The Infrared Structure of Scalar EFTs
Based on 1709.08639, 1804.08629, 1810.XXXXX

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CCWZ formalism of NLSM

In general, $G$ broken to $H$,

- Unbroken generators $T^i$, associated with unbroken group $H$
- Broken generators $X^a$, associated with coset $G/H$
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Callan, Coleman, Wess & Zumino:

$$-i\xi^{-1}\partial_\mu\xi = d_\mu + E_\mu = D_\mu \pi^a X^a + E^i_\mu T^i, \quad \xi = \exp[i\pi^a X^a/f],$$

$$d_\mu \rightarrow h d_\mu h^\dagger, \quad E_\mu \rightarrow h E_\mu h^\dagger - ih\partial_\mu h^\dagger$$

$$\mathcal{L} = \frac{f^2}{2}D_\mu \pi^a D^\mu \pi^a + O(\partial^4).$$

The shift symmetry and Adler’s zero

The transformation of the Goldstones:

\[ \pi^a \rightarrow \pi^a + \varepsilon^a + iT^i_{ab} \alpha^i \pi^b + \cdots \]

Current under the shift symmetry contains a one-particle pole:

\[ \langle \Omega | J^\mu(x) | \pi(p) \rangle = ip^\mu e^{-ip \cdot x}, \]

Current conservation leads to Adler’s zero condition:

\[ \partial_\mu \langle f | J^\mu | i \rangle = 0, \quad \langle f + \pi(p) | i \rangle = p_\mu R^\mu(p). \]

Trivial case: \( G = U(1) \)

\[
\pi \rightarrow \pi + \varepsilon, \quad \mathcal{L} = \frac{1}{2} (\partial \pi)^2 + \mathcal{O}(\partial^4).
\]
Nonlinearity from IR

Trivial case: \( G = U(1) \)

\[ \pi \rightarrow \pi + \varepsilon, \quad \mathcal{L} = \frac{1}{2} (\partial \pi)^2 + \mathcal{O}(\partial^4). \]

General case: several assumptions
- Nonlinear transformation of \( \pi \) is a field-dependent \( H \) rotation:
  \[ |\pi\rangle \rightarrow |\pi\rangle + |\varepsilon\rangle + i \alpha^i(\varepsilon, \pi) T^i |\pi\rangle \]
- Goldstone covariant derivative \( |\mathcal{D}\pi\rangle \) transform as a field-dependent \( H \) rotation:
  \[ |\mathcal{D}\pi\rangle \rightarrow e^{iu^i(\varepsilon, \pi) T^i/f} |\mathcal{D}\pi\rangle. \]
- All of the above can be reduced to the trivial case.
Universal nonlinearity

Solve $\alpha$, $u$ and $D$ simultaneously.

$$D_\mu \pi^a = \left( \frac{\sin \sqrt{T}}{\sqrt{T}} \right)_{ab} \partial_\mu \pi^b,$$

$$T \equiv \frac{1}{f^2} \langle T^i \pi \rangle \langle \pi T^i \rangle.$$
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Then the Lagrangian is

$$\mathcal{L} = \frac{f^2}{2} D_\mu \pi^a D^\mu \pi^a + \mathcal{O}(\partial^4),$$

Low, 1412.2145.

$E_\mu$ can be solved similarly.
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- Structure of nonlinear interaction independent of $G$ at UV
Universal nonlinearity

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- Structure of nonlinear interaction independent of $G$ at UV
- $G$ dependence all included in the value of $f$
Universal nonlinearity

Solve $\alpha$, $u$ and $D$ simultaneously.

$$D_\mu \pi^a = \left( \frac{\sin \sqrt{T}}{\sqrt{T}} \right)_{ab} \partial_\mu \pi^b, \quad T \equiv \frac{1}{f^2} |T_i \pi \rangle \langle \pi | T^i |. \quad \text{Then the Lagrangian is}$$

$$\mathcal{L} = \frac{f^2}{2} D_\mu \pi^a D^\mu \pi^a + \mathcal{O}(\partial^4),$$

Low, 1412.2145.

$E_\mu$ can be solved similarly.

- Structure of nonlinear interaction independent of $G$ at UV
- $G$ dependence all included in the value of $f$
- Can construct interactions of higher orders in momentum, with implications for composite Higgs models

Liu, Low, Yin, 1805.00489, 1809.09126.
Subleading single soft theorem for NLSM

Discovered using the CHY formalism:
When $p_{n+1} \to \tau p_{n+1}$ and $\tau \to 0$,

$$M_{n+1}^{\text{NLSM}}(\mathbb{1}_{n+1}) = \tau \sum_{i=2}^{n-1} s_{n+1,i} M_{n}^{\text{NLSM} \oplus \phi^3}(\mathbb{1}_n|1, n, i) + \mathcal{O}(\tau^2),$$

Cachazo, Cha, Mizera, 1604.03893.

Biadjoint scalars $\phi^{a\tilde{a}}$:
- Transform under $SU(N) \times SU(\tilde{N})$
- Self interaction: $f^{abc} f^{\tilde{a}\tilde{b}\tilde{c}} \phi^{a\tilde{a}} \phi^{b\tilde{b}} \phi^{c\tilde{c}}$
Ingredients for the Ward identity

IR construction:

\[ |\pi\rangle \rightarrow |\pi\rangle + |\varepsilon\rangle + i\alpha^i(\varepsilon, \pi) |T^i\pi\rangle, \]
\[ |D_\pi\rangle \rightarrow e^{iu^i(\varepsilon, \pi)T^i/f} |D_\pi\rangle. \]

\(\alpha^i(\varepsilon, \pi), u^i(\varepsilon, \pi)\) and \(|D_\pi\rangle\) are solved simultaneously.

\[
D_{\mu}\pi^a = \left( \frac{\sin \sqrt{T}}{\sqrt{T}} \right)_{ab} \partial_\mu \pi^b, \\
\pi^a \rightarrow \pi^a + \left( \sqrt{T} \cot \sqrt{T} \right)_{ab} \varepsilon^b.
\]

The current:

\[
J_\mu^a = \left[ \frac{\sin \sqrt{T} \cos \sqrt{T}}{\sqrt{T}} \right]_{ab} \partial_\mu \pi^b = \partial_\mu \pi^a + \sum_{k=1}^{\infty} \frac{(-4)^k}{(2k + 1)!} (T^k)_{ab} \partial_\mu \pi^b.
\]

Low, Yin, 1709.08639; 1804.08629.
Tree level: subleading single soft theorem

\[ M(\Pi_n) = -i \sum_{k=1}^{[n/2]} \sum_{\{l_m\}} V_{\Pi_{2k+1}}(q_l, \cdots, q_{l_{2k+1}}) \prod_{m=1}^{2k+1} J(l_{m-1} + 1, \cdots, l_m), \]

\[ V(\Pi_{2k+1}) = \frac{-i(-4)^k}{(2k + 1)! f^{2k}} \sum_{j=1}^{2k-1} \left[ \binom{2k}{j} (-1)^j - 1 \right] q \cdot p_{j+1}. \]

Compare with the CHY results:

\[ M_n^{\text{NLSM}}(\Pi_n) = \tau \sum_{i=2}^{n-2} s_{n,i} M_n^{\text{NLSM} \oplus \phi^3}(\Pi_{n-1} | 1, n-1, i) + \mathcal{O}(\tau^2) \]
The Lagrangian of the extended theory

\[ \mathcal{L}^{\text{NLSM} \oplus \phi^3} = \frac{f^2}{8(N^2 - 1)} \text{Tr} \left\{ \partial_\mu U^{-1} (\phi + \pi) \partial^\mu U (\phi + \pi) \right\} \]

\[ + \frac{1}{3} \lambda \phi^{\alpha \bar{\alpha}} \phi^{\beta \bar{\beta}} \phi^{\gamma \bar{\gamma}} f^{abc} f^{\tilde{a} \tilde{b} \tilde{c}} + \frac{6i f^3 \lambda}{(N^2 - 1)^{3/2}} \]

\[ \times \text{Tr}\{ [U(\phi + \pi) - U(\pi)][U(\phi - \pi) - U(-\pi)] \]

\[ - [U(-\phi - \pi) - U(-\pi)][U(\pi - \phi) - U(\pi)] \} \]

where \( U(x) = e^x \),

\[
\begin{align*}
(\pi)_{ab,\bar{a}\bar{b}} & \equiv \left( \frac{2i}{f} \right) \pi^c (T^c)_{ab} \delta_{\bar{a}\bar{b}}, \\
(\phi)_{ab,\bar{a}\bar{b}} & \equiv \left( \frac{2i}{f} \right) \sqrt{N^2 - 1} \phi^{\gamma \bar{\gamma}} (T^c)_{ab} (T^\tilde{c})_{\bar{a}\bar{b}}. 
\end{align*}
\]
Exceptional scalar EFTs

\[ L \supset \partial^m \phi^n, \quad \rho = \frac{m - 2}{n - 2}, \quad M(\tau q) = \mathcal{O}(\tau^\sigma) \]

- Simple CHY representations
- Enhanced shift symmetry
  - DBI:
    \[ \delta \pi = \theta_\mu \left( x^\mu - F^{-d} \pi \partial^\mu \pi \right) \]
  - sGal:
    \[ \delta \pi = \theta^{\mu \nu} \left( \alpha^2 x_\mu x_\nu + \partial_\mu \pi \partial_\nu \pi \right) \]

Cheung et al., 1611.03137.
Extended theory of sGal

CHY gives

\[ M_{sGal}^n = \tau^3 \sum_{a=2}^{n-2} \sum_{c=2}^{n-1} \sum_{d=1}^{n-2} \sum_{c \neq a \neq d} s_{an} s_{cn} s_{dn} M_{sGal}^{\oplus NLSM^2 \oplus \phi^3} (a, c, 1 \mid n - 1, d, a) \]

\[ + \mathcal{O}(\tau^4). \]

Result from the Ward identity:

\[ M_{sGal}^n = \tau^3 \left\{ \sum_{l} \frac{2}{\alpha^2} \rho_{n,l^1} \rho_{n,l^2} \rho_{n,l^3} \prod_{i=1}^{3} J(\{p_{l^i}\}) \right\} \]

\[ - \sum_{l} \frac{1}{6\alpha^4} \sum_{\sigma \in S_5} \rho_{n,l^{\sigma(1)}} \rho_{n,l^{\sigma(2)}} \rho_{n,l^{\sigma(3)}} \left[ q_{l^{\sigma(4)}}^2 q_{l^{\sigma(5)}}^2 - p_{l^{\sigma(4)},l^{\sigma(5)}}^2 \right] \prod_{i=1}^{5} J(\{p_{l^i}\}) \]

\[ + \mathcal{O}(\tau^4). \]
Conclusion

- NLSM can be constructed from IR constraints
- Interesting soft limits beyond Adler’s zero can be derived using the Ward identity
- The Lagrangian of the extended theory can be identified
Backup Slides
The QCD Lagrangian with two light quarks:

\[
\mathcal{L} \supset i \bar{u}^L \not\!\!\!D u^L + i \bar{u}^R \not\!\!\!D u^R + i \bar{d}^L \not\!\!\!D d^L + i \bar{d}^R \not\!\!\!D d^R
\]

\(SU(2)_L \times SU(2)_R\) chiral symmetry:

\[
\begin{pmatrix}
  u^L \\
  d^L
\end{pmatrix} \rightarrow g_L \begin{pmatrix}
  u^L \\
  d^L
\end{pmatrix}, \quad \begin{pmatrix}
  u^R \\
  d^R
\end{pmatrix} \rightarrow g_R \begin{pmatrix}
  u^R \\
  d^R
\end{pmatrix}
\]

The chiral symmetry is spontaneously broken:

\[
\langle \bar{u} u \rangle = \langle \bar{d} d \rangle = V^3,
\]

\(SU(2)_L \times SU(2)_R \rightarrow SU(2)_V\).

Given the symmetry breaking pattern, we can describe the Nambu-Goldstone Bosons without QCD.
The Sigma model

Introducing the scalar $\Sigma_{ij}$ that transform under $SU(2)_L \times SU(2)_R$:

$$\mathcal{L} = |\partial_\mu \Sigma|^2 + m^2 |\Sigma|^2 - \frac{\lambda}{4} |\Sigma|^4.$$
The Sigma model

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$$\mathcal{L} = |\partial_\mu \Sigma|^2 + m^2 |\Sigma|^2 - \frac{\lambda}{4} |\Sigma|^4.$$

Decouple the radial part and only keep the NGBs:

$$\Sigma \rightarrow U = \exp \left[ 2i \frac{\pi^a T^a}{F_\pi} \right]$$

NLSM:

$$\mathcal{L} = \frac{F_\pi^2}{4} \text{Tr} \left[ D_\mu U (D_\mu U)^\dagger \right] + \mathcal{O}(D^4).$$
Nonlinear transformation

\[ U \rightarrow g_L U g_R^\dagger, \quad \pi^a \rightarrow \pi^a + \epsilon^a + iT_{ab}^i \alpha^i \pi^b + \cdots \]
For there to be Adler’s zero, the remainder $R$ needs to be regular: no “pole digrams” should appear.
The “closure condition” is necessary for solutions to exist in the IR construction of NLSM:

\[
(T^i)_{ab}(T^i)_{cd} + (T^i)_{ac}(T^i)_{db} + (T^i)_{ad}(T^i)_{bc} = 0.
\]
Flavor-ordering:

\[ M^{a_1 \cdots a_n}(p_1, \cdots, p_n) \]
\[ \equiv \sum_{\sigma \in S_{n-1}} \text{Tr}(X^{a_n}X^{a_{\sigma(1)}} \cdots X^{a_{\sigma(n-1)}}) M_{\sigma}(p_1, \cdots, p_{n-1}), \]
The CHY representation:

\[ M_n^{NLSM \oplus \phi^3}(\alpha|\beta) = \oint d\mu_n C_n(\alpha)\left(C(\beta)(Pf A_{\beta})^2\right). \]

Compared to pure NLSM:

\[ M_n^{NLSM}(\alpha) = \oint d\mu_n C_n(\alpha)(Pf' A_n)^2. \]

Cachazo, He, Yuan, 1412.3479.

Biadjoint scalars interacting with Goldstones. How?
Subleading single soft from Ward identity

\[ \partial_\mu \langle f | J_\mu | i \rangle = 0, \quad \langle f + \pi(p) | i \rangle = p_\mu R^\mu(p). \]

To go beyond Adler’s zero, simply need to calculated \( R \).
- Current algebra: hard to go beyond leading order in \( 1/f \).
- CCWZ: seemingly different for different coset.
- Do not have anything to relate to.

Compared with single soft theorem in gauge theory:

\[
M_{n+1}^{a_1a_2\cdots a_n+1} = \sum_{i=1}^{n} igf^{a_ia_{n+1}b} \left( \frac{1}{\tau} \frac{p_i \cdot \varepsilon_{n+1}}{p_i \cdot p_{n+1}} - i \frac{\varepsilon_{n+1}^\nu p^\rho_{n+1}}{p_i \cdot p_{n+1}} J_{i\nu\rho} \right) \\
\times M_n^{a_1\cdots a_{i-1}ba_{i+1}\cdots a_n} + \mathcal{O}(\tau)
\]
Derivation of the current

\[ \mathcal{J}_\mu^a = \left[ \frac{\sin \sqrt{\mathcal{T}} \cos \sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}} \right]^{ab} \partial_\mu \pi^b = \partial_\mu \pi^a + \sum_{k=1}^{\infty} \frac{(-4)^k}{(2k + 1)!} (\mathcal{T}^k)_{ab} \partial_\mu \pi^b. \]

The Ward identity:

\[ i \partial^\mu \langle \Omega | \mathcal{J}_\mu^a(x) \prod_{i=1}^n \pi^{a_i}(x_i) | \Omega \rangle \]

\[ = \sum_{r=1}^n \langle \Omega | \pi^{a_1}(x_1) \cdots \left[ \sqrt{\mathcal{T}} \cot \sqrt{\mathcal{T}} \right]_{a_r a_r} (x_r) \delta^{(4)}(x - x_r) \cdots \pi^{a_n}(x_n) | \Omega \rangle. \]
Ward identity/soft theorem

\[ M^{a_1 \cdots a_n}(p_1, \cdots, p_n, q) = q \cdot R^{a_1 \cdots a_n}(p_1, \cdots, p_n; q), \]

The remainder function \( R \) is regular:

\[ R^{a_1 \cdots a_n}_{\mu}(p_1, \cdots, p_n; q) = \frac{1}{\sqrt{Z}} \sum_{k=1}^{\infty} \frac{-i(-4)^k}{(2k + 1)!} \langle 0 | \int d^4x \: e^{-iq \cdot x} [T^k(x)]_{ab} \: \partial_\mu \pi^b(x) | \pi^{a_1} \cdots \pi^{a_n} \rangle. \]

Inserting the current twice: leading double soft theorem
Semi-on-shell amplitudes:

\[ J^{a_1 \cdots a_n, a}(p_1, \cdots, p_n) = \langle 0 | \pi^a(0) | \pi^{a_1}(p_1) \cdots \pi^{a_n}(p_n) \rangle . \]

\[ q^2 J(\Pi_n) = i \sum_{k=1}^{[n/2]} \sum_{\{l_m\}} V_{\Pi_{2k+1}}(q_{l_1}, \cdots, q_{l_{2k+1}}) \prod_{m=1}^{2k+1} J(l_{m-1} + 1, \cdots, l_m) \]
$M_5^{\text{NLSM} \oplus \phi^3} (\mathbb{I}_5 | 1, 5, 2)$

$M_5^{\text{NLSM} \oplus \phi^3} (\mathbb{I}_5 | 1, 5, 3)$
Identification of Feynman vertices

- Flavor conservation: no vertex with one $\phi$
- Soft constraint: $\phi^2$ vertices exactly the same as NLSM
- The structure of the current leads to $\phi^3$ vertices:

$$\mathcal{V}^{\text{NLSM}} \oplus \phi^3 (\mathbb{I}_{2k+1}|1, 2k + 1, j) = i \frac{(-4)^k}{2 (2k + 1)! f^{2k}} \left[ \binom{2k}{j - 1} (-1)^{j-1} - 1 \right].$$

Recovers the self-interaction of $\phi$. 
Cayley parameterization

Field redefinition: $\pi \rightarrow \pi + O(\pi^2)$.

$$\mathcal{L} = \frac{f^2}{8} \text{Tr} \left( \partial_\mu U \partial^\mu U^{-1} \right), \quad U = \frac{1 + i\pi^a T^a/f}{1 - i\pi^a T^a/f} = 1 + 2 \sum_{k=1}^{\infty} \left( \frac{i}{f} \pi^a T^a \right)^k$$

$$V(\mathbb{I}_{2k+2}) = \frac{i(-1)^k}{f^{2k}} \left( \sum_{i=0}^{k} p_{2i+1} \right)^2$$

$$V^{\text{NLSM} \oplus \phi^3}(\mathbb{I}_{2k+1}|1, 2k+1, j) = \begin{cases} \frac{i(-1)^k}{f^{2k}} & \text{for even } j, \\ 0 & \text{for odd } j. \end{cases}$$

- Simplified Feynman rules
- Single soft theorem emerges by directly evaluating diagrams
- Mixed theory emerges at triple soft limit
Double soft theorem of NLSM from Ward identity

\[ i \partial_\mu^\mu \langle J_\mu^a(x) J_\nu^b(y) \pi^{a_1}(x_1) \cdots \pi^{a_n}(x_n) \rangle = \]
\[ \sum_{j=1}^n \delta^{(4)}(x - x_j) \langle J_\nu^b(y) \pi^{a_1}(x_1) \cdots [F_1(T)]_{a_ja}(x) \cdots \pi^{a_n}(x_n) \rangle \]
\[ - \delta^{(4)}(x - y)(T^i)_{ab} \langle J_\nu^i(x) \pi^{a_1}(x_1) \cdots \pi^{a_n}(x_n) \rangle , \]

\[ \lim_{p^\mu \to 0} \lim_{q^\nu \to 0} M^{a_1a_2\cdots a_n}(p, q, p_1, \cdots, p_n) \]
\[ = \frac{1}{2f^2} \sum_k (T^i)_{ab} (T^i)_{akc} \frac{p_k \cdot (p - q)}{p_k \cdot (p + q)} M^{a_1\cdots a_n}(p_1, \cdots, p_n) . \]
Exceptional scalar EFTs

- Can be studied using Ward identities
- Subleading single soft limits, Berends-Giele recursion relations
  e.g. for DBI:

\[ M(p_1, \cdots, p_n, \tau q) \]

\[ = \tau^2 q_\mu q_\nu \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{l} \frac{(2k - 2)!}{2^{2k-2}[(k - 1)!]^2} (-F^{-4})^k \]

\[ \times \sum_{\sigma \in S_{2k+1}} \left[ q_{l\sigma(1)}^{\mu} q_{l\sigma(2)}^{\nu} \prod_{m=1}^{k-1} q_{l\sigma(2m+1)} \cdot q_{l\sigma(2m+2)} \right] \prod_{i=1}^{2k+1} J(\{p_{lj}\}) . \]
Extended theory of sGal

CHY gives

\[ M_{s\text{Gal}} = \tau^3 \sum_{a=2}^{n-2} \sum_{c=2}^{n-1} \sum_{d=1}^{n-2} s_{an} s_{cn} s_{dn} M_{s\text{Gal}}^{\oplus \text{NLSM}} M_{s\text{Gal}}^{\oplus \phi^3} (a, c, 1|n - 1, d, a) + \mathcal{O}(\tau^4). \]

Result from the Ward identity:

\[ M_{s\text{Gal}} = \tau^3 \left\{ \sum_{l} \frac{2}{\alpha^2} p_{n,l^1} p_{n,l^2} p_{n,l^3} \prod_{i=1}^{3} J(\{p_{l^j}\}) \right. \]

\[ \left. - \sum_{l} \frac{1}{6\alpha^4} \sum_{\sigma \in S_5} p_{n,l^{\sigma(1)}} p_{n,l^{\sigma(2)}} p_{n,l^{\sigma(3)}} \left[ q_{l^{\sigma(4)}}^2 q_{l^{\sigma(5)}}^2 - p_{l^{\sigma(4)}}^2 p_{l^{\sigma(5)}}^2 \right] \prod_{i=1}^{5} J(\{p_{l^j}\}) \right\} + \mathcal{O}(\tau^4). \]
The total number of flavor indices of a vertex must be even.
The allowed vertices with 2 flavor indices are $\Sigma^2 \pi^2$ and $\tilde{\Sigma}^2 \pi^2$, the Feynman rules of which are the same as the one for the $\pi^4$ vertex.
The allowed vertices with 4 flavor indices are $\phi^2 \pi^2$ and $\Sigma \tilde{\Sigma} \phi \pi$, the Feynman rules of which are the same as the one for the $\pi^4$ vertex.
The allowed vertices with 6 flavor indices are $\phi^3$, $\phi^3 \pi^2$, $\Sigma \tilde{\Sigma} \phi^2 \pi$, and $\Sigma^2 \tilde{\Sigma}^2 \phi$. 
Fixing the Feynman vertices

- The Ward identity can fix $\phi^3$ and $\phi^3\pi^2$, but not $\Sigma\bar{\Sigma}\phi^2\pi$ or $\Sigma^2\bar{\Sigma}^2\phi$
- 8 free parameters left after the constraint of the Ward identity is applied
- All vertices fixed after requiring the amplitudes of the extended theory to satisfy Adler’s zero for $\Sigma/\bar{\Sigma}$, and enhanced Adler’s zero for $\pi$