Self-similar solution of a telegraph-type equation for heat propagation

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Outline

- **Motivation** *(infinite propagation speed with the diffusion/heat equation)*
- **A way-out** *(Cattaneo equ., different derivation of the telegraph eq. & properties*
- **Derivation of a self-similar telegraph- type equation**
- **Two different kind of solutions & properties**

Ordinary diffusion/heat conduction equation

$$\mathbf{q} = -k\nabla U(x,t), \quad \nabla \mathbf{q} = -\gamma \frac{\partial U(x,t)}{\partial t}$$

U(x,t) temperature distribution

urier law + conservation law

- $$\begin{cases} u_t(x,t) - k u_{xx}(x,t) = 0 & -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,t=0) = \delta(x) \end{cases}$$

  lic PDA, no time-reversal sym.

- *strong maximum principle ~ solution is smeared out in time*

$$\Phi(x,t) = \int \frac{1}{\sqrt{4\pi k t}} exp\left(-\frac{x^2}{4kt}\right)$$

- *the fundamental solution:*
- *general solution is:*

$$u(x,t) = \int \Phi(x-y,t) g(y) dy$$

$$u(x,0) = g(x) \ \ for -\infty < x < \infty \ \ and \ \ 0 < t < \infty$$

- *kernel is non compact = inf. prop. speed*
- *Problem from a long time ☹*
- *But have self-similar solution ☺*

$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

The wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad u(x,0) = g(x), \quad \frac{\partial u(x,0)}{\partial t} = h(x)$$

- *hyperbolic PDA with finite wave propagation speed, time reversal symmetry*
- *the general d'Alambert solution is*

$$u(x,t) = \frac{1}{2}\left[g(x-ct) + g(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\, d\xi$$

which is a sum of two travelling waves

Different derivations of the telegraph equ.

- *Electrodynamics*
- *Random walk for diffusion*
- *Ad-hoc derivation from hydrodynamics for particle diffusion*
- *Cattaneo heat conduction equ.*

Electrodynamics

- *Taking the Maxwell Equation for a medium*

$$\mathbf{D} = \epsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}, \qquad \mathbf{j} = \sigma \mathbf{E}$$

$$\rho = 0$$

$$\text{curl}\,\mathbf{H} = \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi\sigma}{c}\mathbf{E} \qquad (16.2)$$

$$\text{curl}\,\mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad (16.3)$$

$$\text{div}\,\mathbf{H} = 0 \qquad (16.4)$$

$$\text{div}\,\mathbf{E} = 0 \qquad (16.5)$$

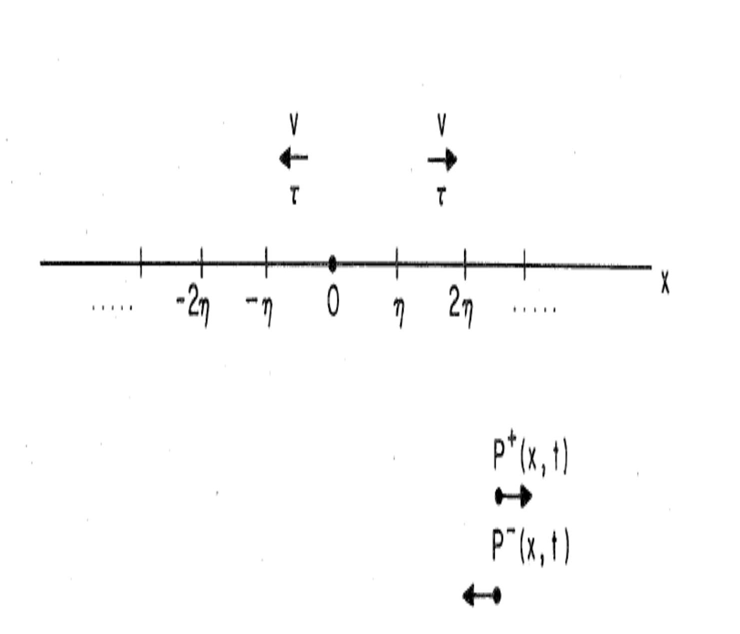
- *Curl Eq 16.2 and plug into 16.3 use 16.4*

$$\Delta \mathbf{H} = \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{H} + \frac{4\pi\mu\sigma}{c^2}\frac{\partial}{\partial t}\mathbf{H}$$

we have the telegraph eq.

other derivation is possible for two realistic wires with R,L,C

Random walk for diffusion



deriv. is only for 1 dim, P probability

$$P(x,t) = P^{+}(x,t) + P^{-}(x,t)$$

$$P^{+}(x,t) = a\, P^{+}(x-\eta,\ t-\tau) + b\, P^{-}(x-\eta,\ t-\tau)$$

$$P^{-}(x,t) = a\, P^{-}(x+\eta,\ t-\tau) + b\, P^{+}(x+\eta,\ t-\tau)$$

$a = b = 1/2$ **symmetric prob to right and left**

$$P\,(x,t) = \frac{1}{2} P(x+\eta,\ t-\tau) + \frac{1}{2} P(x-\eta,\ t-\tau)$$

Expansion of the right-hand side in the powers of η and τ yields

$$P\,(x,t) = \frac{1}{2}\Big[\ P(x,t) + \eta \frac{\partial P}{\partial x} - \tau \frac{\partial P}{\partial t} + \frac{1}{2}\eta^2 \frac{\partial^2 P}{\partial x^2} - \eta\tau \frac{\partial^2 P}{\partial x \partial t} + \frac{1}{2}\tau^2 \frac{\partial^2 P}{\partial t^2} +$$

$$P(x,t) - \eta \frac{\partial P}{\partial x} - \tau \frac{\partial P}{\partial t} + \frac{1}{2}\eta^2 \frac{\partial^2 P}{\partial x^2} + \eta\tau \frac{\partial^2 P}{\partial x \partial t} + \frac{1}{2}\tau^2 \frac{\partial^2 P}{\partial t^2}\Big] + 0\{(\eta+\tau)^3\}$$

or

$$\frac{\partial P}{\partial t} = \frac{1}{2}\frac{\eta^2}{\tau}\frac{\partial^2 P}{\partial x^2} + \frac{1}{2}\tau \frac{\partial^2 P}{\partial t^2} + 0\Big\{\frac{(\eta+\tau)^3}{\tau}\Big\} \quad .$$

Ad hoc derivation from a basic set of hydrodynamical equations

$$\frac{\partial u_i}{\partial x_i} = 0$$

stationary continuity eq.

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\rho^{-1}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i + g_i$$

Navier-Stokes eq.

An instantaneous equation for conservative concentration, S, is written as

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x_i}(u_i S) = D\nabla^2 S \quad ,$$

where D is the molecular diffusivity.

where $(u_i S)$ is the turbulent flux

Idea no direct study of N-S but the diffusion

Reynolds decomposition average + fluct.

$$u_i = \overline{u}_i + u'_i$$

$$p = \overline{p} + p'$$

$$g_i = \overline{g}_i + g'_i$$

$$S = \overline{S} + S'$$

just plug them back

Ad hoc derivation from a basic set of hydrodinamical equations

After some tedious algebra, and with neglecting terms we get

$$\frac{\partial^2 \overline{S}}{\partial t^2} + \tau^{-1} \frac{\partial \overline{S}}{\partial t} = \frac{\partial}{\partial x_i} \left(\overline{u'_i u'_j} \frac{\partial \overline{S}}{\partial x_j} \right) .$$

For stationary and isotropic turbulence,

$$\overline{u'_i u'_j} = \overline{u'^2} \delta_{ij} \equiv w^2 \delta_{ij}$$

Therefore,

$$\frac{\partial^2}{\partial t^2} \overline{S} + \tau^{-1} \frac{\partial \overline{S}}{\partial t} = w^2 \nabla^2 \overline{S} .$$

This is the telegraph equation in three-dimensional space.

the gradient is equal with a time scale times the turb. flux

$$\frac{\partial}{\partial x_j} \overline{u'_i u'_j S'} \equiv \tau^{-1} \overline{u'_i S'} \quad (\tau \geqq 0)$$

the time average of the velocity tensor is a scalar

Cattaneo heat conduction equ.

$$\tau\frac{\partial \mathbf{q}}{\partial t}+\mathbf{q}=-k\nabla T(x,t)$$

Cattaneo heat conduction law

$$\nabla \mathbf{q}=-\gamma\frac{\partial T(x,t)}{\partial t}$$

Energy conservation law

$$\frac{\partial^2 T(x,t)}{\partial t^2}+\frac{1}{\tau}\frac{\partial T(x,t)}{\partial t}=c^2\nabla^2 T(x,t)$$

T(x,t) temperature distribution
q heat flux
k effective heat conductivity
γ heat capacity
τ relaxation time

$c=\sqrt{k/\tau\gamma}$ is the sound of the transmitted heat wave.

General properties of the solution

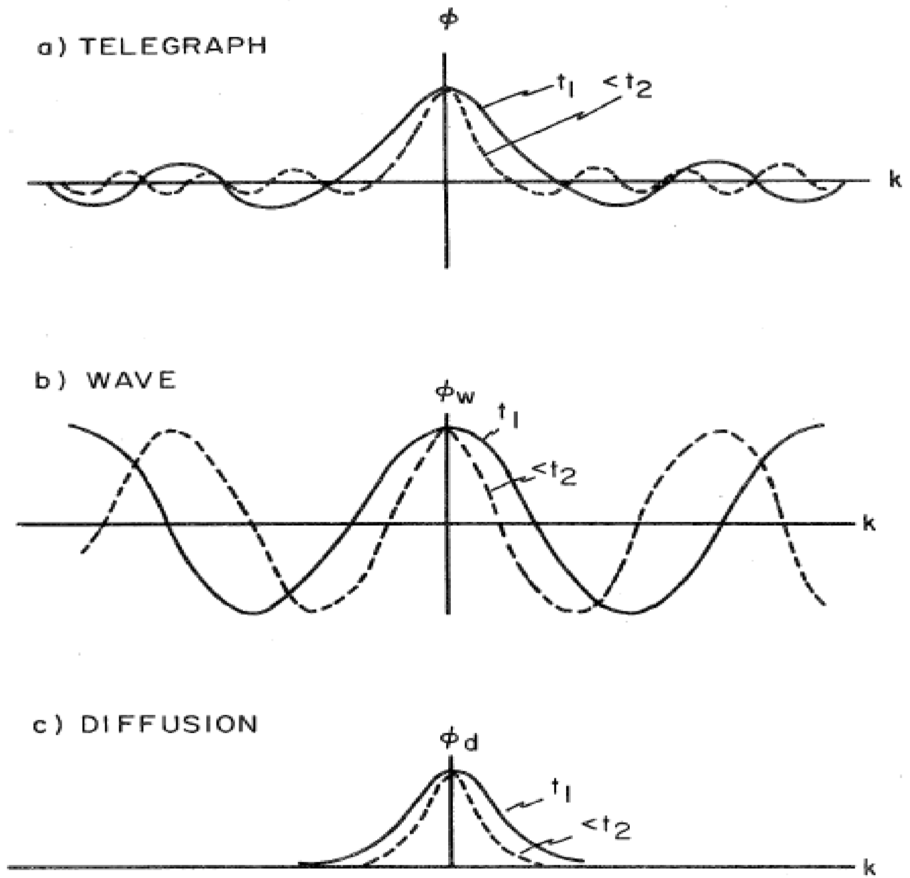


Figure 1. Behaviors in wavenumber space
a) Telegraph equation
b) Wave equation
c) Diffusion equation

decaying travelling waves

$$T(x,t) \propto e^{-\lambda t} f(x-ct)$$

$$T(x,t) = e^{-\lambda t} I_0 \left(\frac{\lambda}{2c} \sqrt{(c^2t^2 - x^2)} \right)$$

Bessel function

Problem:

1) no self-similar diffusive solutions $T(x,t) = t^{-\alpha} f(\eta)$ $\eta = \frac{x}{t^\beta}$

2) oscillations, T<0 ?

Our telegraph-like equation and self-similar solution

Modifying the Cattaneo law and keeping in mind the Ansatz

$$T(x,t) = t^{-\alpha} f(\eta) \quad with \quad \eta = \frac{x}{t^{\beta}}$$

we got an equation

$$\epsilon \frac{\partial^2 T(x,t)}{\partial t^2} + \frac{\alpha}{t}\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}$$

There are differential eqs. for $f(\eta)$ **only for** $\alpha = \beta = +1$
or $\alpha = -2$ and $\beta = +1$

Solution for Case I

$\alpha = \beta = +1$

$$\epsilon(\eta^2 f(\eta))'' - a(\eta f(\eta))' = f''(\eta)$$

a total difference = conserved quantity

$$\epsilon(\eta^2 f(\eta))' - a\eta f(\eta) = f'(\eta) + c_1$$

There are two different cases:

$c_1 = 0$ or $c_1 \neq 0.$

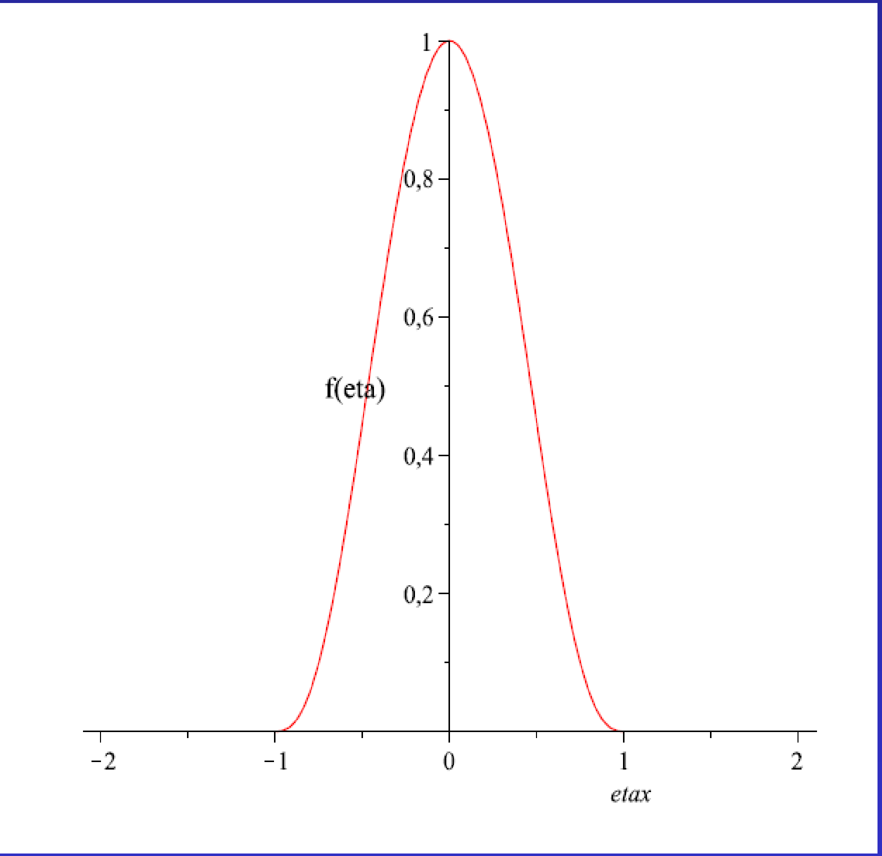
Case I

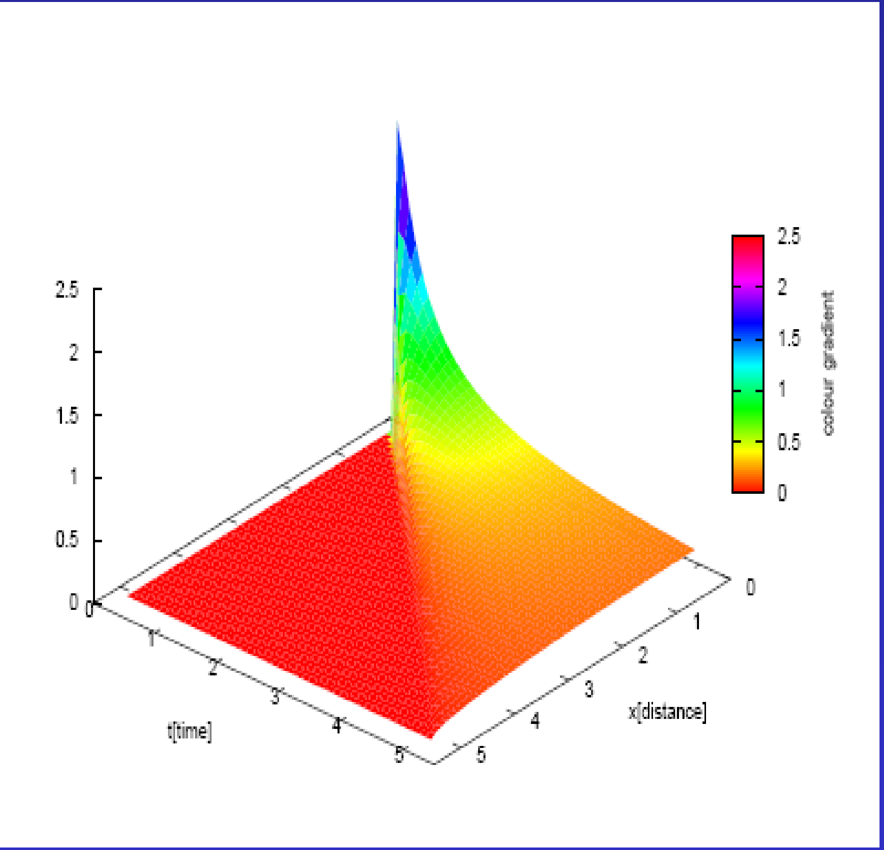
$c_1 = 0$

$a = 4.1, \epsilon = 1.$

$$f(\eta) = (1 - \epsilon\eta^2)^{\frac{a}{2\epsilon} - 1}$$

$$T(x,t) = \frac{1}{t}\left(1 - \epsilon\frac{x^2}{t^2}\right)^{\frac{a}{2\epsilon} - 1}$$





2 Important new feature: the solution is a product of 2 travelling wavefronts

$if\ a > 4\epsilon,\ f'(\eta) = 0$

no flux conservation problem

$$T(x,t) \sim U(x - ct)U(x + ct)$$

Case I

$c_1 \neq 0.$

$$f(\eta) = (\epsilon\eta^2 - 1)^{\frac{a}{2\epsilon}-1}\left[c_1\{signum(\epsilon\eta^2-1)\}^{\frac{a}{2\epsilon}-1}\{-signum(\epsilon\eta^2-1)\}^{\frac{a}{2\epsilon}-1} \eta F(1/2, a/2/\epsilon; 3/2; \epsilon\eta^2) + c_2\right]$$

- *where F(a,b;c;z) is the hypergeometric function*

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad a,b,c,z \ \epsilon\, C$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \qquad (a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) \quad n = 1,2,3,\ldots$$

- *some elementary functions can be expressed via F*
- *In our case if* $\frac{a}{2\epsilon}$ *Integer or Half-Integer are important*

the 4 basic cases:

$$\frac{a}{2\epsilon} = 0, \quad F\left(0, \frac{1}{2}; \frac{3}{2}; \epsilon\eta^2\right) = 1$$

$$\frac{a}{2\epsilon} = 1, \quad F\left(1, \frac{1}{2}; \frac{3}{2}; \epsilon\eta^2\right) = \frac{1}{2\sqrt{\epsilon}\eta} \ln\left(\frac{1+\sqrt{\epsilon}\eta}{1-\sqrt{\epsilon}\eta}\right)$$

$$\frac{a}{2\epsilon} = \frac{1}{2}, \quad F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \epsilon\eta^2\right) = \frac{arccos(\sqrt{\epsilon}\eta)}{\sqrt{\epsilon}\eta}$$

$$\frac{a}{2\epsilon} = \frac{3}{2}, \quad F\left(\frac{3}{2}, \frac{1}{2}; \frac{3}{2}; \epsilon\eta^2\right) = \frac{1}{(1-\epsilon\eta^2)}$$

Case I

with the following recursion all the other cases can be evaluated

$$(c-a)F(a-1,b;c;z)+(2a-c-az+z)F(a,b;c;z)+a(z-1)F(a+1,b;c;z)=0$$

two examples for negative parameters

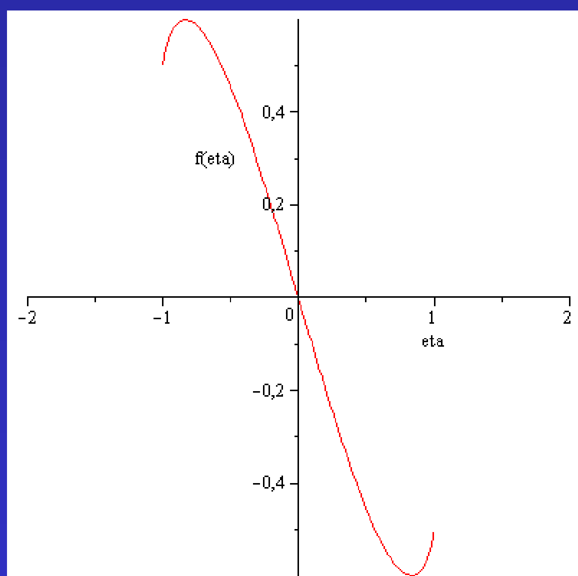
$$\frac{a}{2\epsilon}=-2,\quad \epsilon>0\quad F\left(-1,\frac{1}{2};\frac{3}{2};\epsilon\eta^2\right)=1-\frac{\epsilon\eta^2}{3}$$

$$\frac{a}{2\epsilon}=-\frac{1}{2},\quad \epsilon>0,\quad F\left(\frac{-1}{2},\frac{1}{2};\frac{3}{2};\epsilon\eta^2\right)=\frac{1}{2}\left\{(1/2-\epsilon\eta^2)\frac{arcsin(\sqrt{\epsilon}\eta)}{\sqrt{\epsilon}\eta}-\frac{\epsilon\eta^2-1}{2(1-\epsilon\eta^2)^{1/2}}\right\}$$

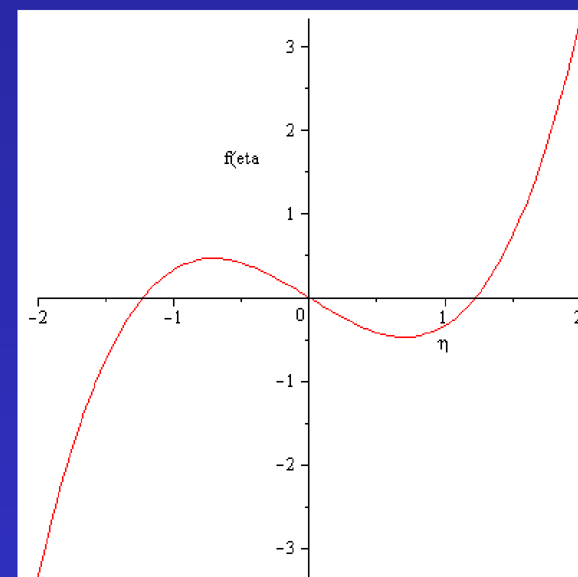
for non integer/half-integer values an inifinte series comes out

Case I

- ***what can we learn from this?***
- *Well, in some cases the **domain and range of** $f(\eta)$ compact*
- *Hyperbolic feature, like the wave-equation*
- *in some cases the **domain and range of** $f(\eta)$ non-compact*
- *parabolic feature*



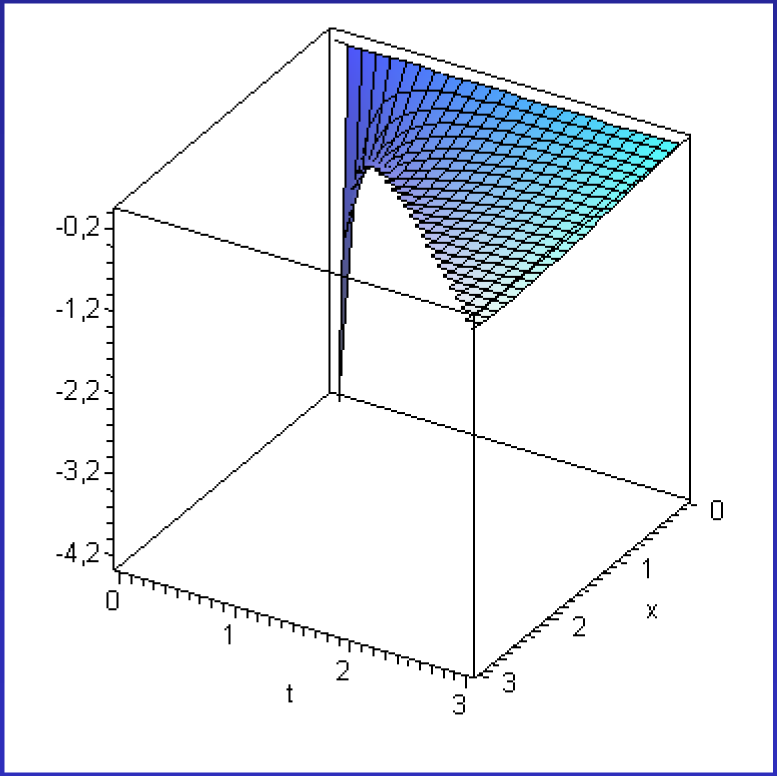
$$\eta\,\mathrm{hypergeom}\left(\left[\frac{1}{2}, 2\right], \left[\frac{3}{2}\right], \eta^2\right)\left(\eta^2 - 1\right)$$

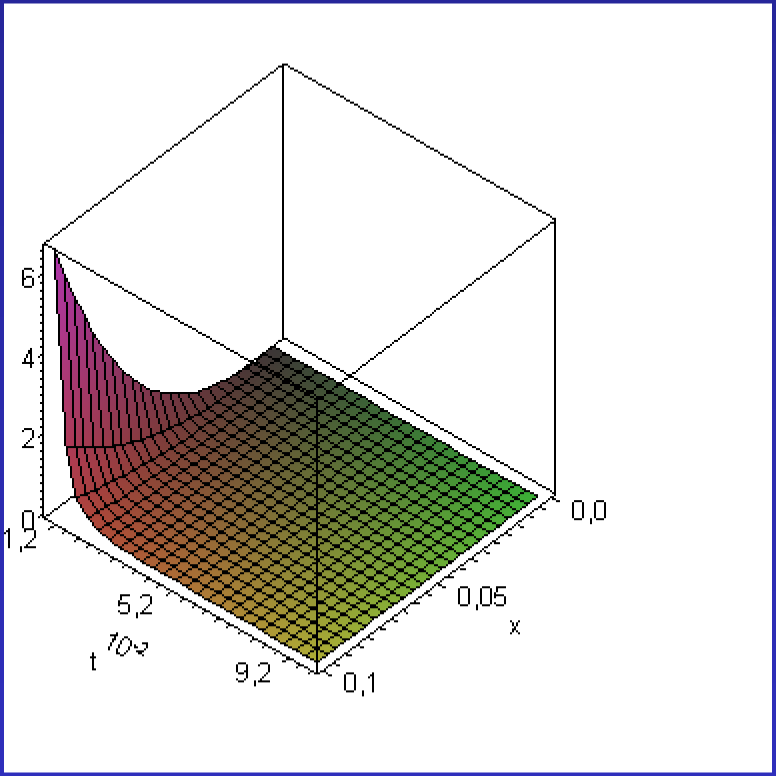


$$\frac{1}{\mathrm{signum}(\eta^2-1)^{5/2}}\left(\left(-\mathrm{signum}(\eta^2-1)\right)^{5/2}\eta\,\mathrm{hypergeom}\left(\left[\frac{1}{2}, \frac{5}{2}\right], \left[\frac{3}{2}\right], \eta^2\right)\left(\eta^2-1\right)^{3/2}\right)$$

only when a $\frac{a}{2\epsilon}$ is half integer

The solution T(x,t)





$$\left(\left(\frac{1}{t}\right)\cdot\left(\frac{x}{t}\right)\mathrm{hypergeom}\left(\left[\frac{1}{2},2\right],\left[\frac{3}{2}\right],\left(\frac{x}{t}\right)^2\right)\left(\left(\frac{x}{t}\right)^2-1\right), x=0..3, t=0.1..3\right)$$

$$\left(\frac{1}{\mathrm{signum}\left(\left(\frac{x}{t}\right)^2-1\right)^{5/2}}\left(\left(\frac{1}{t}\right)\cdot\left(-\mathrm{signum}\left(\left(\frac{x}{t}\right)^2-1\right)\right)^{5/2}\left(\frac{x}{t}\right)\mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{5}{2}\right],\left[\frac{3}{2}\right],\left(\frac{x}{t}\right)^2\right)\left(\left(\frac{x}{t}\right)^2-1\right)^{3/2}\right), x=0...1, t=0.01..0.1\right)$$

Solution for Case II

$\alpha = -2$ and $\beta = +1$

$$f''(\eta)[\epsilon\eta^2 - 1] - f'(\eta)\eta[2\epsilon + a] + 2f(\eta)[\epsilon + a] = 0.$$

$$f(\eta) = \left[c_1 P_{\frac{-a}{2\epsilon}}^{\frac{a}{2\epsilon}+2}(\sqrt{\epsilon}\eta) + c_2 Q_{\frac{-a}{2\epsilon}}^{\frac{a}{2\epsilon}+2}(\sqrt{\epsilon}\eta)\right](\epsilon\eta^2 - 1)^{\frac{a}{4\epsilon}+1},$$

$P_{-a/2\epsilon}^{\frac{4\epsilon+a}{2\epsilon}}(\sqrt{\epsilon}\eta)$ and $Q_{-a/2\epsilon}^{\frac{4\epsilon+a}{2\epsilon}}(\sqrt{\epsilon}\eta)$ are the associated Legendre functions

For the regular solution:

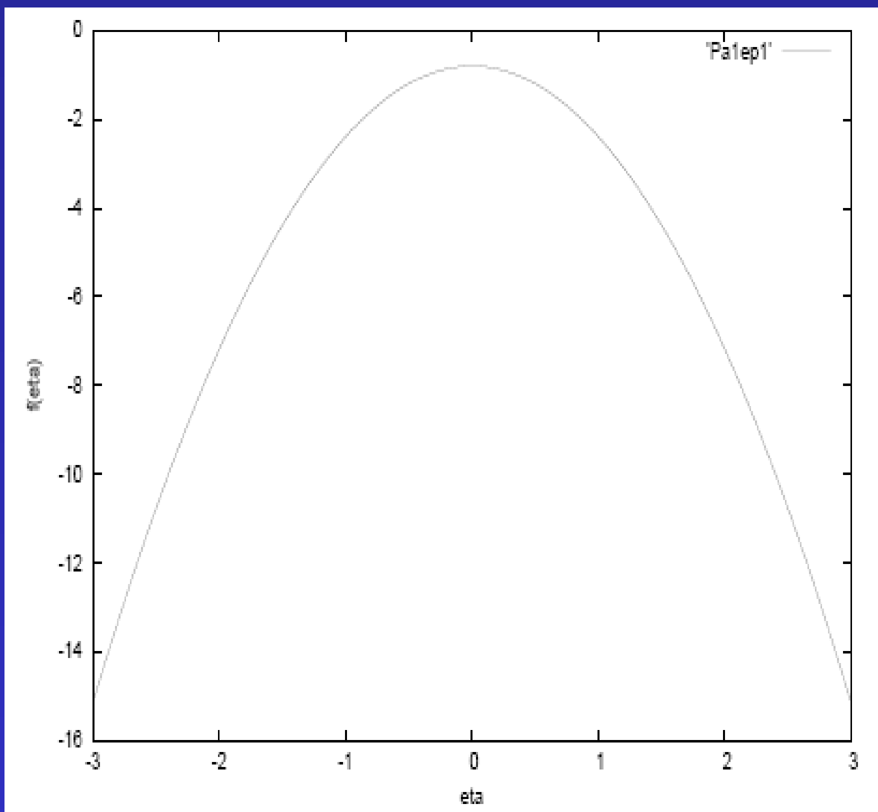
$$P_\nu^\mu(\sqrt{\epsilon}\eta) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+\sqrt{\epsilon}\eta}{1-\sqrt{\epsilon}\eta}\right)^{\mu/2} F\left(-\nu, \nu+1; 1-\mu; 1/2 - \sqrt{\epsilon}\eta/2\right) \quad -1 < \sqrt{\epsilon}\eta < 1$$

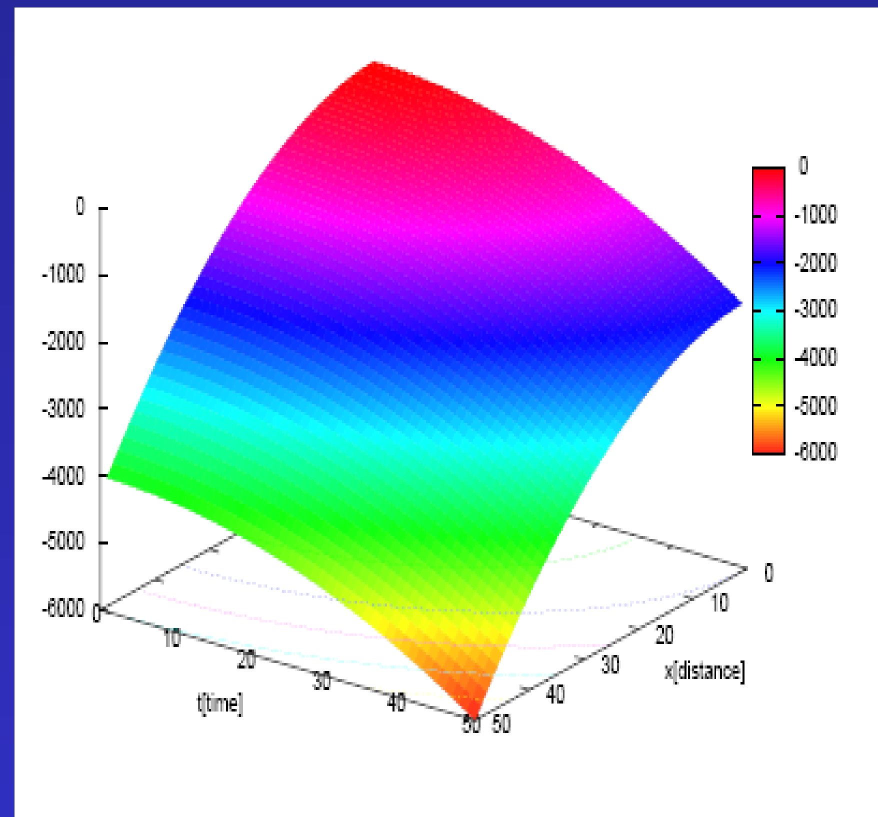
After some algebra of the hypergeometric function we get:

$$P_{-\frac{a}{2\epsilon}}^{\frac{a}{2\epsilon}+2}(\sqrt{\epsilon}\eta)(\epsilon\eta^2 - 1)^{\frac{a}{4\epsilon}+1} = -2\frac{([a+\epsilon]\eta^2 + 1)(\frac{1}{2})^{(-\frac{a}{2\epsilon})}}{\Gamma(\frac{-a}{2\epsilon})}.$$

A second order polinomial ☹

The regular solution



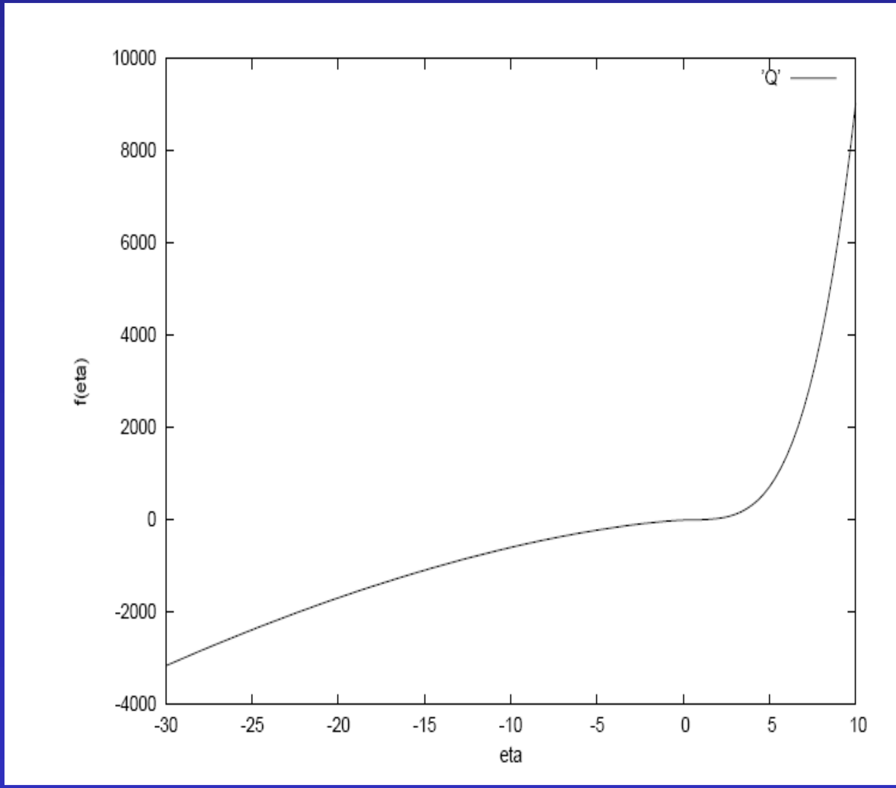


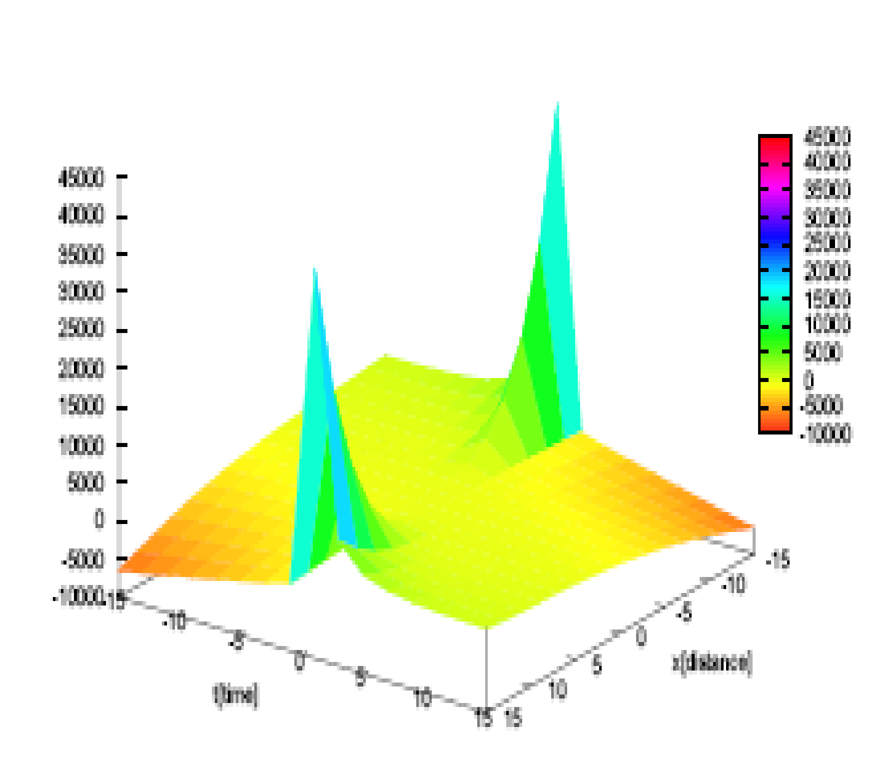
$$f(\eta) = P^{\frac{a}{2\epsilon}+2}_{-\frac{a}{2\epsilon}} \sqrt{\epsilon}\eta)[\epsilon\eta^2 - 1]^{\frac{a}{4\epsilon}+1}$$

$$T(x,t) = t^2 P^{\frac{a}{2\epsilon}+2}_{-\frac{a}{2\epsilon}}(\sqrt{\epsilon}x/t)[\epsilon(x/t)^2 - 1]^{\frac{a}{4\epsilon}+1} \quad for\ a=1,\ \epsilon=1$$

The irregular solution

Till now it is not possible to write it in closed form





$$f(\eta) = Q^{\frac{a}{2\epsilon}+2}_{-\frac{a}{2\epsilon}}(\sqrt{\epsilon}\eta)[\epsilon\eta^2 - 1]^{\frac{a}{4\epsilon}+1}$$

$$T(x,t) = t^2 Q^{\frac{a}{2\epsilon}+2}_{-\frac{a}{2\epsilon}}(\sqrt{\epsilon}x/t)[\epsilon(x/t)^2 - 1]^{\frac{a}{4\epsilon}+1} \; for \; a = 1, \; \epsilon = 1$$

both P and Q are non-compact solutions = parabolic property

Summary and Outlook

we presented various derivations and
interpretation of the telegraph equation

As a new feature we presented a new telegraph-type
equation with self-similar solutions
It has both parabolic and hyperbolic properties

further work is in progress to clear out the dark points
and improve physical interpretation