Nonextensive thermostatistics and Lesche stability

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- Nonextensive thermostatistics
  - Rényi, Tsallis and more
- Experimental robustness – a distinction?
- Extensivity and additivity – compositions

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Extensivity and additivity

Jaynes, 1957 (information theoretical, predictive, bayesian):

The measure of information is unique under general physical conditions.

(Shannon, 1948)

– Extensivity (density is meaningful)

$$\lim_{N \to \infty} \frac{X(N)}{N} < \infty$$

– Additive density (independent probabilities – independent systems)

$$s(f_1 f_2) = s(f_1) + s(f_2)$$

$$\Rightarrow \quad s_S(f) = -k \ln f$$

(unique solution)

Rényi, 1963

– Additive total entropy

$$g\left(\int f_1 f_2 s(f_1 f_2)\right) = g\left(\int f_1 s(f_1)\right) + g\left(\int f_2 s(f_2)\right)$$

$g\left(\int f \,\_\right)$ generalized average

(solutions)

$$\begin{cases} S_{BG}(f) = \int f \ln f \\ S_R(f) = \dfrac{1}{1-q} \ln\left(\int f^q\right) \end{cases}$$

$$S_T(f) = \frac{1}{q-1} \int (f - f^q)$$

Tsallis entropy

$$S_T = \frac{1 - e^{(1-q)S_R}}{q-1}$$

Tsallis-Rényi relation

$$\frac{1}{q-1} \ln\left(1 - (q-1) S_T\right) = S_R$$

Statistical physics of power law tails

Tsallis, 1988 - entropy and distribution

(distribution: Pareto, Zipf, etc.
entropy: Daróczi, Rényi, Hartley, Havrda-Charvat, etc…)

Non-additive but extensive (Touchette 2002, Tsallis 2006):

$$s_T(f_1 f_2) = s_T(f_1) + s_T(f_2) + (q-1)s_T(f_1)s_T(f_2)$$

$$S_T(f_1 f_2) = S_T(f_1) + S_T(f_2) + (q-1)S_T(f_1)S_T(f_2)$$

Several problems, several versions:

$$S_T(f) = \frac{1}{1-q}\int (f - f^q)$$

$$S_{nT}(f) = \frac{1}{1-q}\left(1 - \frac{1}{\int f^q}\right)$$

Landsberg-Vedral 1998

$$\overline{A}_1 = \int fA$$ Tsallis (1988)

$$\overline{A}_2 = \int f^q A$$ Curado-Tsallis (1992)

$$\overline{A}_3 = \int f_{esc} A = \int \frac{f^q}{\int f^q} A$$ Tsallis-Mendes-Plastino (1998)

escort probabilities

Problems:

non-additive averages (T2),

thermodynamic stability (T1,T3),

zeroth law (T1,T2,T3), q equilibration,

- microscopic origin of non-extensivity?
- alternatives:
  - incomplete distributions (Wang),
  - fluctuating temperature, superstatistics (Wilk, Beck, Cohen)
  - other entropies, etc…

Triumph: ~~Rényi~~

Lesche stability

a general validation criteria in thermostatistics.

Non-extensive entropies: which one to use?

Experimental robustness:

“A physically meaningful function of a probability distribution should not change drastically if the underlying distribution function is slightly changed.”

Lesche stability (Lesche, 1982):

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall n > n_0)(\forall t, p \in V_n)\left(\|r - p\| < \delta \Rightarrow \frac{|S(r) - S(p)|}{\max_n S} < \varepsilon \right)$$

$$\max_n S = \sup\{S(p) \mid p \in V_n\} < \infty$$

$$V_n = \{p \in D \mid p_i = 0 \;\; if \;\; i > n\}$$

a kind of uniform continuity

Rényi entropy is instable (Lesche 1982)

Proof idea (discrete, q>1):

Use special probability distributions

$$p_i = \frac{1}{n-1}(1-\delta_{i1}) \Rightarrow p = \left(0, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right) \in l_n$$

$$p'_i = \frac{\delta}{2}\delta_{i1} + \frac{1}{n-1}\left(1-\frac{\delta}{2}\right)(1-\delta_{i1}) \Rightarrow p' = \left(\frac{\delta}{2}, \frac{1}{n-1}\left(1-\frac{\delta}{2}\right), \ldots, \frac{1}{n-1}\left(1-\frac{\delta}{2}\right)\right) \in l_n$$

… and apply the definition (?)…

$$\frac{\left|S_R(p') - S_R(p)\right|}{\max S_R} = \frac{1}{\ln n}\frac{1}{1-q}\left|\ln\left(\frac{(n-1)^{1-q}}{(\delta/2)^q + (n-1)^{1-q}(1-\delta/2)^q}\right)\right| \le \frac{\ln(n-1)}{\ln n} \to 1$$

Abe 2002: Tsallis entropy is stable (normalized Tsallis is instable)
hot discussion

Abe 2008: q-expectation value is Lesche unstable
discussion,…

$$\bar{A}_3 = \int f_{esc} A = \int \frac{f^q}{\int f^q} A$$

However $S_T(S_R)$ is smooth:

$$S_T = \frac{1-e^{(1-q)S_R}}{q-1}$$

Tsallis-Rényi relation

$$\frac{1}{q-1}\ln\left(1-(q-1)S_T\right) = S_R$$

The previous proof is wrong.

Clarification of the relation of continuity, uniform continuity and Lesche stability gives that:

Rényi and Tsallis entropies are continuous and stable if 1<q and are not continuous and instable for finite uniform distributions if 1>q.

The q-expectation values of an $A \in l^\infty$ physical quantity are continuous and stable if 1<q and are not continuous and instable for practically all physical quantities in case of finite uniform distributions.

Matolcsi-Ván: arXiv:0910.1918

Nonextensive thermostatistics –

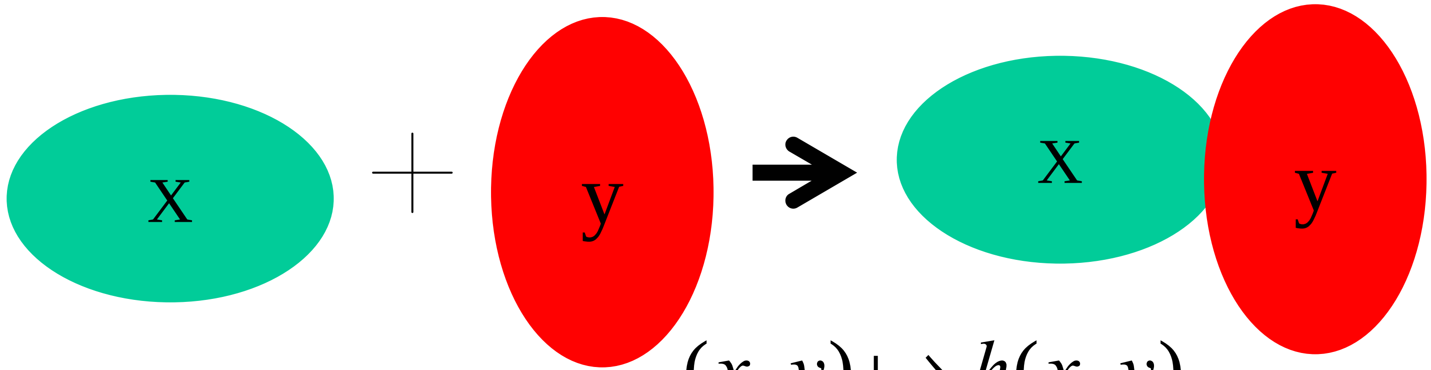
a *universality* behind power law tails.

?

Questions

- What is non-additivity?
  - physical and mathematical generalizations
  - averages or densities?
- What is the relation of non additivity and non-extensivity?
- Is there a what kind of thermostatistics?
  - universality
  - MEP, distributions, averages, temperature, …
- What is behind power law tails?
  - nonequilibrium, fractals or long range interactions?

Non-additivity – a composition



$$(x, y) \mapsto h(x, y)$$

Extensive composition:

$$\lim_{N\to\infty} \underbrace{h\left(h\left(\ldots h\left(h\right.\right.\right.}_{N}\left(\frac{x}{N}, \frac{x}{N}\right), \frac{x}{N}\Big)\ldots, \frac{x}{N}\Big), \frac{x}{N}\Big) < \infty \qquad \left(\lim_{N\to\infty} \frac{X(N)}{N} < \infty\right)$$

associative
symmetric

$$h(h(x, y), z) = h(x, h(y, z))$$
$$h(x, y) = h(y, x)$$

Associativity – generalized additivity

$$h(h(x,y),z) = h(x,h(y,z))$$

$$\Rightarrow \quad \exists L : h_\infty(x,y) = L^{-1}(L(x)+L(y))$$

(*L – formal logarithm*

(unique up to a constant multiplier)

Constructive:

$$L(x) = \int_0^x \left(\frac{\partial h}{\partial y}(z,0) \right)^{-1} dz$$

E.g. Tsallis rule is associative:

$$h(S_1,S_2) = S_1 + S_2 + aS_1S_2$$

$$L(x) = \int_0^x \frac{1}{1+az} dz = \frac{1}{a}\ln(1+ax)$$

$$\Rightarrow \quad L(S_T) = \frac{1}{a}\ln(1+aS_T) = S_R$$

Universality of Tsallis:

– additivity, Tsallis additivity and hyperTsallis additivity

$$h(x,y) = \boxed{\boxed{\boxed{h_0 + cx + c_1 y} + axy + a_1 x^2 + a_2 y^2} + bxy^2 + b_1 x^2 y + b_2 y^2 x^2} + \ldots$$

$$h(0,0) = 0$$

$$h(x,0) = x$$

1) $\boxed{h_\infty(x,y) = x + y}$ additive

2) $\boxed{h_\infty(x,y) = x + y + axy}$ Tsallis-additive

3) $\boxed{h_\infty(x,y) = \dfrac{x + y + axy}{1 - \dfrac{b}{2}xy}}$ hiperTsallis-additive

Summary

Dream of non-extensive thermostatistics:
universality behind observed power law tails.

Rényi and Tsallis are intimately related. One may use them equivalently.

Experimental robustness can and should be properly formulated.

Most general nonadditivity and thermodynamic limit gives Tsallis rule as a universal - first approximation.

There are higher order approximations.

Thank you for the attention!

Extensives but non-additive - associative thermostatistics

– Boltzmann-Gibbs entropy
– associative kinetic energy
– MEP– equilibrium distributions
– Boltzmann equation – equilibration (Bíró)
– Heavy-ion collisions

Quark coalescence + flow + Cooper-Fry freezeout

p_T tails (Ürmössy)

$$\boxed{s(f_1 f_2) = s(f_1) + s(f_2)}$$

$$\frac{\partial}{\partial f_1} s(f_1 f_2) = f_2 s'(f_1 f_2) = s'(f_1)$$

$$\frac{\partial}{\partial f_2} s(f_1 f_2) = f_1 s'(f_1 f_2) = s'(f_2)$$

$$s'(f_1 f_2) = \frac{s'(f_1)}{f_2} = \frac{s'(f_2)}{f_1}$$

$$f\, s'(f) = const. = -\kappa$$

$$s(f) = -\int \frac{\kappa}{f} df = -\kappa \ln f + \cancel{C}$$

$$s(f) = -\kappa \ln f$$