Correlations in the High Temperature QCD Dirac Spectrum

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Banks-Casher relation

spectrum of $D(A)$ $\Longleftrightarrow$ $\langle\overline{\psi}\psi\rangle$ (chiral order parameter)

$$\begin{aligned}
\langle\overline{\psi}\psi\rangle &= \lim_{m\to 0}\,\lim_{V\to\infty}\,\frac{1}{V}\,\langle\sum_x \mathrm{Tr}\,(D+m)^{-1}\rangle \\
&= \lim_{m\to 0}\,\int_0^\infty d\lambda\,\rho(\lambda)\,\frac{1}{\lambda^2+m^2} \\
&= \pi\,\rho(0),
\end{aligned}$$

where $\rho(\lambda)$ is the spectral density per unit volume of D

Chirally symmetric and broken phase

$\rho(0) \neq 0$	$\rho(0) = 0$
χ-symmetry broken	$\chi-$symmetry intact

$\rho(0) \neq 0$ — χ-symmetry broken

- $T < T_c$
- $T > T_c$ and $\mathrm{Re}(P) < 0$ ("unphysical" P-loop sector)

$\rho(\lambda) \approx \Sigma$

Random matrix theory (RMT) $\Rightarrow$ statistical description of low eigenvalues.

$\rho(0) = 0$ — $\chi-$symmetry intact

- $T > T_c$ and $ReP > 0$ ("physical" Polyakov sector)

$\rho(\lambda) \approx C \cdot \lambda^{\alpha}$
$\rho(\lambda) \approx C \cdot (\lambda - G)^{\alpha}$

No simple statistical description of low eigenvalues.

Simulations

- $SU(2)_c$ quenched.
- Overlap Dirac operator.
- $T = 2.6 \cdot T_c$
- $N_t = 4$; $L = 12 - 32$, 8 different spatial volumes.
- $N_t = 6$; $L = 18 - 32$, 8 different spatial volumes.
- Only $Q = 0$ topological sector used.
  (at hight T few $Q \neq 0$ confs.)
- Antiperiodic b.c. for the fermions in the time direction,
  P-loop > 0 sector $\Rightarrow$ $\rho(0) = 0$
  P-loop < 0 sector $\Rightarrow$ $\rho(0) \neq 0$

Is there any difference between the eigenmodes?

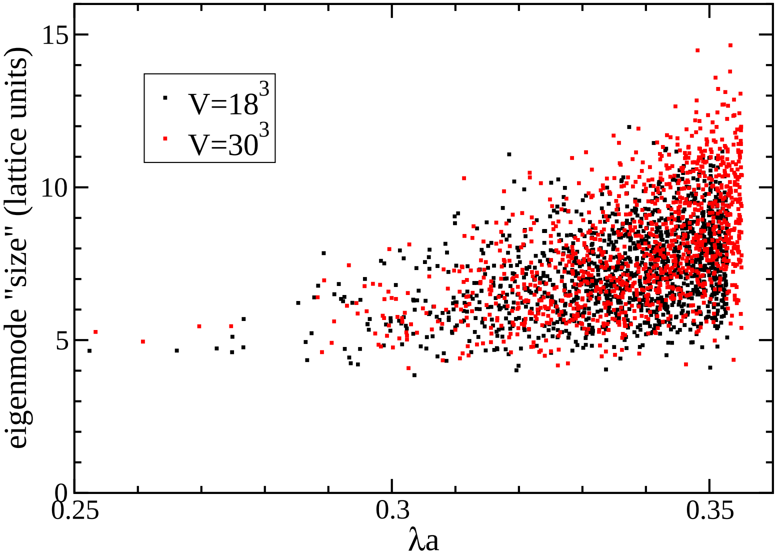
Localization

- How to measure spatial "extension" of (normalized) eigenmodes?
- $$\mathscr{V} = \left[\sum_x (\psi^\dagger \psi(x))^2 \right]^{-1}.$$
- If ψ is constant in volume V and zero elsewhere $\Rightarrow \mathscr{V} = V$.
- $\mathscr{V}$ measures "volume" occupied by eigenmode.
- $\Rightarrow$ Define 3-dimensional "size":

$$d = \left[\frac{\mathscr{V}}{N_t} \right]^{1/3}.$$

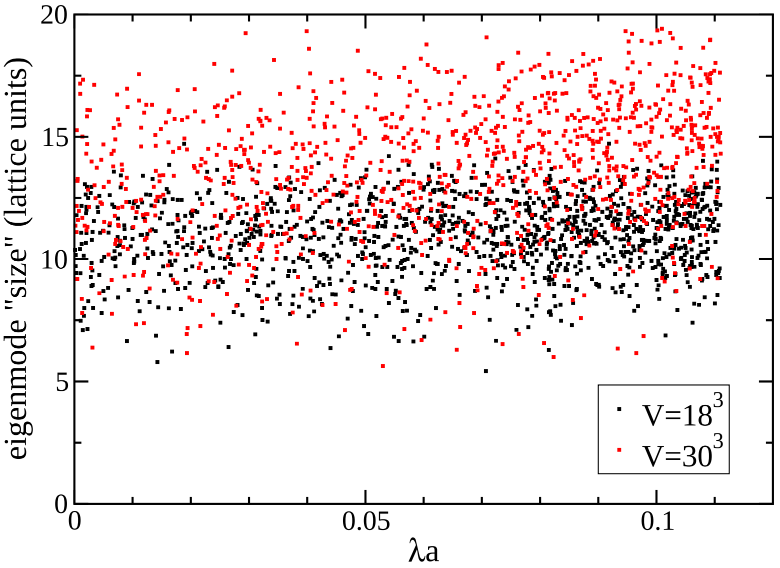
Localization of low eigenmodes of D:

$\rho(0) = 0$ sector (Polyakov loop in physical sector)



Localization of low eigenmodes of D:

$\rho(0) \neq 0$ sector (Polyakov loop in un-physical sector)



Differences in eigenmodes

$\rho(0) \neq 0$	$\rho(0) = 0$
Eigenmodes extended	Eigenmodes localized
Extension grows with box size	Extension independent of box size

$\rho(0) = 0$ simple picture

- Low eigenvalues produced independently in many small subvolumes of $\approx d^3$.
- Avg. number of "small" eigenvalues in volume d^3: $\ll 1$.
- $\Rightarrow$ Small eigenvalues occur independently.

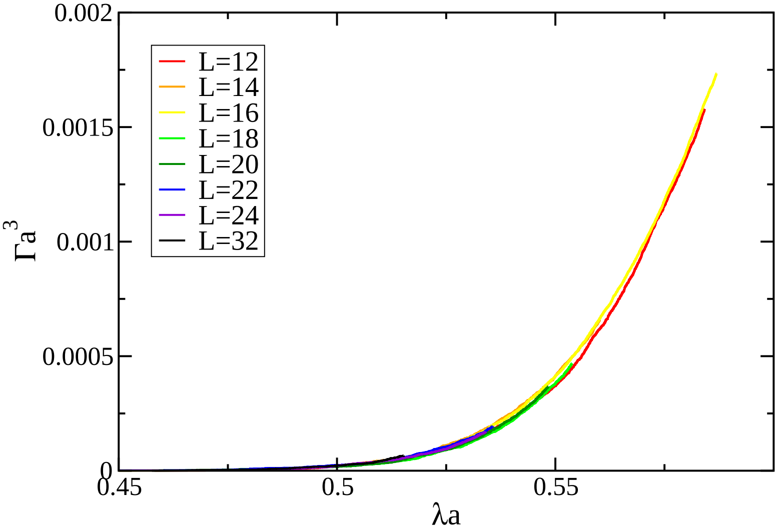
Assumptions for low eigenvalues in the $\rho(0) = 0$ phase

- For any given temperature $\rho(\lambda)$ is well-defined, i.e. the spectral density scales with the 3-volume.
- $\rho(\lambda) \approx C \cdot \lambda^{\alpha}$.
- The number of eigenvalues in any two disjoint intervals are independent random variables.

  $\Rightarrow$ The spectral density encodes all statistical properties of the spectrum.

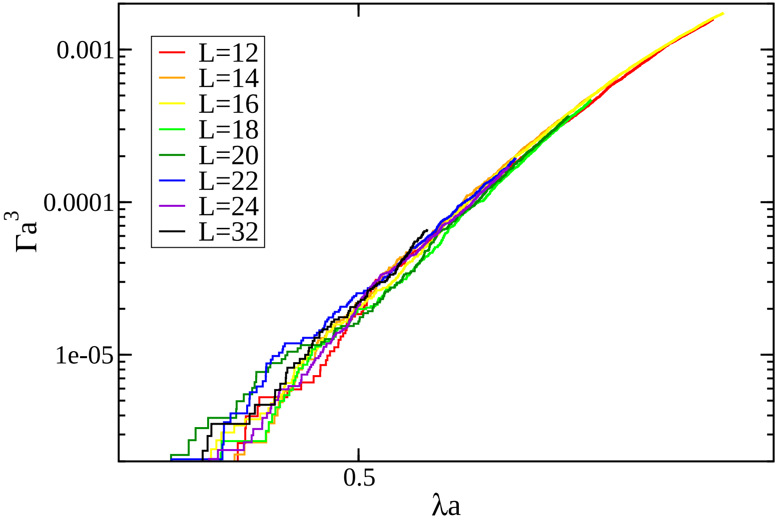
Scaling of the spectral density with the 3-volume

Integrated spectral density: $\Gamma(\lambda) = \int_0^\lambda dx \, \rho(x)$



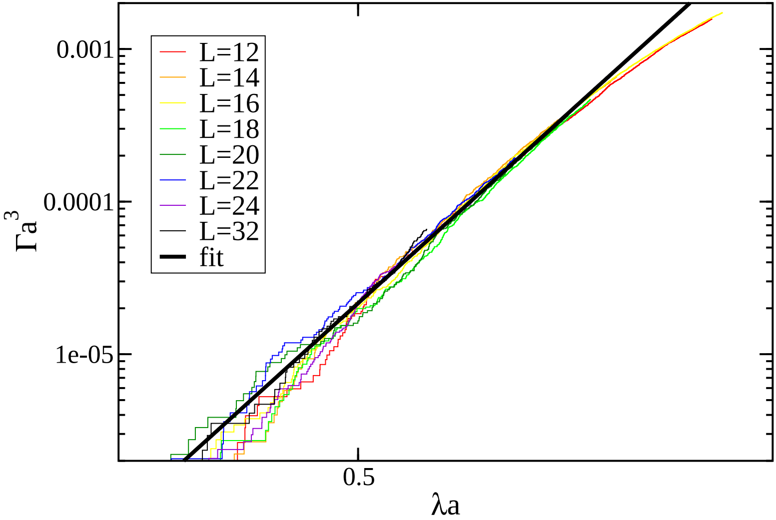
Scaling of the spectral density with the 3-volume

$\Gamma(\lambda) = \int_0^{\lambda} dx\, \rho(x)$ log-log scale



Spectral density around zero

$$\Gamma(\lambda) = \frac{C}{\alpha+1} \cdot \lambda^{\alpha+1} \qquad \text{fit}$$



Correlations in the spectrum

Assume that different regions in the spectrum are un-correlated.

- Number of eigenvalues in any two disjoint intervals are independent random variables.
- It is not the correlations among different eigenvalues! (Usually computed in RMT.)
- Hard to test directly.
- Assumption $\Rightarrow$ Fully determines:
  - Statistical properties of low eigenvalues.
  - Volume dependence of statistics.
- All statistical information encoded in spectral density.
- $\Rightarrow$ Easily verifiable analytic predictions.

Distribution of smallest eigenvalue

Let's first compute $\mathscr{P}_\emptyset(\lambda_1, \lambda_2)$, the probability that there is no eigenvalue in the $[\lambda_1, \lambda_2]$ interval:

Average number of eigenvalues in a small $\Delta x(\to 0)$ interval: $V\rho(x)\Delta x$

- Probability of ≥ 2 eigenvalue: $\to 0$
- Probability of 0 eigenvalue: $1 - V\rho(x)\Delta x$
- Probability of 1 eigenvalue: $V\rho(x)\Delta x$

No correlation $\Rightarrow$ Probabilities factorize:

$$\mathscr{P}_\emptyset(\lambda, x + \Delta x) = \mathscr{P}_\emptyset(\lambda, x) \cdot [1 - V\rho(x)\Delta x]$$

$$\Rightarrow \quad \frac{d\mathscr{P}_\emptyset(\lambda, x)}{dx} = -V\rho(x) \cdot \mathscr{P}_\emptyset(\lambda, x) = -VCx^\alpha \cdot \mathscr{P}_\emptyset(\lambda, x)$$

Solution with initial value

$$\mathscr{P}_\emptyset(\lambda,\lambda) = 1 :$$

$$\mathscr{P}_\emptyset(\lambda,x) = \exp\left[-\frac{VC}{\alpha+1}\left(x^{\alpha+1} - \lambda^{\alpha+1}\right)\right]$$

Probability density of smallest eigenvalue, $p_1(\lambda)$:

$$\text{Probability of} \begin{cases} \text{no eigenvalue in } [0,\lambda]: & \mathscr{P}_\emptyset(0,\lambda) \\ \text{one eigenvalue in } [\lambda,\lambda+d\lambda]: & V\rho(\lambda)\,d\lambda \end{cases}$$

$$\mathscr{P}_\emptyset(0,\lambda)\cdot V\rho(\lambda)\,d\lambda = p_1(\lambda)\,d\lambda$$

$$\Rightarrow \quad p_1(\lambda) = \exp\left(-\frac{CV}{\alpha+1}\lambda^{\alpha+1}\right) CV\lambda^{\alpha}$$

Further analytic predictions

Distribution of the second smallest eigenvalue, $p_2(\lambda)$:

Probability of
$$\begin{cases} \text{Smallest eigenvalue in } [x, x+\Delta x]: & p_1(x)\Delta x \\ \text{No eigenvalue in } [x+\Delta x, \lambda]: & \mathscr{P}_\emptyset(x,\lambda) \\ \text{One eigenvalue in } [\lambda, \lambda + d\lambda]: & V\rho(\lambda) d\lambda \end{cases}$$

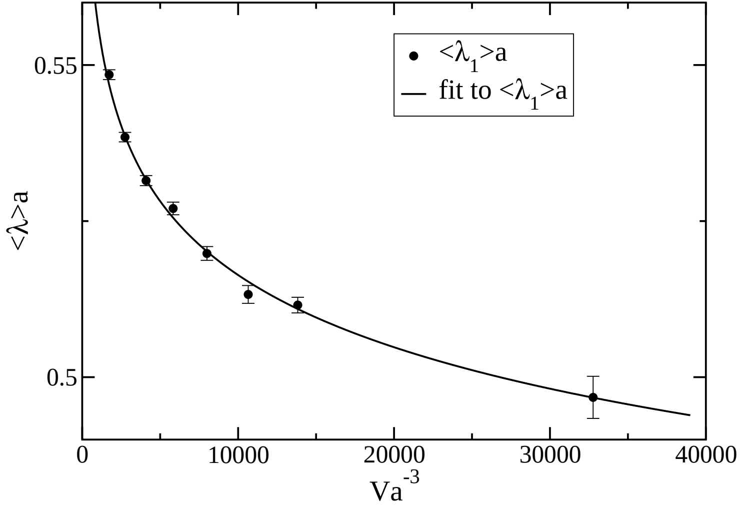
$$\Rightarrow \quad p_2(\lambda) = \int_0^\lambda dx\, p_1(x) \cdot \mathscr{P}_\emptyset(x,\lambda) \cdot V\rho(\lambda)$$
$$= \frac{C^2V^2}{\alpha+1} \exp\left(-\frac{CV}{\alpha+1}\lambda^{\alpha+1}\right) \lambda^{2\alpha+1}.$$

Average smallest eigenvalues

$$\langle\lambda_1\rangle = (CV\mu)^{-\mu} \cdot \Gamma(1+\mu)$$
$$\langle\lambda_2\rangle = (CV\mu)^{-\mu} \cdot \Gamma(2+\mu)$$

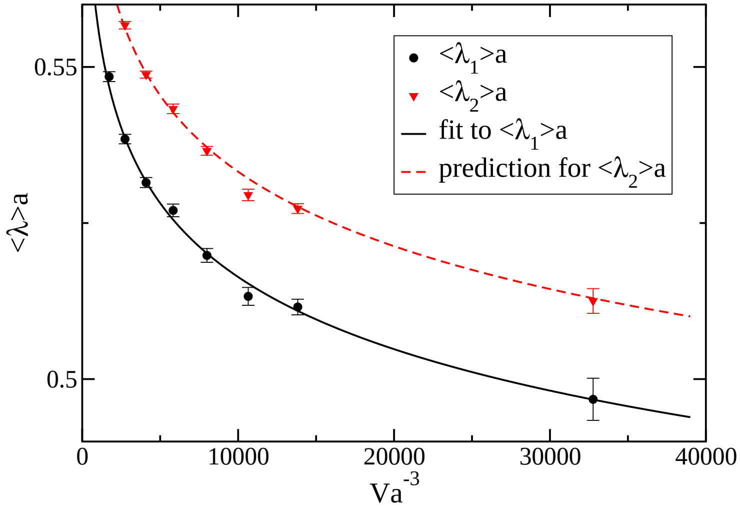
$$\mu = \frac{1}{\alpha+1}$$

Fit: $\langle\lambda_1\rangle = (CV\mu)^{-\mu} \cdot \Gamma(1+\mu)$ Fit parameters: μ, C



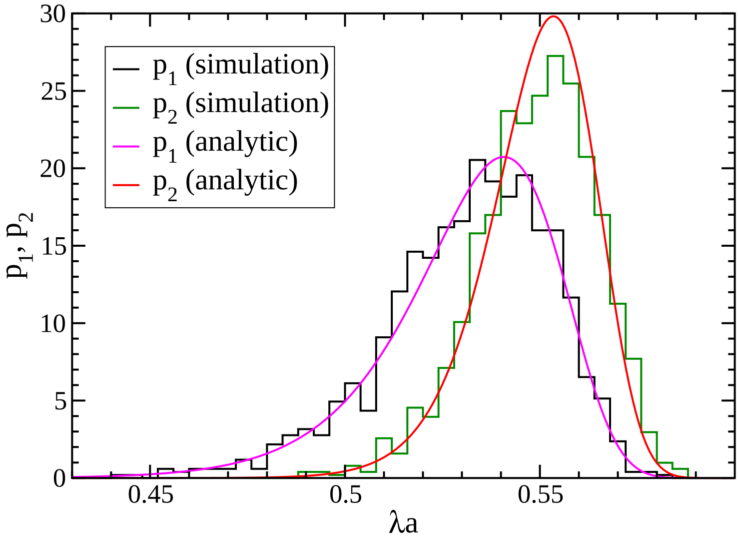
Volume dependence of $\langle\lambda_2\rangle$ $(N_T = 4)$

$\langle\lambda_2\rangle = (CV\mu)^{-\mu} \cdot \Gamma(2+\mu)$ No free parameter!



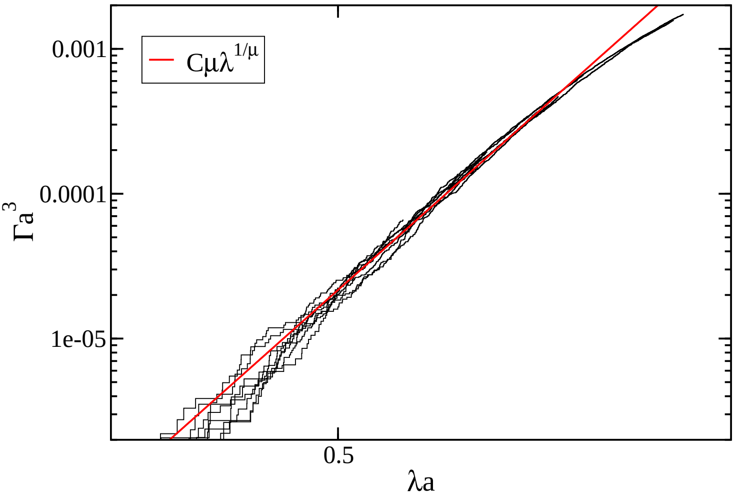
Distribution of the smallest two eigenvalues ($V = 16^3$)

Not a fit, $p_1(\lambda)$,$p_2(\lambda)$ with previously fitted parameters μ, C.



Integrated spectral density, $\Gamma(\lambda)$

Not a fit, $\Gamma(\lambda) = \mu C \lambda^{\frac{1}{\mu}}$ with previously fitted parameters μ, C.



Is it really incompatible with RMT?

- Isn't it possible that this works also for $\rho(0) \neq 0$?
- RMT analytic predictions are different.
  E.g. distribution of smallest eigenvalue:

  After a

  $$z = \frac{CV}{\alpha+1} \cdot \lambda^{\alpha+1}$$

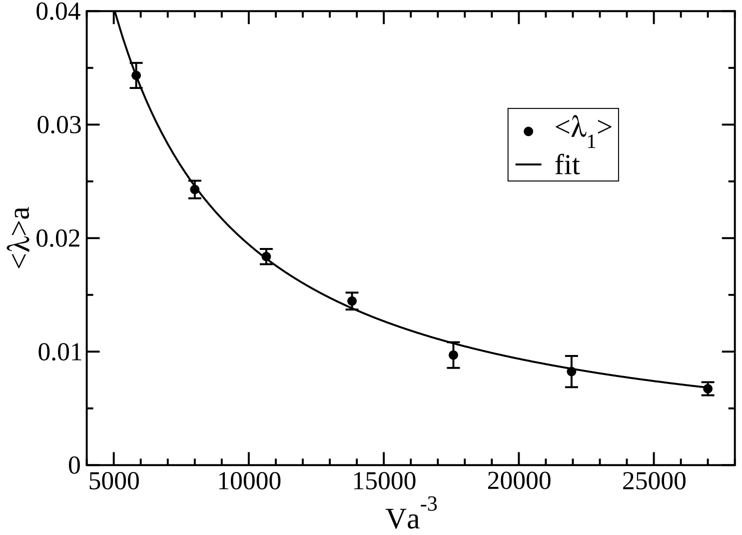
  transformation (in RMT: $\alpha = 0, C = \Sigma$):

  RMT: $p_1(z) = \frac{2+z}{4} \exp(-\frac{z}{2} - \frac{z^2}{8})$

  Here: $p_1(z) = \exp(-z)$
- Can we see the difference numerically?
- Same analysis for the $\rho(0) \neq 0$ case:
  in un-physical P-loop sector.

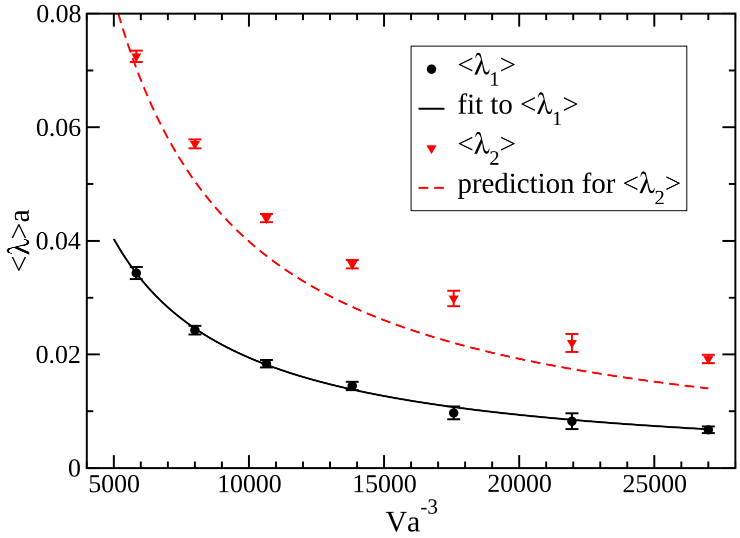
Volume dependence of $\langle \lambda_1 \rangle$ $(N_T = 6, P < 0)$

Fit: $\langle \lambda_1 \rangle = (CV\mu)^{-\mu} \cdot \Gamma(1+\mu)$ Best fit parameter: $\mu = 1.05(3)$



Volume dependence of $\langle\lambda_2\rangle$

$\langle\lambda_2\rangle = (CV\mu)^{-\mu} \cdot \Gamma(2+\mu)$ No free parameter!



Summary

- Main assumption:
  number of eigenvalues in disjoint intervals are uncorrelated
- Chirally symmetric phase ($\rho(0) = 0$):
  good description of the low end of the Dirac spectrum
- Chirally broken phase ($\rho(0) \neq 0$):
  random matrix theory is the proper description.

Further questions:

- Are these results generally true if $\rho(0) = 0$?
  ($SU(3)_c$, dynamical fermions, other fermion representations...?)
- Possible connection to Anderson localization?
  A.M. Garcia-Garcia and J.C. Osborn, Phys. Rev. D75 (2007) 034503.
- Possible connection to “generalized” random matrix models?
  Jackson and Verbaarschot, Phys. Rev. D53 (1996) 7223.