ANALYTIC RIHD SOLUTIONS:

An Embedding-Reduction Approach

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OUTLINE

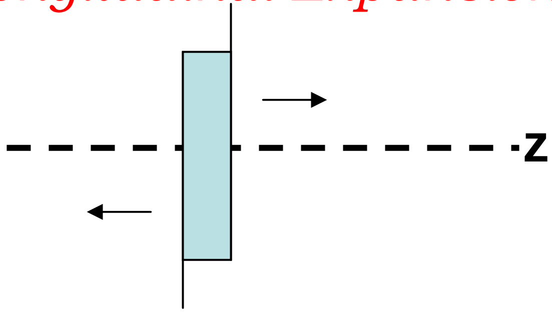
- **Brief Motivation**
- **The Embedding-Reduction Approach**
- **Solutions with Longitudinal and Transverse Elliptic Flow**
- **Discussion on New Developments**

JL & Koch, arXiv:0905.3406 [Phys.Rev.C];
Lin & JL , arXiv:0909.2284;
JL, in progress.

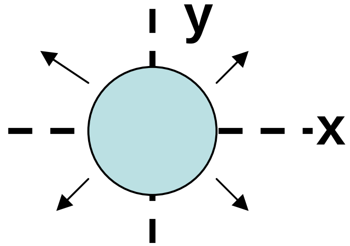
Collective Flow @ RHIC

$$\hat{D}_t \vec{v} \sim -\vec{\nabla} p$$

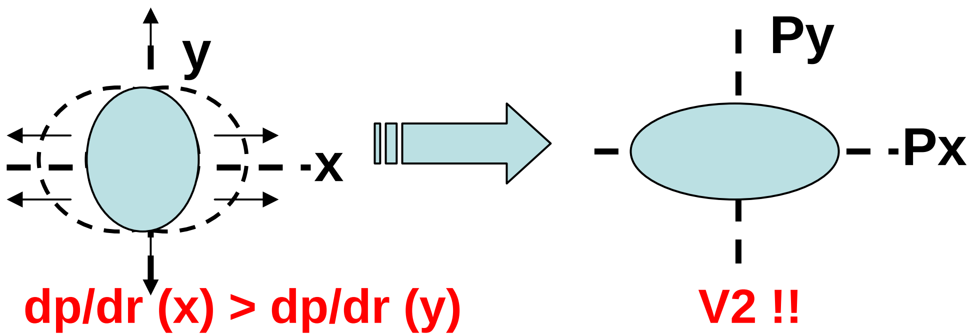
- *Longitudinal Expansion*



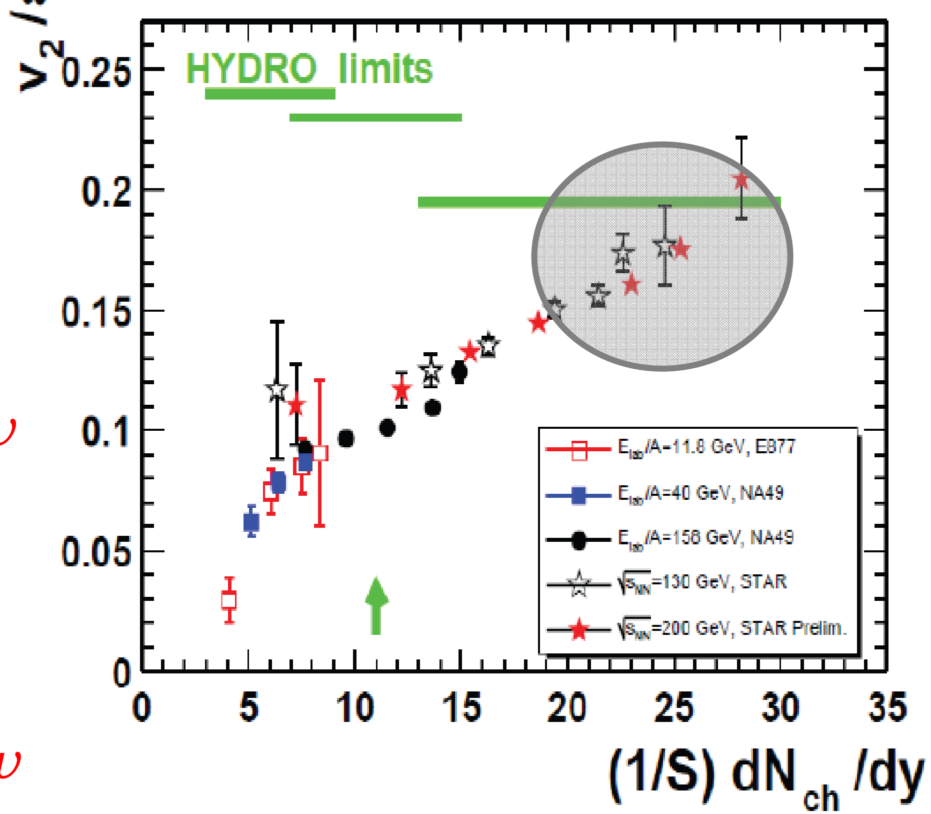
- *Transverse Expansion: Radial Flow*



- *Transverse Expansion: Elliptic Flow*



arXiv:0809.2949



Triumph of Relativistic Ideal Hydrodynamics (RIHD) at RHIC !!

STATUS & PROSPECT

- Very successful numerical hydrodynamics at RHIC
- Viscous corrections small at RHIC but necessary
- Many progresses in analytic solutions for (1+1)D RIHD
- Efforts toward analytic solutions with transverse radial and elliptic flows
- Finding new analytic solutions in (1+3)D RIHD are interesting and important:
  - for HIC experiments;
  - for academic curiosity.

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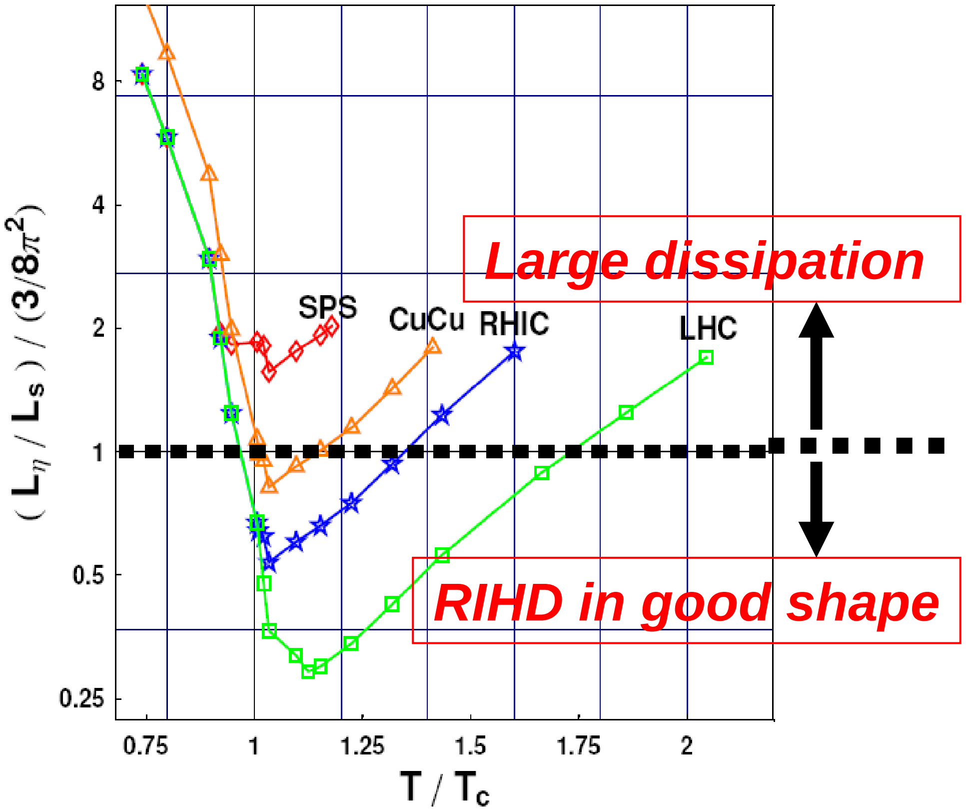


JL & Koch, arXiv:0909.3105

Expect even more success of RIHD at LHC ……

RIHD IN BRIEF

- **Conservation of energy and momenta: 4 eqns**

$$T^{mn}{}_{;n} = 0 \quad , \quad T^{mn} = (\epsilon + p)u^m u^n - p g^{mn}$$

- **4-velocity field: 3 independent velocity components**

$$u^m \cdot u_m = 1 \dashrightarrow \gamma(1, \vec{v}) \, , \; \gamma = 1/\sqrt{1 - \vec{v}^2}$$

- **Equation of State (EoS): 1 independent matter field**

$$p = \nu(\epsilon + p)$$

Alternatively: epsilon[T(x)], p[T(x)]

Note ------ we here discuss RIHD without conserved charges.

For RIHD with conserved charges (e.g. baryonic):
one more matter field, n(x) or \mu(x) ;
one more equation for charge conservation

What's "Reduction"-I

- Take the (1+1)D RIHD example for demonstration:

$$\frac{[\partial_t + v\partial_z]\epsilon}{\epsilon + p} = -\partial_z v - \gamma_v^2[\partial_t + v\partial_z]\left(\frac{v^2}{2}\right),$$

$$\gamma_v^2[\partial_t + v\partial_z]v = -\frac{\partial_z p}{\epsilon + p} - \frac{v\partial_t p}{\epsilon + p}.$$

- Use EoS in a slightly different way:

$$w = \epsilon + p \dashrightarrow d\epsilon = \frac{1}{1+c_s^2}dw, \quad dp = \frac{c_s^2}{1+c_s^2}dw$$

- Recast the equations into neat forms:

$$\partial_t[\ln(w)] + v\partial_z[\ln(w)] = -\frac{\gamma_v^2}{1-\xi}[\partial_x v + v\partial_t v]$$

$$v\partial_t[\ln(w)] + \partial_z[\ln(w)] = -\frac{\gamma_v^2}{\xi}[v\partial_x v + \partial_t v]$$

$$\partial_t[\ln(w)] = \mathcal{X}[v, \partial_t v, \partial_z v] = \mathcal{X}[t,z]$$

$$\partial_z[\ln(w)] = \mathcal{Y}[v, \partial_t v, \partial_z v] = \mathcal{Y}[t,z]$$

$$\partial_z \mathcal{X}[t,z] - \partial_t \mathcal{Y}[t,z] = 0$$

WHAT'S "REDUCTION"–II

- Take the (1+1)D RIHD example for demonstration:

$$\partial_t[\ln(w)] = \mathcal{X}[v,\, \partial_t v,\, \partial_z v] = \mathcal{X}[t,z]$$
$$\partial_z[\ln(w)] = \mathcal{Y}[v,\, \partial_t v,\, \partial_z v] = \mathcal{Y}[t,z]$$

$$\partial_z \mathcal{X}[t,z] - \partial_t \mathcal{Y}[t,z] = 0$$

$$w(t,z) = w_0 \cdot e^{\int_{t_0}^{t} dt' \mathcal{X}[t',z] + \int_{z_0}^{z} dz' \mathcal{Y}[t,z']}$$

Reduction:
a set of two 1st-order equations for matter and velocity field
→ → a single 2nd-order equation for velocity field only

$$\begin{aligned}
&[(1-\xi) - \xi v^2](1-v^2)\left(\partial_t^2 v\right) + [(2-3\xi) - \xi v^2](2v)(\partial_t v)^2 \\
&\quad - [\xi - (1-\xi)v^2](1-v^2)\left(\partial_z^2 v\right) - [(3\xi - 1) \\
&\quad + (\xi - 1)v^2](2v)\left(\partial_z v\right)^2 + 2(1-2\xi)(1-v^2)v(\partial_t \partial_z v) \\
&\quad + 2(1-2\xi)(1+3v^2)(\partial_t v)(\partial_z v) = 0\,. \qquad \text{(B10)}
\end{aligned}$$

Nagy-Csörgő-Csanád (NCC) solutions

$$v = \tanh[\lambda \eta] = \frac{(t+z)^\lambda - (t-z)^\lambda}{(t+z)^\lambda + (t-z)^\lambda}$$

$$\frac{2\lambda(1-v^2)^2}{(t^2-z^2)^2} \cdot (1-2\xi) \cdot (1-\lambda) \cdot [(tz)v^2 - (t^2+z^2)v + (tz)] = 0$$

What's "Embedding"-I

- (1+3)D RIHD is more difficult: 3 velocities
  →→ *embedding known lower dimensional solutions*
- Example: embedding
  1-D longitudinal Hwa-Bjorken & transverse radial flows

$$(t,x,y,z) \longrightarrow (\tau,\eta,\rho,\phi)$$

$$u^\tau = \gamma(\cosh\eta - v_z \sinh\eta) \qquad u^\eta = \frac{\gamma}{\tau}(v_z \cosh\eta - \sinh\eta)$$

$$u^\rho = \gamma(v_x \cos\phi + v_y \sin\phi) \qquad u^\phi = \frac{\gamma}{\rho}(v_y \cos\phi - v_x \sin\phi)$$

EMBEDDING

$$v_z = z/t = \tanh\eta \longrightarrow u^\eta = 0$$

$$v_x = v_\rho \cos\phi$$
$$v_y = v_\rho \sin\phi \longrightarrow u^\phi = 0$$

$$u^m = \bar{\gamma}(1,0,\bar{v}_\rho,0) \qquad \bar{\gamma} \equiv 1/\sqrt{1-\bar{v}_\rho^2}$$

What's "Embedding"–II

- (1+3)D RIHD equations after embedding:

$$u^m = \bar{\gamma}(1, 0, \bar{v}_\rho, 0)$$

$$T^{\tau\lambda}{}_{;\lambda} = T^{\tau\tau}{}_{,\tau} + \frac{T^{\tau\tau}}{\tau} + \frac{p}{\tau} + T^{\tau\rho}{}_{,\rho} + \frac{T^{\tau\rho}}{\rho} = 0,$$

$$T^{\eta\lambda}{}_{;\lambda} = \frac{1}{\tau^2} p_{,\eta} = 0,$$

$$T^{\rho\lambda}{}_{;\lambda} = T^{\rho\rho}{}_{,\rho} + \frac{T^{\rho\rho}}{\rho} - \frac{p}{\rho} + T^{\tau\rho}{}_{,\tau} + \frac{T^{\tau\rho}}{\tau} = 0,$$

$$T^{\phi\lambda}{}_{;\lambda} = \frac{1}{\rho^2} p_{,\phi} = 0.$$

Now two equations with two fields depending on two variables, the pressure and one velocity field →→ *the reduction is possible !*

EMBEDDING-REDUCTION

- (1+3)D RIHD equations after embedding → two remains,

$$D_a \cdot \mathcal{K}_{,\tau} + D_b \cdot \mathcal{K}_{,\rho} = D_1 \cdot \mathcal{K},$$
$$D_b \cdot \mathcal{K}_{,\tau} + D_c \cdot \mathcal{K}_{,\rho} = D_2 \cdot \mathcal{K}.$$

$$p = \frac{\nu\rho\tau}{\bar{\gamma}^2}\mathcal{K}$$

All the D-coefficients involve only the velocity and its two 1st-derivatives

$$(\ln \mathcal{K})_{,\tau} = \mathcal{F}[\tau, \rho],$$
$$(\ln \mathcal{K})_{,\rho} = \mathcal{G}[\tau, \rho]$$

→

$$\frac{\partial}{\partial\rho}\mathcal{F}[\bar{v}_\rho(\tau, \rho)] - \frac{\partial}{\partial\tau}\mathcal{G}[\bar{v}_\rho(\tau, \rho)] = 0.$$

The F,G involve only on the velocity and its two 1st-derivatives

- (1+3)D RIHD equations after embedding & reduction
  →→ a single 2nd equation for velocity field only!

$$p = \text{constant} \times \frac{\rho\tau}{\kappa\bar{\gamma}^2} \times e^{[\int^\tau d\tau'\mathcal{F}[\tau',\rho] + \int^\rho d\rho'\mathcal{G}[\tau,\rho']]}.$$

- Two concrete examples as a check:

Hwa-Bjorken flow

$$\mathcal{F}_{\text{Bj.}} = \frac{\nu}{\nu - 1}\frac{1}{\tau}, \quad \mathcal{G}_{\text{Bj.}} = \frac{1}{\rho}$$

3-D Hubble flow

$$\mathcal{F}_{\text{Hu.}} = \frac{3}{\tau} + \frac{\nu - 5/2}{1 - \nu}\frac{2\tau}{\tau^2 - \rho^2},$$
$$\mathcal{G}_{\text{Hu.}} = \frac{1}{\rho} + \frac{\nu - 5/2}{1 - \nu}\frac{-2\rho}{\tau^2 - \rho^2}.$$

APPLICATION: AN EXAMPLE

- (1+3)D RIHD equations after embedding & reduction

→→

$$\frac{\partial}{\partial\rho}\mathcal{F}[\bar{v}_\rho(\tau,\rho)] - \frac{\partial}{\partial\tau}\mathcal{G}[\bar{v}_\rho(\tau,\rho)] = 0.$$

- Transverse radial velocity: power law ansatz:

$$\bar{v}_\rho = A\cdot\tau^{B}\cdot\rho^{C}$$

$$\frac{\partial\mathcal{F}}{\partial\rho} - \frac{\partial\mathcal{G}}{\partial\tau} = \frac{\bar{v}_\rho}{\nu(1-\nu)\rho^2\tau^2\left(1-\bar{v}_\rho^2\right)^3}\times\mathcal{I}[\tau,\rho] = 0,$$

$$\begin{aligned}\mathcal{I}[\tau,\rho] = {} & [-f_2(C+1)A^4]\tau^{4B+2}\rho^{4C} \\ & +[g_2(B+1)A^4]\tau^{4B}\rho^{4C+2} \\ & +[(g_1-f_1)A^3]\tau^{3B+1}\rho^{3C+1} \\ & +[(2f_2+f_4(3C+1))A^2]\tau^{2B+2}\rho^{2C} \\ & +[(-2g_2-g_4(3B+1))A^2]\tau^{2B}\rho^{2C+2} \\ & +[(f_1+f_3-g_1-g_3)A]\tau^{B+1}\rho^{C+1} \\ & +[(f_2+f_4)(C-1)]\tau^2 \\ & +[-(g_2+g_4)(B-1)]\rho^2.\end{aligned}$$

NEW (1+3)D SOLUTIONS

- **Two trivial solutions: Hwa-Bjorken ; 3-D Hubble**

- **New Solution:** $$p = \frac{\text{constant}}{(\tau\rho)^{1/(1-\nu)}(\rho^2 - \tau^2)^{(1-3\nu)/(2\nu-2\nu^2)}}$$

$$v_x = \frac{x}{t} \cdot \frac{t^2 - z^2}{x^2 + y^2}, \qquad v_y = \frac{y}{t} \cdot \frac{t^2 - z^2}{x^2 + y^2}, \qquad v_z = \frac{z}{t}$$

- **New Solution: with EoS e=3p**

$$v_x = \frac{x}{t} \cdot \left(\frac{t^2 - z^2}{x^2 + y^2}\right)^{2/3}, \quad v_y = \frac{y}{t} \cdot \left(\frac{t^2 - z^2}{x^2 + y^2}\right)^{2/3}$$

$$v_z = \frac{z}{t}, \quad p = \text{constant} \times \frac{\left(\rho^{2/3} - \tau^{2/3}\right)^{2/3}}{(\rho\tau)^{4/3}}$$

- **New Solution: with EoS e=p**

$$v_x = \frac{-x}{t}, \quad v_y = \frac{-y}{t}, \quad v_z = \frac{z}{t},$$

$$p = \text{constant} \times (\tau^2 - \rho^2).$$

DISCUSSIONS

JL & Koch, arXiv:0905.3406 [Phys.Rev.C]

Possible new directions to extend the approach:

- alternative longitudinal embedding with newly found (1+1)D solutions
- transverse radial flow beyond power law ansatz

✓ *more general embedding*

✓ *elliptic flow: using E-R with special coordinates*

✓ *elliptic flow: linearized perturbation on top of radial flow*

More General Embedding

Lin & JL , arXiv:0909.2284
See also talk by Shu Lin

- **2-D Hubble Embedding:**

$$(t, x, y, z) \longrightarrow (\tau_\perp, \eta_\perp, \phi_\perp, z)$$

$$v_x = \frac{x}{t} = \tanh \eta_\perp \cos \phi_\perp$$
$$v_y = \frac{y}{t} = \tanh \eta_\perp \sin \phi_\perp \longrightarrow u^m = \bar{\gamma}(1, 0, 0, \bar{v}_z)$$

- **General Scaling Embedding:**

$$v_x = \alpha_x \frac{x}{t}, \; v_y = \alpha_y \frac{y}{t}, \; v_z = \alpha_z \frac{z}{t}$$

$$\mathcal{M}^{\mu\nu} \partial_\nu (\ln \mathbf{p}) = B^\mu \longrightarrow \partial_\nu (\ln \mathbf{p}) = \mathcal{M}^{-1}_{\mu\nu} B^\mu$$

1-D, 2-D, 3-D Hubble flows

IV. $\alpha_x = \alpha_y$ and $1 + \alpha_x + \alpha_y + \alpha_z = 0$ with $\nu = 1/2$;
V. $\alpha_x = 0$ and $1 + \alpha_x + \alpha_y + \alpha_z = 0$ with $\nu = 1/2$.

ELLIPTIC FLOW –I

- Using special hyperbolic coordinates:

$$(x, y) \longrightarrow (\xi, \eta)$$

$$x = a\sqrt{(\xi^2 - 1)(1 - \eta^2)}\ ,\ y = a\xi\eta$$

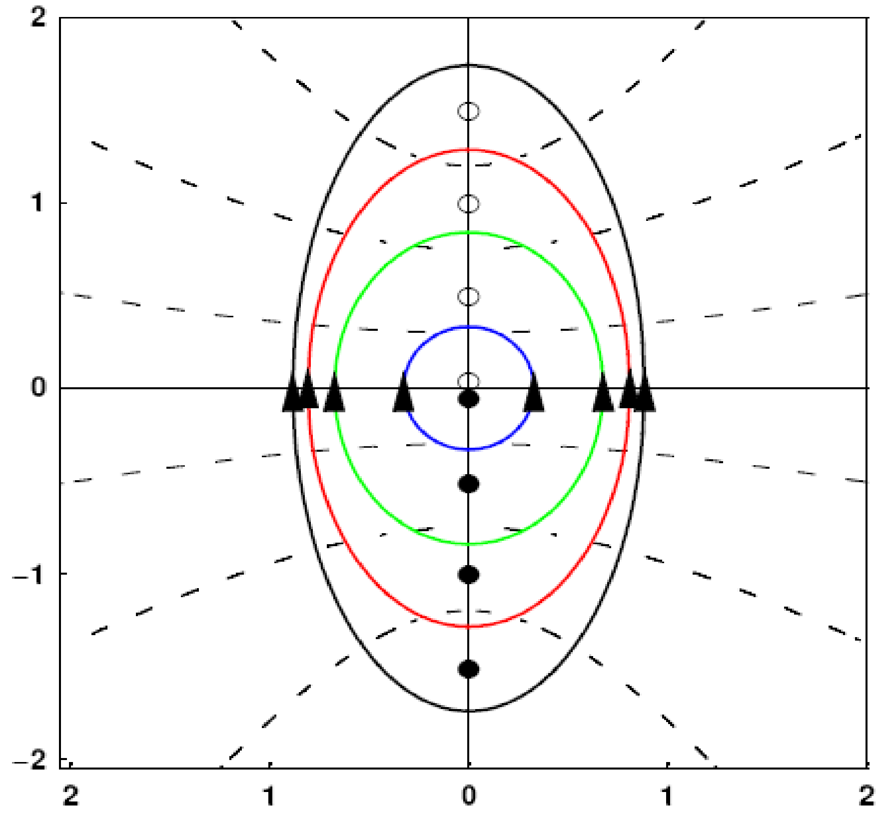
- Velocity embedding is fine:

$$u \longrightarrow \bar{\gamma}(1, u_\xi, 0, 0)$$

- Harder to reduce hydro eqns:

$$ds_\perp^2 = H_\xi^2 d\xi^2 + H_\eta^2 d\eta^2$$

$$H_\xi = a\frac{\sqrt{\xi^2 - \eta^2}}{\sqrt{\xi^2 - 1}}\quad ,\quad H_\eta = a\frac{\sqrt{\xi^2 - \eta^2}}{\sqrt{1 - \eta^2}}$$



ELLIPTIC FLOW –II

- At RHIC radial flow is way stronger than the elliptic

$$u^r = \frac{r}{R} u_o [1 + u_2 \cos(2\phi_s)] \Theta(R_o - r)$$

u_2 <<1

- Linearizing hydro equations on top of known analytic solutions with longitudinal and transverse radial flows

$$p = P_0(\tau, \rho) + \delta p(\tau, \rho, \phi)$$
$$v_\rho = V_\rho(\tau, \rho) + \delta v_\rho(\tau, \rho, \phi)$$
$$v_\phi = \delta v_\phi(\tau, \rho, \phi)$$

From known solutions

Furthermore: \phi-dependence is readily solved by Fourier decomposition and matching

---- THE END ----

Thank you !
Thanks to the organizers,
particularly Tamas and Marton!