 + L^3 \sum_{\mathcal{K}} \int \dfrac{d^4k}{(2\pi)^4} \, \varrho_{\mathcal{K}}(k) \, e^{-\beta k_0}\right)}.$$

Is there a lower bound in this formula?

Generic structure

Structure of η/s for small ϱ is

$$\frac{\eta_J}{s} \sim \frac{\int f_1 \varrho^2}{\int f_2 \varrho} \xrightarrow{rescaling} \frac{\langle \varrho^2 \rangle}{\langle \varrho \rangle}.$$

$\langle \varrho^2 \rangle \geq \langle \varrho \rangle^2 \quad \Rightarrow \quad$ we expect $\eta \gtrsim s^2$ up to rescaling factors.

Quasiparticle vs. non-quasiparticle systems:

- large peak in $\varrho \quad \Rightarrow \quad \varrho^2$ even larger $\quad \Rightarrow \quad \eta/s$ large
- ϱ small everywhere $\quad \Rightarrow \quad \varrho^2$ even smaller $\quad \Rightarrow \quad \eta/s$ small

in non-quasiparticle systems η/s is naturally small!

Lower bound – mathematical approach

More exactly:

- need a sum rule in each energy channel $\int \frac{dk_0}{2\pi}\, \varrho_Q(k_0) = U_Q$
- minimize η by tuning ϱ with respecting the sum rules and keeping the entropy density constant.
- technically: Lagrange multiplicators
- two cases analytical: small/large s. The minimum values:

$$\left.\frac{\eta}{s}\right|_{min} \sim \frac{\mathcal{F}(L^3 s)}{N_Q (LT)^4}, \qquad \mathcal{F} = \begin{cases} x & \text{for small } x \\ e^x/x & \text{for large } x \end{cases}$$

N_Q: number of effective quantum channels (species).

$\Rightarrow$ There is a lower bound at finite s, but it is not universal.

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Simplifications

For the concrete model calculations we use simplifications

- We use generalized quasiparticle systems:

$$f = T\int \frac{d^4k}{(2\pi)^4}\ \varrho_{QP}(k)\,(\mp)\ln\left(1 \pm e^{-\beta k_0}\right).$$

- we omit the effect of $\mathcal{J}_Q$ and define a "reduced" viscosity coefficient as

$$\bar{\eta} = \frac{\beta}{15}\int \frac{d^4k}{(2\pi)^4}\ \frac{(\mathbf{k}^2)^2}{k_0^2}\, e^{-\beta k_0}\ \varrho_{\mathcal{K}}^2(k).$$

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Assume that the lowest lying states can be approximated with Breit-Wigner form:

$$\varrho(q) = \frac{2\Gamma}{(q_0 - \varepsilon_q)^2 + \Gamma^2}.$$

In the small width limit $\varrho(q)^2 \approx \frac{2}{\Gamma} 2\pi\delta(q_0 - \varepsilon_q)$. We find:

$$\frac{\bar{\eta}_{QP}}{s_{QP}} = \begin{cases} \dfrac{540}{\mathcal{A}_\pm \pi^4} \dfrac{T}{\Gamma}, & \text{if } \varepsilon_k = k \quad \mathcal{A}_\pm = (1, 8/7) \\ 30\pi \dfrac{T^2}{\Gamma m}, & \text{if } \varepsilon_k = m + \frac{k^2}{2m} \\ 16\pi \dfrac{m^2}{\Gamma T}, & \text{if } \varepsilon_k = \frac{k^2}{2m} \end{cases}.$$

The value of Γ

- in conformal case $\Gamma \sim T \quad \Rightarrow \quad \frac{\eta_J}{s} \sim \text{constant}$
  lower limit may come from infinte coupling, $1/4\pi$.
- massive case at low temperature: the width is the consequence of scattering on thermal particles
  $\Rightarrow$ abundance is $e^{-M/T}$ (M: energy of scattering state)

$$\Rightarrow \quad \frac{\eta_J}{s} \sim Te^{M/T} \to \infty.$$

$\Rightarrow$ in the quasiparticle case the conformal case represents a lower bound

Off-shell effects

We consider three examples of off-shell effects which can appear in physical systems

- wave function renormalization
- large width
- low temperature non-quasiparticle systems

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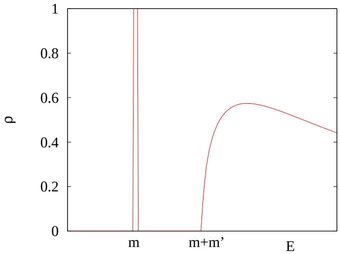
- In the real case the quasiparticle peak never stands alone, there is always a continuum at higher energies.



- But the *complete* spectral function must obey the sum rule, like

$$1 = \int \frac{dk_0}{2\pi}\,\varrho(k_0) = \mathrm{peak} + \mathrm{continuum}.$$

Therefore the peak must be reduced $\varrho_{QP} \to Z\varrho_{QP}$.

- At small temperatures the continuum contribution for η/s is much smaller than the QP contribution $\Rightarrow$

$$\frac{\eta}{s} \longrightarrow Z\frac{\eta}{s}$$

$\Rightarrow$ wave function renormalization directly reduces the η/s ratio.

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At high temperatures the quasiparticle peak melts into the continuum, forming a broad spectral function $\Rightarrow$ the width can be larger than the peak position.
A simplified example

$$\varrho(k_0, k) = \frac{2\pi}{E_2 - E_1} \Theta(E_1 < k_0 < E_2)$$

step function, where $E_{1,2}(k) = \sqrt{k^2 + m_{1,2}^2}$. At small temperatures $(T < m_1)$

$$\frac{\eta}{s} = 6\pi \frac{T}{\Delta m}$$

$\Rightarrow$ by broadening the distribution the viscosity to entropy density ratio has no lower bound

Realistic example

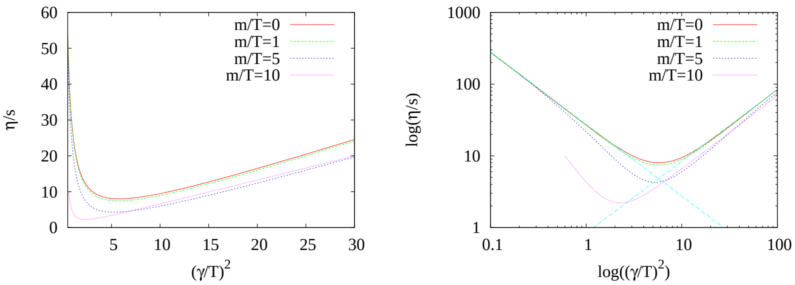
Self-energy corrections to a free particle leads to a Breit-Wigner-like form

$$\varrho = \frac{-2k_0 \,\mathrm{Im}\,\Sigma}{(k^2 - m^2 - \,\mathrm{Re}\,\Sigma)^2 + \,\mathrm{Im}\,\Sigma^2}$$

If we neglect the real part of the self energy we arrive at

$$\varrho_{BW}(k) = \frac{1}{\mathcal{N}} \frac{4\gamma_k^2 k_0}{(k_0^2 - E_k^2)^2 + \gamma_k^4}.$$

$\mathcal{N} \sim 1$ normalization factor



- for small γ/T the quasiparticle picture holds
- for large γ/T the $\eta/s \sim \gamma^2/T^2$
- turnover depends on the mass, for $m \to \infty$ the $\sim \gamma^2$ behaviour until zero!

$\Rightarrow$ in realistic cases $m \sim T$ find a lower bound at high temperature near $\gamma \sim T$.

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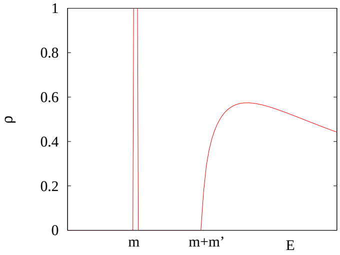
Non-quasiparticle excitations in real models

Simplest interacting field theories at zero temperature has a spectral function (DoS) like

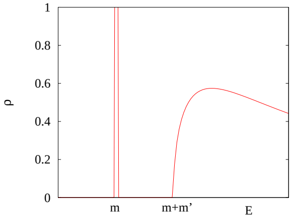


- the Dirac-delta at m represents a stable particle
- the continuum at $m+m'$ represents a multiparticle state.
- infinite lifetime $\Rightarrow$ gas

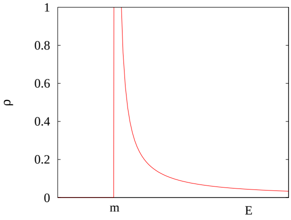
What could spoil this simple picture?

Zero mass excitations

- If there are in the system zero mass particles, then $m' = 0$
  $\Rightarrow$ cut and the delta-peak melt together
- 1-loop threshold behavior linear $\Rightarrow$ $\varrho \sim 1/(E - E_{\rm thr})$



$\longrightarrow$



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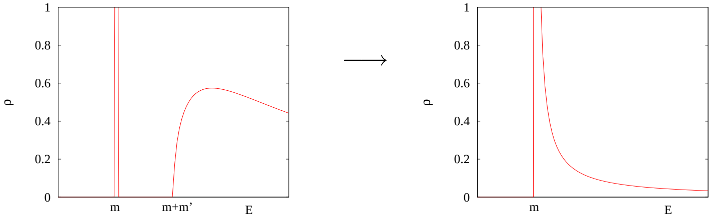
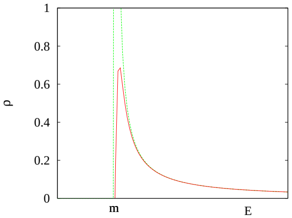


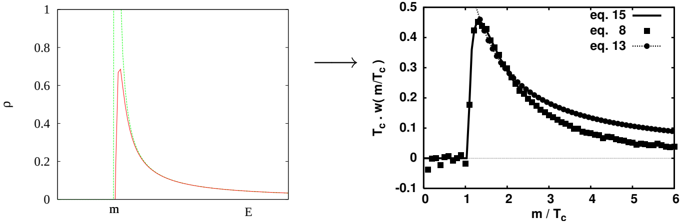
- This is not normalizable! $\Rightarrow$ IR divergences near the threshold, which smear out the $1/x$ singularity

Zero mass excitations



- Interpretation: no single charged particle (electron, quark), it is always surrounded by soft gauge bosons

Zero mass excitations



- Interpretation: no single charged particle (electron, quark), it is always surrounded by soft gauge bosons
- In QCD: from fitting to MC pressure data one obtains similar distribution of quasiparticle masses

(T.S.Biro, P.Levai, P.Van, J.Zimanyi, Phys.Rev.C75:034910,2007)

The η/s ratio at low temperatures

$\varrho^{\#} e^{-\beta q_0}$ enhances the lowest lying states $\Rightarrow$ power expand near the threshold:

$$\varrho(q) = \mathcal{C} q_0 \Theta(q - M)(q^2 - M^2)^w .$$

C is dimensionfull: $[\mathcal{C}] = [E]^{-2(1+w)}$
$\eta_J \sim \mathcal{C}^2$ and $f \sim \mathcal{C}$ $\Rightarrow$ $\mathcal{C}$ remains in the ratio.
In the massive and massless case we find

$$\frac{\eta_J}{s} \sim \mathcal{C} M^w T^{2+w} \quad \text{and} \quad \mathcal{C} T^{2(1+w)} \xrightarrow{T\to 0} \quad 0$$

for an integrable threshold $(w > -1)$

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Conclusions

- $1/4\pi$ lower bound for η/s is true only for quasiparticle and conformal theories
- in general the lower bound in a given environment depend on several factors;for small s

$$\frac{\eta}{s} \gtrsim \frac{s}{N_Q L T^4}$$

- there exist several models where $\eta/s < 1/4\pi$ is possible
  - quasiparticle with multiparticle continuum $\Rightarrow$ wave function renormalization
  - finite temperature, broad spectral function
  - low temperature systems with zero mass excitation $\Rightarrow$ there even $\eta/s =$ is conceivable
- in QCD any of these effects can be important

Hydrodynamics

System with local collective flow: QM description?
Construction of the system: at $t = -\infty$ equilibrium in rest ($\bar{u} = (1, 0, 0, 0)$), then a modified time evolution corresponding to the flow u – denote $\Delta u = u - \bar{u}$:

$$H^{(0)} = \bar{u}_\mu P^{(0)\mu} = u_\mu P^\mu = H + \Delta u_\mu T^{0\mu} \quad \Rightarrow \quad \delta H = -\int d^3x\, \Delta u_\mu T^{0\mu}$$

$$\delta H = \int_{-\infty}^{t} dt'\, \partial_0 \delta H = -\int_{-\infty}^{t} dt' d^3x\, \left[\partial_0 u_\mu T^{0\mu} + \Delta u_\mu \partial_0 T^{0\mu}\right]$$

With energy-momentum conservation $\partial_0 T^{0\mu} = -\partial_i T^{i\mu}$ and partial integration

$$\delta H = -\int_{-\infty}^{t} dt' d^3\mathbf{x}\, \partial_\mu u_\nu T^{\mu\nu}$$

- Linear response theory ($\rightarrow$ the flowing and the original systems are not too far $\Rightarrow$ nonrelativistic):

$$\delta \langle X(t) \rangle = i \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' d^3\mathbf{x}'' \, \langle [X(t), T^{\mu\nu}(x'')] \rangle \, \partial_\mu u_\nu .$$

- Hydrodynamical approximation: $\partial u \approx$ const. $\Rightarrow$ $\delta \langle X \rangle$ time independent.
- spatial rotational symmmetry of the ground state $\Rightarrow$ $\delta \langle \pi_{ij} \rangle \equiv \delta \left\langle T_{ij} - \frac{1}{3}\delta_{ij} T^{k}_{.k} \right\rangle = \frac{\eta}{2} \left[\partial_k v_\ell + \partial_\ell v_k - \frac{2}{3}\delta_{k\ell}\partial v \right]$,

the coefficient from above (denote $C(x) = \langle [T_{12}(0), T_{12}(x)] \rangle$

$$\eta = i \int_{-\infty}^{0} dt \int_{-\infty}^{t} dt' d^3\mathbf{x}' \, C(x') = \lim_{\omega \to 0} \frac{C(\omega, \mathbf{k} = 0)}{\omega}$$

Kubo formula

Zero temperature limit of viscosity

The viscosity η and the entropy have a common form

$$F_{n,m} = \frac{3a^5}{2\pi^3 T} \int \frac{d^4k}{(2\pi)^4} \Theta(k_0)\Theta(k^2-\sigma^2) e^{-k_0/T} (ak_0)^n \varrho^m(k),$$

since

$$\eta = N^2\Delta^2 F_{0,2}, \qquad s = 2F_{1,1}.$$

After reducing the integrals

$$a^3 F_{n,m} = C(a\sigma)^n (a^2\sigma T)^{3/2} e^{-\sigma/T} \int_0^\infty dz\, e^{-z} \left(\frac{2(wz)^{5/2}}{1+(wz)^5}\right)^m,$$

where $w = 2\Delta^{-2/5} T/\sigma$ rescaled temperature.
BOTH η and s goes to zero at zero temperature