AN EXACT HYDRODYNAMIC SOLUTION FOR THE ELLIPTIC FLOW

Emmanuel N. Saridakis

Nuclear and Particle Physics Section
Physics Department
University of Athens

In collaboration with Robi Peschanski

Institut de Physique Théorique, CEA,IPhT

Goal

- Understanding the dynamical mechanisms of elliptic flow and extraction of exact solutions

- Tools:

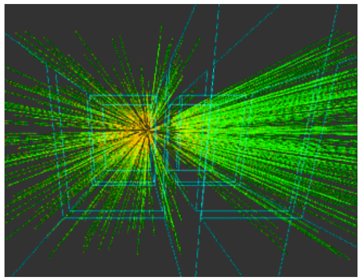
  The formulation of the transverse flow in terms of a hydrodynamic potential

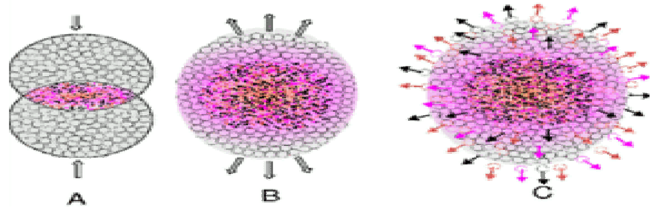
Talk Plan

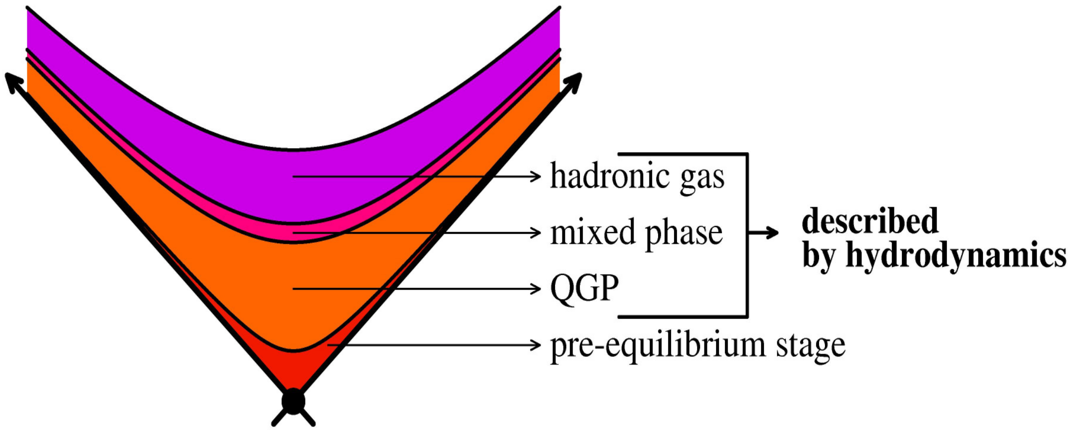
- 1) Introduction (why hydrodynamics?)
- 2) Assumptions and potential formulation of perfect fluid relativistic hydrodynamics
- 3) General and exact solution for spatial anisotropy and elliptic flow
- 4) Applications: Comparison with experiment and simulations
- 5) Assumptions Check
- 6) Conclusions-Prospects

Introduction

- Evidence that hydrodynamics of nearly perfect fluid may be relevant for the description of medium created in heavy-ion collisions







- Almost perfect fluid (low viscosity)

Hydrodynamic Equations of Motion

$$T^{\mu\nu} = (e+p)u^{\mu}u^{\nu} - p\eta^{\mu\nu}, \quad (u_{\nu}u^{\nu} = 1)$$

- E.O.M's : $\partial_{\mu} T_{\nu}{}^{\mu} = 0 \Rightarrow$

$$(e+p)u^{\mu}\partial_{\mu}u_{\nu} = -u_{\nu}u^{\mu}\partial_{\mu}p + \partial_{\nu}p$$

Assumptions

- 1) Quasi-stationary: Transverse flow smooth enough to be driven only by temperature change. Thus, Bernoulli equation:

$$Tu_0 = T_0$$

- 2) No explicit dependence on x_3
  (Transversally isentropic)

Final Hydrodynamic Equations

$$e + p = Ts\,,\;\; de = Tds\,,\;\; dp = sdT$$

- So:

$$\partial_{x_1}\left[(Ts)u_1 u_0\right] + \partial_{x_2}\left[(Ts)u_2 u_0\right] = 0$$

$$\partial_{x_1}\left[(Ts)u_1^2\right] + s\partial_{x_1}T + \partial_{x_2}\left[(Ts)u_1 u_2\right] = 0$$

$$\partial_{x_1}\left[(Ts)u_1 u_2\right] + s\partial_{x_2}T + \partial_{x_2}\left[(Ts)u_2^2\right] = 0$$

- Finally:

$$\partial_{x_1}\left(su_1\right) + \partial_{x_2}\left(su_2\right) = 0$$

$$\partial_{x_1}\left(Tu_2\right) - \partial_{x_2}\left(Tu_1\right) = 0$$

Hydrodynamic Potential $\chi(l,\varphi)$

$$\partial_{x_1}\left(Tu_2\right)-\partial_{x_2}\left(Tu_1\right)=0\Rightarrow\quad \begin{array}{l} Tu_2=\partial_{x_2}\Phi \\ Tu_1=\partial_{x_1}\Phi \end{array}$$

Hydrodynamic Potential $\chi(l,\varphi)$

$$\partial_{x_1}\left(Tu_2\right)-\partial_{x_2}\left(Tu_1\right)=0 \Rightarrow \quad \begin{matrix} Tu_2=\partial_{x_2}\Phi \\ Tu_1=\partial_{x_1}\Phi \end{matrix}$$

- Hodograph Transform:

(kinematical variables) $\rightarrow$ (thermodynamical variables)

$$(x_1, x_2) \rightarrow (l, \varphi)$$

$$l = \frac{1}{2}\log\left[1-\left(\frac{T}{T_0}\right)^2\right] = \frac{1}{2}\log\left[\frac{u_\perp^2}{1+u_\perp^2}\right] \qquad u_\perp^2 \equiv u_1^2 + u_2^2$$

$$\tan\varphi = \frac{u_2}{u_1}$$

Khalatnikov-Kamenshchik equation

$$\left(1-\frac{e^{2l}}{c_s^2}\right)\frac{\partial^2\chi}{\partial\varphi^2}+\left(1-e^{2l}\right)\frac{\partial^2\chi}{\partial l^2}+\left(1-\frac{1}{c_s^2}\right)e^{2l}\frac{\partial\chi}{\partial l}=0$$

- Solvable!

$$c_s^2=\frac{dp}{d\varepsilon}=\frac{sdT}{Tds}$$

Potential equation

$$\left(1-\frac{e^{2l}}{c_s^2}\right)\frac{\partial^2\chi}{\partial\varphi^2}+\left(1-e^{2l}\right)\frac{\partial^2\chi}{\partial l^2}+\left(1-\frac{1}{c_s^2}\right)e^{2l}\frac{\partial\chi}{\partial l}=0$$

- Solvable!

$$c_s^2=\frac{dp}{d\varepsilon}=\frac{sdT}{Tds}$$

$$x_1^2+x_2^2=\frac{e^{-2l}}{T_0^2}\left[\left(\frac{\partial\chi}{\partial l}\right)^2+\left(\frac{\partial\chi}{\partial\varphi}\right)^2\right]$$

$$\arctan\frac{x_2}{x_1}=\varphi+\arctan\left[\frac{\partial\chi}{\partial\varphi}\left(\frac{\partial\chi}{\partial l}\right)^{-1}\right]$$

- (1+1) [Khalatnikov, 1954]
- Transverse (cosmology) [Khalatnikov-Kamenshchik, 2004]

Comparison with longitudinal case

- Now (Quasi-stationary, transverse-isentropic):

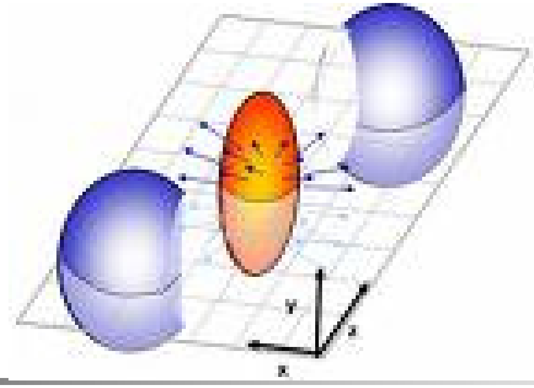
$$\left(1-\frac{e^{2l}}{c_s^2}\right)\frac{\partial^2\chi}{\partial\varphi^2}+\left(1-e^{2l}\right)\frac{\partial^2\chi}{\partial l^2}+\left(1-\frac{1}{c_s^2}\right)e^{2l}\frac{\partial\chi}{\partial l}=0$$

- (1+1) (longitudinal):

$$\frac{\partial^2\chi(\theta,y)}{\partial y^2}-c_s^2\frac{\partial^2\chi(\theta,y)}{\partial\theta^2}-[1-c_s^2]\frac{\partial\chi(\theta,y)}{\partial\theta}=0$$

θ : 'Temperature'
y : Rapidity

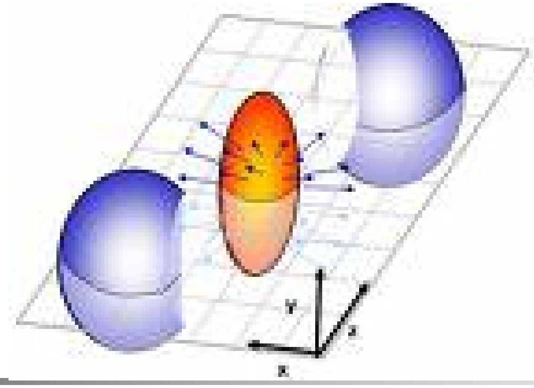
Elliptic Flow



- Cause $\Rightarrow$ Effect

(Initial Spatial Anisotropy) (Entropy Anisotropy)

Elliptic Flow



- **Cause** $\Rightarrow$ **Effect**

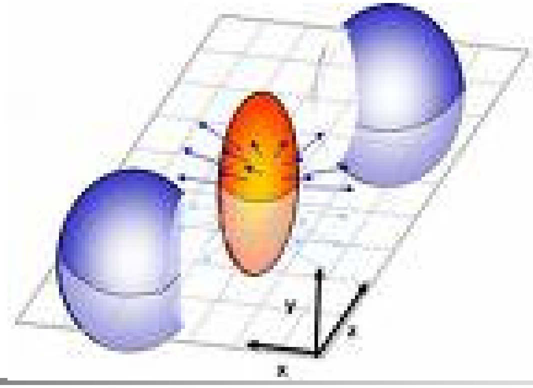
(Initial Spatial Anisotropy) (Entropy Anisotropy)

Eccentricity

$$\varepsilon = \frac{\langle x_2^2 - x_1^2 \rangle}{\langle x_2^2 + x_1^2 \rangle}$$

Elliptic Flow coefficient

$$v_2 = \frac{\int d\varphi \cos(2\varphi) \frac{dS}{d\varphi}}{\int d\varphi \frac{dS}{d\varphi}}$$



Elliptic Flow

- **Cause** $\Rightarrow$ **Effect**

(Initial Spatial Anisotropy) (Entropy Anisotropy)

Eccentricity

$$\varepsilon = \frac{\left\langle x_2^2 - x_1^2 \right\rangle}{\left\langle x_2^2 + x_1^2 \right\rangle}$$

$$\varepsilon = \frac{\int d\varphi \left\{ \cos 2\varphi \left[\left(\frac{\partial \chi}{\partial \varphi} \right)^2 - \left(\frac{\partial \chi}{\partial l} \right)^2 \right] + 2 \sin 2\varphi \frac{\partial \chi}{\partial \varphi} \frac{\partial \chi}{\partial l} \right\}}{\int d\varphi \left[\left(\frac{\partial \chi}{\partial \varphi} \right)^2 + \left(\frac{\partial \chi}{\partial l} \right)^2 \right]}$$

Elliptic Flow coefficient

$$v_2 = \frac{\int d\varphi \cos(2\varphi) \frac{dS}{d\varphi}}{\int d\varphi \frac{dS}{d\varphi}}$$

$$\frac{dS}{d\varphi} = \frac{sT}{T_0 \left(1 - e^{2l} / c_s^2\right)} \left[\frac{\partial \chi}{\partial l} - \frac{\partial^2 \chi}{\partial l^2} \right]$$

[E.N.S, R.Peschanski '09]

General and exact solution

$$\chi(l,\varphi)=c_0\beta_0(l)+\sum_{p=1}^{\infty}\beta_p(l)\cos(2p\varphi)$$

$$\beta_0'(l)=\left(1-e^{2l}\right)^{\frac{1-\frac{1}{c_s^2}}{2}}$$

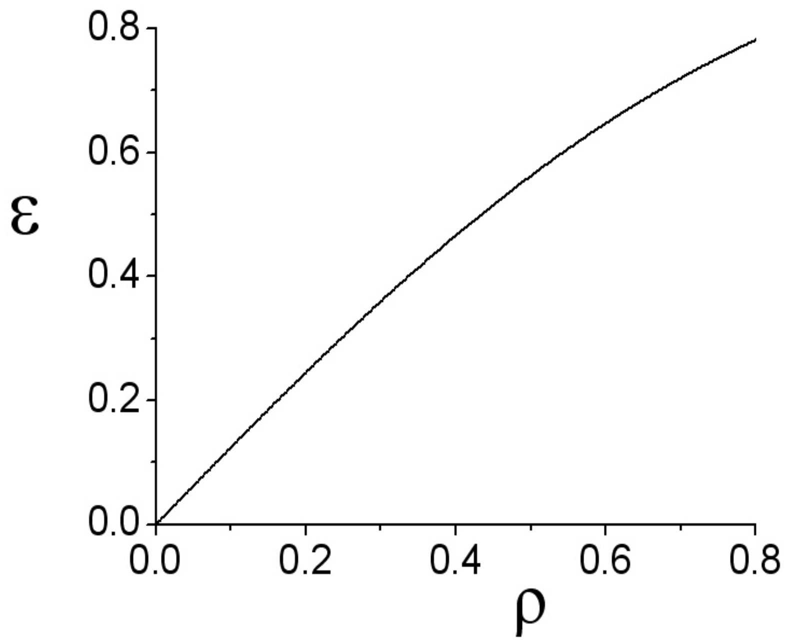
$$\beta_p(l)=c_p^{(1)}(-1)^{p+1}e^{2pl}\;{}_2F_1\left(p+\frac{1}{4}\left(c_s^{-2}-1\right)-\sqrt{\frac{\left(c_s^{-2}-1\right)^2}{16}+p^2c_s^{-2}},\,p+\frac{1}{4}\left(c_s^{-2}-1\right)+\sqrt{\frac{\left(c_s^{-2}-1\right)^2}{16}+p^2c_s^{-2}},1+2p;e^{2l}\right)+$$

$$+c_p^{(2)}\;G_{2,2}^{2,0}\left(e^{2l}\left|\begin{matrix}\frac{5-c_s^{-2}}{4}-\sqrt{\frac{\left(c_s^{-2}-1\right)^2}{16}+p^2c_s^{-2}}, & \frac{5-c_s^{-2}}{4}+\sqrt{\frac{\left(c_s^{-2}-1\right)^2}{16}+p^2c_s^{-2}}\\ -p & p\end{matrix}\right.\right)$$

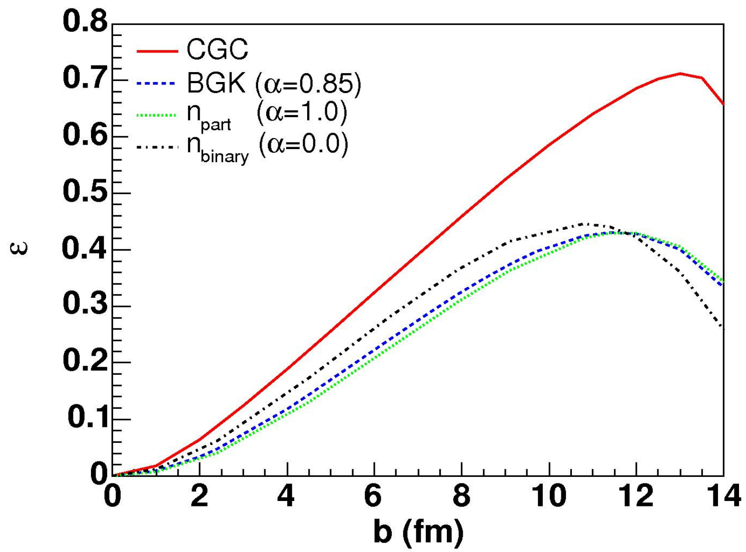
$$\frac{c_1^{(1)}}{c_0}\equiv\rho \qquad \frac{c_1^{(2)}}{c_1^{(1)}}\equiv\lambda$$

Centrality dependence ('Callibration')

- For fixed $T \Rightarrow \dfrac{\rho}{\rho_{\max}} \propto \dfrac{b}{b_{\max}} \propto 1-c \propto 1-\dfrac{N}{N_{\max}}$



[E.N.S, R.Peschanski '09]

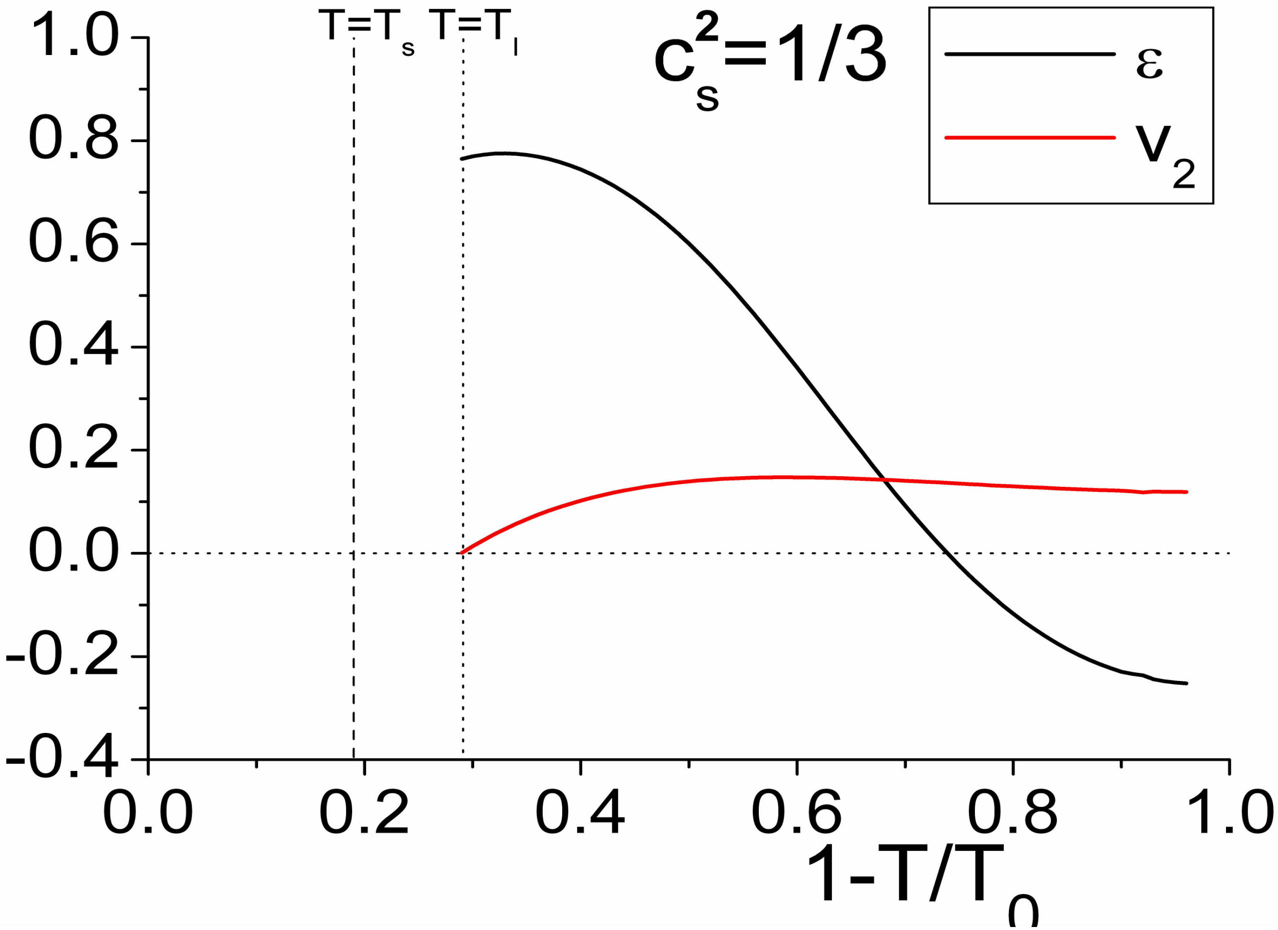


[Hirano, *et al* '06]

Initial conditions

- Source at a given temperature T_I
- $$\varepsilon(T_I) = \max$$ (fixes ρ)
- $$v_2(T_I) = 0$$ (fixes λ)

Temperature dependence

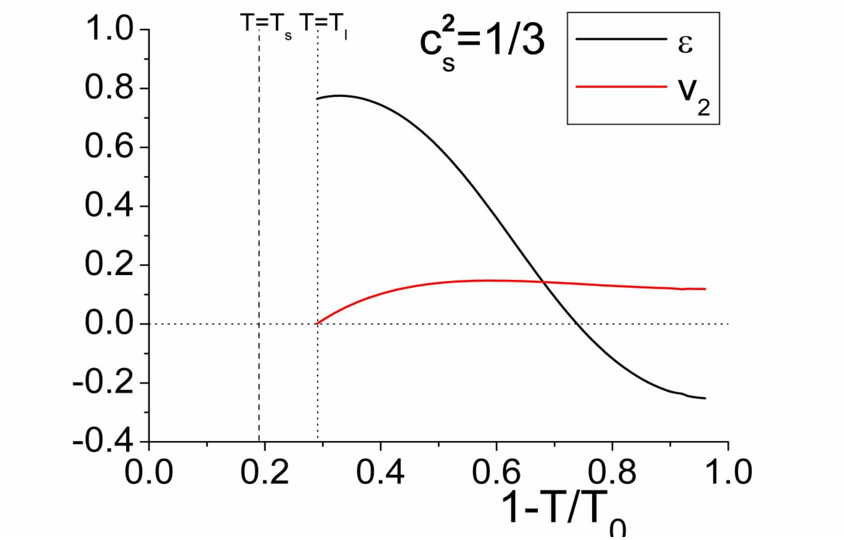


Time dependence

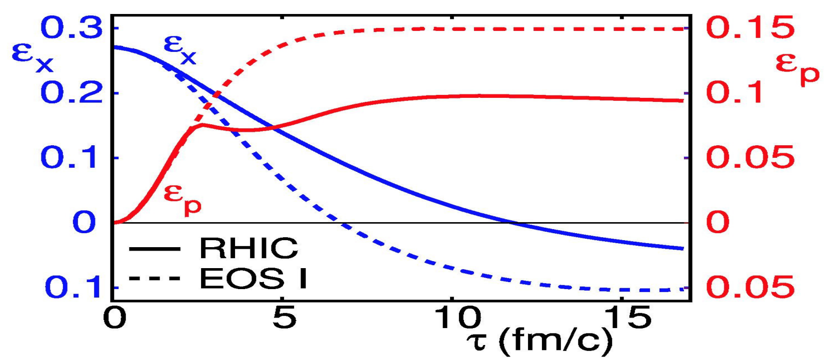
- Temperature evolution $\Leftrightarrow$ Time evolution

$$\frac{T}{T_0} = \left(\frac{\tau_0}{\tau}\right)^{c_s^2}$$

$$\tau \equiv \sqrt{x_0^2 + x_3^2}$$



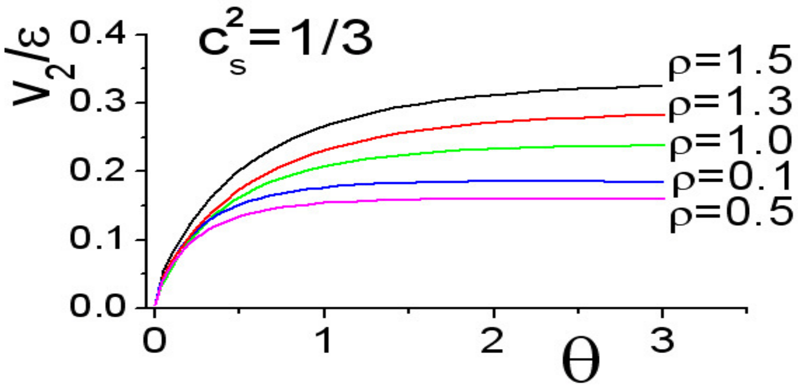
[E.N.S, R.Peschanski '09]

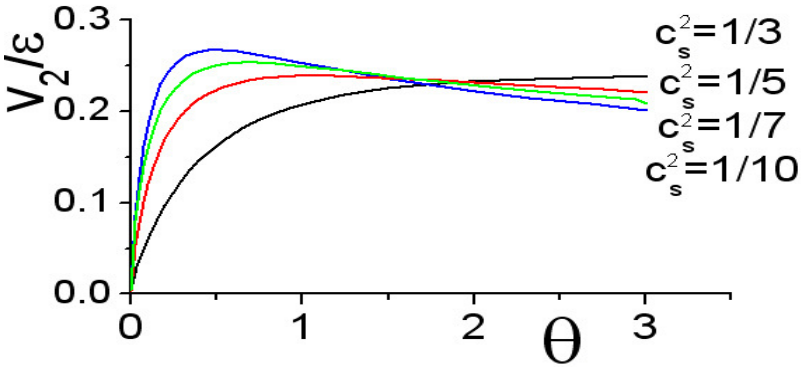


[Kolb, Heinz '03]

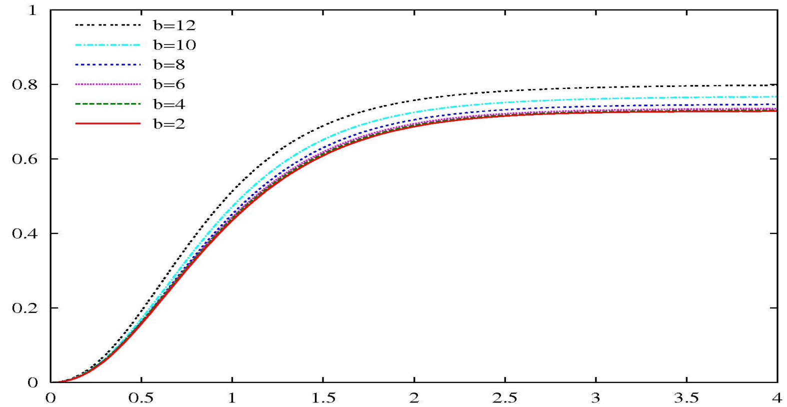
Time dependence

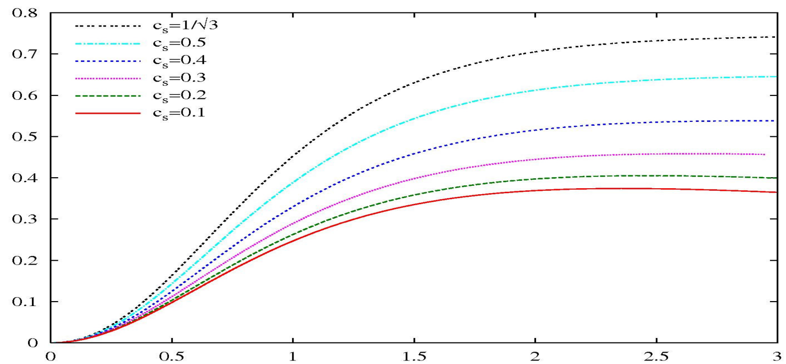
$$\theta \equiv \left(\frac{T_0}{T}\right)^{c_s^{-2}} - \left(\frac{T_0}{T_I}\right)^{c_s^{-2}}$$





[E.N.S, R.Peschanski '09]





[Bhalerao, *et al* '05]

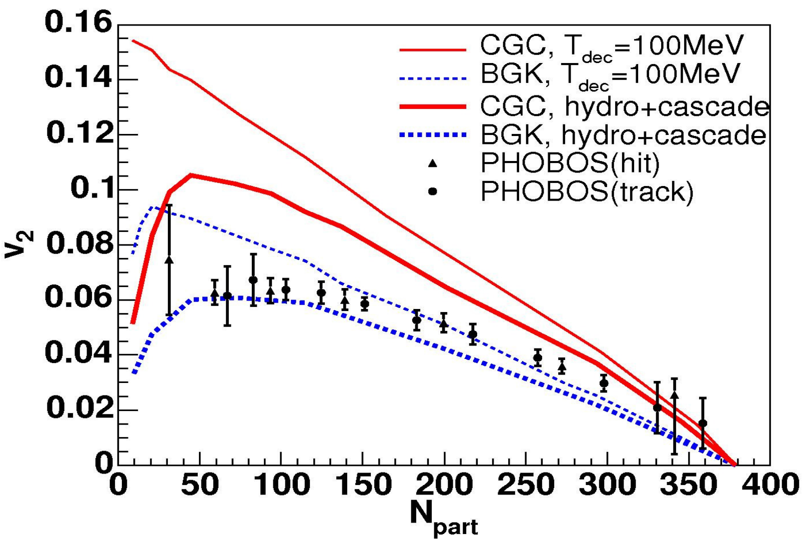
Centrality dependence

- For fixed T,

$$\frac{\rho}{\rho_{\max}} \propto 1-c \propto 1-\frac{N}{N_{\max}}$$

$$\Rightarrow v_{2final} \propto \rho \propto 1-c$$

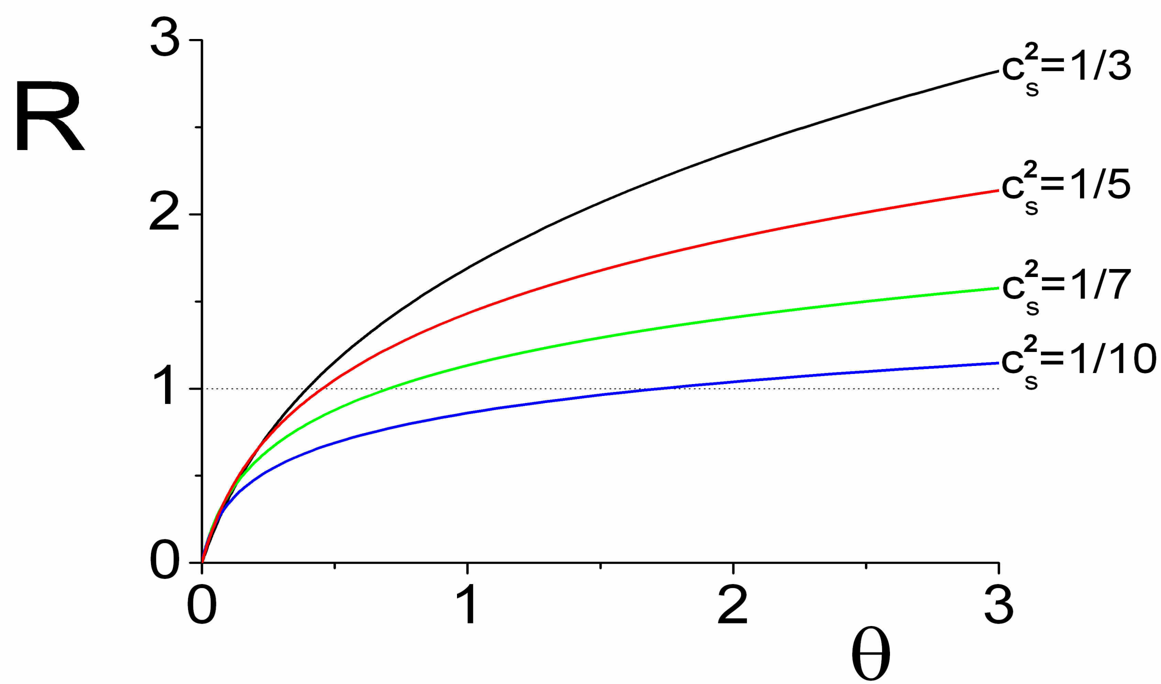
[E.N.S, R.Peschanski '09]



[Hirano, *et al* '06]

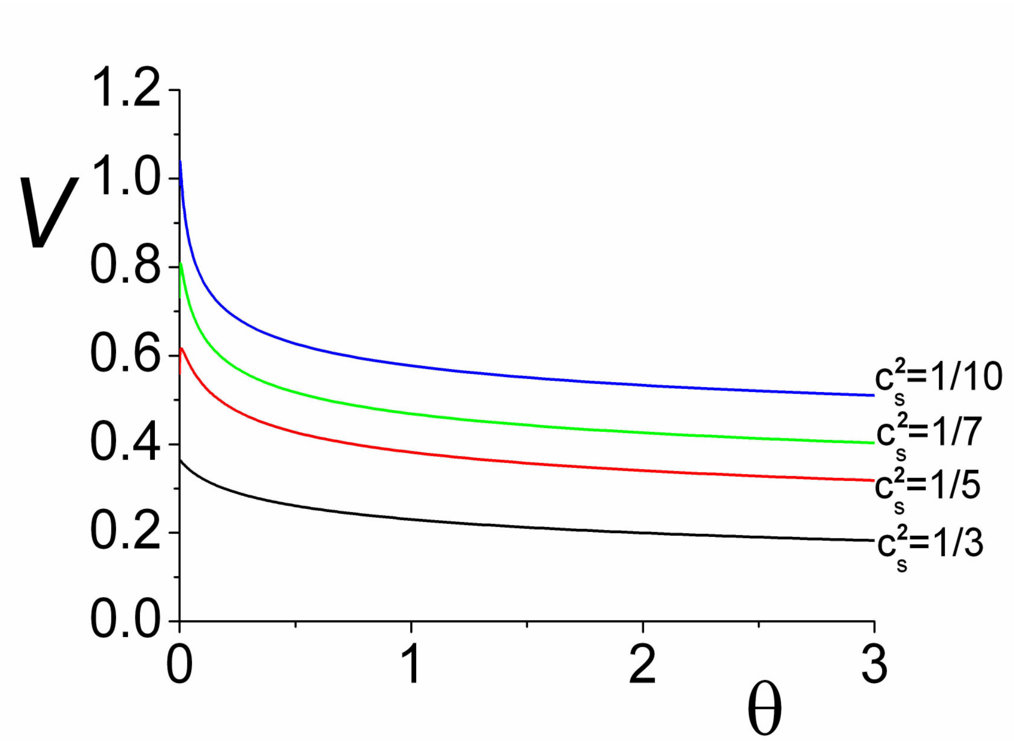
Assumptions check

- 1) Transversally isentropic: $R = \frac{\partial_{x_\perp}(su_\perp)}{\partial_\tau(su_0)} >> 1$ Transverse over time typical entropy gradient



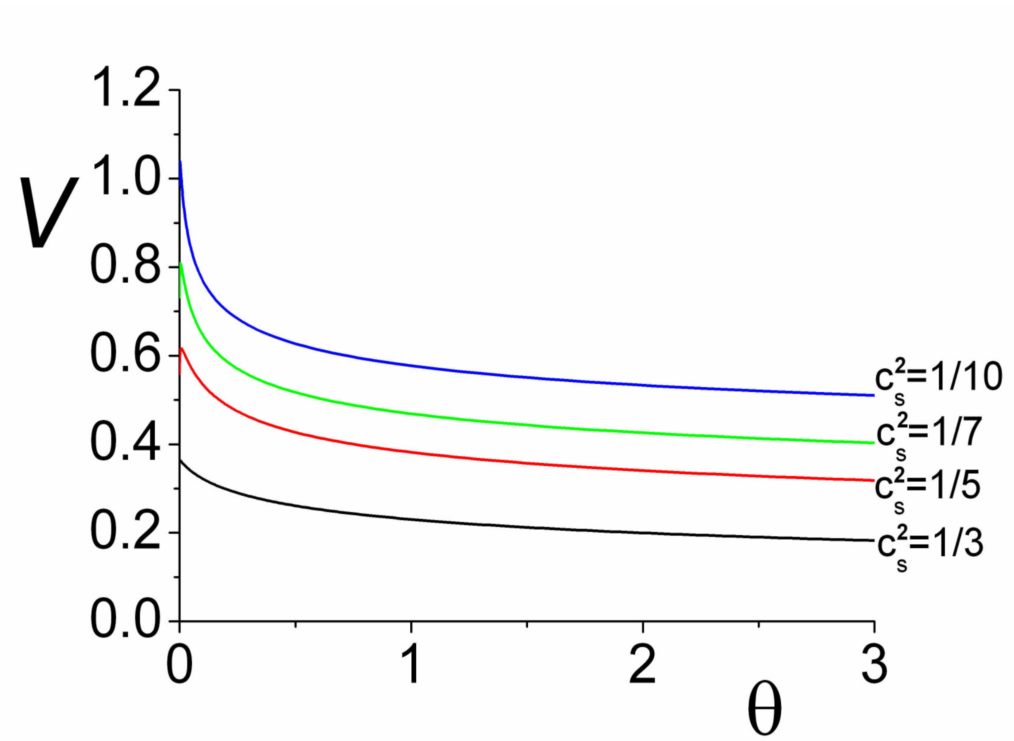
Assumptions check

- 2) Quasi-stationarity: $V = \dfrac{\partial_T \; x_\perp(T)}{\partial_T \; \tau(T)} << 1$ Transverse over time, temperature-dependent expansion rate



Assumptions check

- 2) Quasi-stationarity: $V = \dfrac{\partial_T \ x_\perp(T)}{\partial_T \ \tau(T)} << 1$ Transverse over time, temperature-dependent expansion rate



$$R = \frac{\partial_{x_\perp}(su_\perp)}{\partial_\tau(su_0)} = \frac{\dfrac{\partial_T(su_\perp)}{\partial_T(su_0)}}{\dfrac{\partial_T \ x_\perp(T)}{\partial_T \ \tau(T)}} = \frac{1}{V}\frac{\partial_T(su_\perp)}{\partial_T(su_0)}$$

The two assumptions are connected

Conclusions

- i) Hydrodynamic potential χ allows to bypass the non-linearities of the initial equations.

- ii) Under transverse isentropicity and quasi-stationarity we obtain the general solution for spatial eccentricity and elliptic flow coefficient.

- iii) Time-evolution is acquired through temperature-evolution.

- iv) Assumptions are verified qualitatively, but not exactly

Outlook

- i) Describe the $p_\perp$-dependence of elliptic flow.
- ii) Extend to more physical initial conditions (beyond fixed-temperature)
- iii) Extend to weak viscosity.
- iv) The rather simple mechanisms, may facilitate AdS/CFT approach of elliptic flow.