The Infrared Structure of Scalar EFTs

Based on 1709.08639, 1804.08629, 1810.XXXXX

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CCWZ formalism of NLSM

In general, G broken to H,

- Unbroken generators T^i, associated with unbroken group H
- Broken generators X^a, associated with coset G/H

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Callan, Coleman, Wess & Zumino:

$$-i\xi^{-1}\partial_\mu\xi = d_\mu + E_\mu = \mathcal{D}_\mu\pi^a X^a + E^i_\mu T^i, \qquad \xi = \exp[i\pi^a X^a/f],$$

$$d_\mu \to h d_\mu h^\dagger, \qquad E_\mu \to h E_\mu h^\dagger - ih\partial_\mu h^\dagger$$

$$\mathcal{L} = \frac{f^2}{2}\mathcal{D}_\mu\pi^a\mathcal{D}^\mu\pi^a + \mathcal{O}(\partial^4).$$

Phys. Rev. **177**, 2239 (1969); **177**, 2247 (1969).

The shift symmetry and Adler's zero

The transformation of the Goldstones:

$$\pi^a \to \pi^a + \varepsilon^a + iT^i_{ab}\alpha^i\pi^b + \cdots$$

Current under the shift symmetry contains a one-particle pole:

$$\langle\Omega|\mathcal{J}^\mu(x)|\pi(p)\rangle = ifp^\mu e^{-ip\cdot x},$$

Current conservation leads to Adler's zero condition:

$$\partial_\mu\langle f|\mathcal{J}^\mu|i\rangle = 0, \qquad \langle f + \pi(p)|i\rangle = p_\mu R^\mu(p).$$

Phys. Rev. **137**, B1022 (1965).

Nonlinearity from IR

Trivial case: $G = U(1)$

$$\pi \to \pi + \varepsilon, \qquad \mathcal{L} = \frac{1}{2}(\partial\pi)^2 + \mathcal{O}(\partial^4).$$

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General case: several assumptions

- Nonlinear transformation of π is a field-dependent H rotation:

$$|\pi\rangle \to |\pi\rangle + |\varepsilon\rangle + i\alpha^i(\varepsilon, \pi)|T^i\pi\rangle$$

- Goldstone covariant derivative $|\mathcal{D}\pi\rangle$ transform as a field-dependent H rotation:

$$|\mathcal{D}\pi\rangle \to e^{iu^i(\varepsilon,\pi)T^i/f}|\mathcal{D}\pi\rangle.$$

- All of the above can be reduced to the trivial case.

Universal nonlinearity

Solve α, u and $\mathcal{D}$ simultaneously.

$$\mathcal{D}_\mu \pi^a = \left(\frac{\sin\sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}} \right)_{ab} \partial_\mu \pi^b, \qquad \mathcal{T} \equiv \frac{1}{f^2} | T^i \pi \rangle \langle \pi T^i |.$$

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Then the Lagrangian is

$$\mathcal{L} = \frac{f^2}{2} \mathcal{D}_\mu \pi^a \mathcal{D}^\mu \pi^a + \mathcal{O}(\partial^4),$$

Low, 1412.2145.

E_μ can be solved similarly.

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Low, 1412.2145.

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- Structure of nonlinear interaction independent of G at UV
- G dependence all included in the value of f
- Can construct interactions of higher orders in momentum, with implications for composite Higgs models

Liu, Low, Yin, 1805.00489, 1809.09126.

Subleading single soft theorem for NLSM

Discovered using the CHY formalism:
When $p_{n+1} \to \tau p_{n+1}$ and $\tau \to 0$,

$$M_{n+1}^{\text{NLSM}}(\mathbb{I}_{n+1}) = \tau \sum_{i=2}^{n-1} s_{n+1,i}\; M_n^{\text{NLSM}\oplus\phi^3}(\mathbb{I}_n|1,n,i) + \mathcal{O}(\tau^2),$$

Cachazo, Cha, Mizera, 1604.03893.

Biadjoint scalars $\phi^{a\tilde{a}}$:

- Transform under $SU(N) \times SU(\tilde{N})$
- Self interaction: $f^{abc} f^{\tilde{a}\tilde{b}\tilde{c}} \phi^{a\tilde{a}} \phi^{b\tilde{b}} \phi^{c\tilde{c}}$

Ingredients for the Ward identity

IR construction:

$$
\begin{aligned}
|\pi\rangle &\to |\pi\rangle + |\varepsilon\rangle + i\alpha^i(\varepsilon,\pi)|T^i\pi\rangle, \\
|\mathcal{D}\pi\rangle &\to e^{iu^i(\varepsilon,\pi)T^i/f}|\mathcal{D}\pi\rangle.
\end{aligned}
$$

$\alpha^i(\varepsilon,\pi)$, $u^i(\varepsilon,\pi)$ and $|\mathcal{D}\pi\rangle$ are solved simultaneously.

$$
\begin{aligned}
\mathcal{D}_\mu\pi^a &= \left(\frac{\sin\sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}}\right)_{ab}\partial_\mu\pi^b, \\
\pi^a &\to \pi^a + \left(\sqrt{\mathcal{T}}\cot\sqrt{\mathcal{T}}\right)_{ab}\varepsilon^b.
\end{aligned}
$$

The current:

$$
\mathcal{J}^a_\mu = \left[\frac{\sin\sqrt{\mathcal{T}}\cos\sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}}\right]^{ab}\partial_\mu\pi^b = \partial_\mu\pi^a + \sum_{k=1}^{\infty}\frac{(-4)^k}{(2k+1)!}(\mathcal{T}^k)_{ab}\partial_\mu\pi^b.
$$

Low, Yin, 1709.08639; 1804.08629.

Tree level: subleading single soft theorem

$$M(\mathbb{I}_n) = -i \sum_{k=1}^{[n/2]} \sum_{\{l_m\}} V_{\mathbb{I}_{2k+1}}(q_{l_1}, \cdots, q_{l_{2k+1}}) \prod_{m=1}^{2k+1} J(l_{m-1}+1, \cdots, l_m) \, ,$$

$$V(\mathbb{I}_{2k+1}) = \frac{-i(-4)^k}{(2k+1)! f^{2k}} \sum_{j=1}^{2k-1} \left[\begin{pmatrix} 2k \\ j \end{pmatrix} (-1)^j - 1 \right] q \cdot p_{j+1} \, .$$

Compare with the CHY results:

$$M_n^{\mathrm{NLSM}}(\mathbb{I}_n) = \tau \sum_{i=2}^{n-2} s_{n,i} \; M_n^{\mathrm{NLSM}\oplus\phi^3}(\mathbb{I}_{n-1}|1, n-1, i) + \mathcal{O}(\tau^2)$$

The Lagrangian of the extended theory

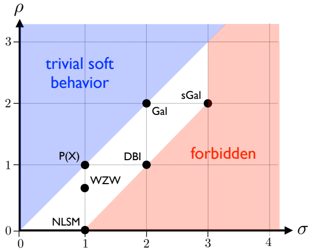
$$
\begin{aligned}
\mathcal{L}^{\mathrm{NLSM}\oplus\phi^3} &= \frac{f^2}{8(N^2-1)}\mathrm{Tr}\left\{\partial_\mu U^{-1}\left(\underline{\phi}+\underline{\pi}\right)\partial^\mu U\left(\underline{\phi}+\underline{\pi}\right)\right\} \\
&+\frac{1}{3}\lambda\phi^{a\tilde{a}}\phi^{b\tilde{b}}\phi^{c\tilde{c}}f^{abc}f^{\tilde{a}\tilde{b}\tilde{c}}+\frac{6if^3\lambda}{(N^2-1)^{3/2}} \\
&\times\mathrm{Tr}\{[U(\underline{\phi}+\underline{\pi})-U(\underline{\pi})][U(\underline{\phi}-\underline{\pi})-U(-\underline{\pi})] \\
&-[U(-\underline{\phi}-\underline{\pi})-U(-\underline{\pi})][U(\underline{\pi}-\underline{\phi})-U(\underline{\pi})]\}
\end{aligned}
$$

where $U(x)=e^x$,

$$
\begin{aligned}
(\underline{\pi})_{ab,\tilde{a}\tilde{b}} &\equiv \left(\frac{2i}{f}\right)\pi^c(T^c)_{ab}\delta_{\tilde{a}\tilde{b}}, \\
(\underline{\phi})_{ab,\tilde{a}\tilde{b}} &\equiv \left(\frac{2i}{f}\right)\sqrt{N^2-1}\phi^{c\tilde{c}}(T^c)_{ab}(T^{\tilde{c}})_{\tilde{a}\tilde{b}}.
\end{aligned}
$$

Exceptional scalar EFTs

$$\mathcal{L} \supset \partial^m \phi^n, \qquad \rho = \frac{m-2}{n-2}, \qquad M(\tau q) = \mathcal{O}(\tau^\sigma)$$



Cheung et al., 1611.03137.

- Simple CHY representations
- Enhanced shift symmetry
  - DBI: $\delta\pi = \theta_\mu \left(x^\mu - F^{-d} \pi \partial^\mu \pi \right)$
  - sGal: $\delta\pi = \theta^{\mu\nu}(\alpha^2 x_\mu x_\nu + \partial_\mu \pi \partial_\nu \pi)$

Extended theory of sGal

CHY gives

$$M_n^{\text{sGal}} = \tau^3 \sum_{a=2}^{n-2} \sum_{\substack{c=2\\c\neq a}}^{n-1} \sum_{\substack{d=1\\d\neq a}}^{n-2} s_{an}s_{cn}s_{dn} M_{n-1}^{\text{sGal}\oplus\text{NLSM}^2\oplus\phi^3}(a,c,1|n-1,d,a)$$
$$+\mathcal{O}(\tau^4).$$

Result from the Ward identity:

$$M_n^{\text{sGal}} = \tau^3 \left\{ \sum_I \frac{2}{\alpha^2} p_{n,I^1} p_{n,I^2} p_{n,I^3} \prod_{i=1}^{3} J(\{p_{I_j^i}\}) \right.$$
$$\left. - \sum_I \frac{1}{6\alpha^4} \sum_{\sigma\in S_5} p_{n,I^{\sigma(1)}} p_{n,I^{\sigma(2)}} p_{n,I^{\sigma(3)}} \left[q^2_{I^{\sigma(4)}} q^2_{I^{\sigma(5)}} - p^2_{I^{\sigma(4)},I^{\sigma(5)}} \right] \prod_{i=1}^{5} J(\{p_{I_j^i}\}) \right\}$$
$$+\mathcal{O}(\tau^4).$$

Conclusion

- NLSM can be constructed from IR constraints
- Interesting soft limits beyond Adler's zero can be derived using the Ward identity
- The Lagrangian of the extended theory can be identified

Backup Slides

Pions as pNGB

The QCD Lagrangian with two light quarks:

$$\mathcal{L} \supset i\bar{u}^L \not{D} u^L + i\bar{u}^R \not{D} u^R + i\bar{d}^L \not{D} d^L + i\bar{d}^R \not{D} d^R$$

$SU(2)_L \times SU(2)_R$ chiral symmetry:

$$\begin{pmatrix} u^L \\ d^L \end{pmatrix} \to g_L \begin{pmatrix} u^L \\ d^L \end{pmatrix}, \qquad \begin{pmatrix} u^R \\ d^R \end{pmatrix} \to g_R \begin{pmatrix} u^R \\ d^R \end{pmatrix}$$

The chiral symmetry is spontaneously broken:

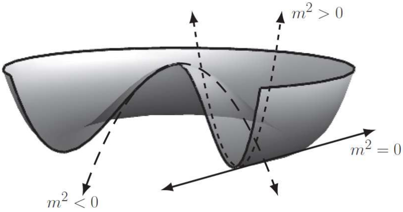
$$\begin{aligned} \langle \bar{u}u \rangle = \langle \bar{d}d \rangle &= V^3, \\ SU(2)_L \times SU(2)_R &\to SU(2)_V. \end{aligned}$$

Given the symmetry breaking pattern, we can describe the Nambu-Goldstone Bosons without QCD

The Sigma model

Introducing the scalar Σ_{ij} that transfrom under $SU(2)_L \times SU(2)_R$:

$$\mathcal{L} = |\partial_\mu \Sigma|^2 + m^2|\Sigma|^2 - \frac{\lambda}{4}|\Sigma|^4.$$



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$$\mathcal{L} = |\partial_\mu \Sigma|^2 + m^2 |\Sigma|^2 - \frac{\lambda}{4}|\Sigma|^4.$$

Decouple the radial part and only keep the NGBs:

$$\Sigma \to U = \exp\left[2i\frac{\pi^a \tau^a}{F_\pi}\right]$$

NLSM:

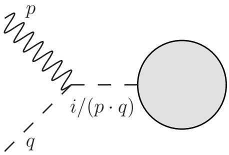
$$\mathcal{L} = \frac{F_\pi^2}{4}\mathrm{Tr}\left[D_\mu U (D_\mu U)^\dagger\right] + \mathcal{O}(D^4).$$

Nonlinear transformation

$$U \to g_L U g_R^\dagger, \qquad \pi^a \to \pi^a + \varepsilon^a + i T^i_{ab} \alpha^i \pi^b + \cdots$$

The regularity condition

For there to be Adler's zero, the remainder R needs to be regular: no "pole digrams" should appear.



The closure condition

The "closure condition" is necessary for solutions to exist in the IR construction of NLSM:

$$(T^i)_{ab}(T^i)_{cd} + (T^i)_{ac}(T^i)_{db} + (T^i)_{ad}(T^i)_{bc} = 0.$$

Flavor-ordering

Flavor-ordering:

$$
\begin{aligned}
& M^{a_1 \cdots a_n}(p_1, \cdots, p_n) \\
\equiv \quad & \sum_{\sigma \in S_{n-1}} \mathrm{Tr}(X^{a_n} X^{a_{\sigma(1)}} \ldots X^{a_{\sigma(n-1)}}) M_\sigma(p_1, \cdots, p_{n-1}) \ ,
\end{aligned}
$$

The extension of NLSM

The CHY representation:

$$M_n^{\mathsf{NLSM}\oplus\phi^3}(\alpha|\beta) = \oint d\mu_n\, \mathcal{C}_n(\alpha)\Big(\mathcal{C}(\beta)\,(\mathrm{Pf}\,\mathsf{A}_{\bar{\beta}})^2\Big).$$

Compared to pure NLSM:

$$M_n^{\mathsf{NLSM}}(\alpha) = \oint d\mu_n\, \mathcal{C}_n(\alpha)(\mathrm{Pf}'\,\mathsf{A}_n)^2.$$

Cachazo, He, Yuan, 1412.3479.

Biadjoint scalars interacting with Goldstones. How?

Subleading single soft from Ward identity

$$\partial_\mu \langle f | \mathcal{J}^\mu | i \rangle = 0, \qquad \langle f + \pi(p) | i \rangle = p_\mu R^\mu(p).$$

To go beyond Adler's zero, simply need to calculated R.

- Current algebra: hard to go beyond leading order in $1/f$.
- CCWZ: seemingly different for different coset.
- Do not have anything to relate to.
  Compared with single soft theorem in gauge theory:

$$M_{n+1}^{a_1 a_2 \cdots a_{n+1}} = \sum_{i=1}^{n} igf^{a_i a_{n+1} b} \left(\frac{1}{\tau} \frac{p_i \cdot \varepsilon_{n+1}}{p_i \cdot p_{n+1}} - i \frac{\varepsilon_{n+1}^{\nu} p_{n+1}^{\rho}}{p_i \cdot p_{n+1}} J_{i\nu\rho} \right) \times M_n^{a_1 \cdots a_{i-1} b a_{i+1} \cdots a_n} + \mathcal{O}(\tau)$$

Derivation of the current

$$\mathcal{J}^a_\mu = \left[\frac{\sin\sqrt{\mathcal{T}}\cos\sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}}\right]^{ab}\partial_\mu\pi^b = \partial_\mu\pi^a + \sum_{k=1}^{\infty}\frac{(-4)^k}{(2k+1)!}(\mathcal{T}^k)_{ab}\partial_\mu\pi^b.$$

The Ward identity:

$$i\partial^\mu\langle\Omega|\mathcal{J}^a_\mu(x)\prod_{i=1}^{n}\pi^{a_i}(x_i)|\Omega\rangle$$

$$= \sum_{r=1}^{n}\langle\Omega|\pi^{a_1}(x_1)\cdots\left[\sqrt{\mathcal{T}}\cot\sqrt{\mathcal{T}}\right]_{a_r a}(x_r)\delta^{(4)}(x-x_r)\cdots\pi^{a_n}(x_n)|\Omega\rangle.$$

Ward identity/soft theorem

$$M^{a_1\cdots a_n a}(p_1,\cdots,p_n,q) = q\cdot R^{a_1\cdots a_n a}(p_1,\cdots,p_n;q),$$

The remainder function R is regular:

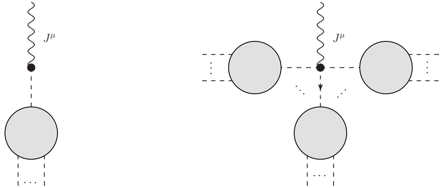
$$\begin{aligned} & R_\mu^{a_1\cdots a_n a}(p_1,\cdots,p_n;q) \\ = \quad & \frac{1}{\sqrt{Z}}\sum_{k=1}^{\infty}\frac{-i(-4)^k}{(2k+1)!}\langle 0|\int d^4x\, e^{-iq\cdot x}[\mathcal{T}^k(x)]_{ab}\,\partial_\mu\pi^b(x)|\pi^{a_1}\cdots\pi^{a_n}\rangle. \end{aligned}$$

Inserting the current twice: leading double soft theorem

Tree level: Berends-Giele recursion relations

Semi-on-shell amplitudes:

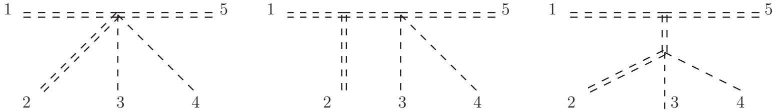
$$J^{a_1 \cdots a_n, a}(p_1, \cdots, p_n) = \langle 0 | \pi^a(0) | \pi^{a_1}(p_1) \cdots \pi^{a_n}(p_n) \rangle.$$



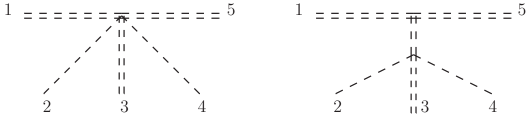
$$q^2 J(\mathbb{I}_n) = i \sum_{k=1}^{[n/2]} \sum_{\{l_m\}} V_{\mathbb{I}_{2k+1}}(q_{l_1}, \cdots, q_{l_{2k+1}}) \prod_{m=1}^{2k+1} J(l_{m-1}+1, \cdots, l_m)$$

Example: 5-pt

$$M_5^{\mathsf{NLSM}\oplus\phi^3}(\mathbb{I}_5|1,5,2)$$



$$M_5^{\mathsf{NLSM}\oplus\phi^3}(\mathbb{I}_5|1,5,3)$$



Identification of Feynman vertices

- Flavor conservation: no vertex with one ϕ
- Soft constraint: ϕ^2 vertices exactly the same as NLSM
- The structure of the current leads to ϕ^3 vertices:

$$
\begin{aligned}
& V^{\mathrm{NLSM}\oplus\phi^3}(\mathbb{I}_{2k+1}|1, 2k+1, j) \\
= \quad & \frac{i}{2}\frac{-(-4)^k}{(2k+1)!f^{2k}}\left[\binom{2k}{j-1}(-1)^{j-1}-1\right] \; .
\end{aligned}
$$

Recovers the self-interaction of ϕ.

Cayley parameterization

Field redefinition: $\pi \to \pi + \mathcal{O}(\pi^2)$.

$$\mathcal{L} = \frac{f^2}{8}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^{-1}\right), \qquad U = \frac{1 + i\pi^a T^a/f}{1 - i\pi^a T^a/f} = 1 + 2\sum_{k=1}^{\infty}\left(\frac{i}{f}\pi^a T^a\right)^k$$

$$V(\mathbb{I}_{2k+2}) = \frac{i(-1)^k}{f^{2k}}\left(\sum_{i=0}^{k} p_{2i+1}\right)^2 .$$

$$V^{\mathrm{NLSM}\oplus\phi^3}(\mathbb{I}_{2k+1}|1, 2k+1, j) = \begin{cases} \frac{i(-1)^k}{f^{2k}} & \text{for even } j, \\ 0 & \text{for odd } j. \end{cases}$$

- Simplified Feynman rules
- Single soft theorem emerges by directly evaluating diagrams
- Mixed theory emerges at triple soft limit

Double soft theorem of NLSM from Ward identity

$$
\begin{aligned}
& i\partial_x^\mu \langle \mathcal{J}_\mu^a(x)\mathcal{J}_\nu^b(y)\pi^{a_1}(x_1)\cdots\pi^{a_n}(x_n)\rangle = \\
& \quad \sum_{j=1}^{n} \delta^{(4)}(x-x_j)\ \langle \mathcal{J}_\nu^b(y)\pi^{a_1}(x_1)\cdots[F_1(\mathcal{T})]_{a_j a}(x)\cdots\pi^{a_n}(x_n)\rangle \\
& \qquad -\delta^{(4)}(x-y)(T^i)_{ab}\ \langle J_\nu^i(x)\pi^{a_1}(x_1)\cdots\pi^{a_n}(x_n)\rangle\ ,
\end{aligned}
$$

$$
\begin{aligned}
& \lim_{p^\mu\to 0}\lim_{q^\nu\to 0} M^{aba_1\cdots a_n}(p,q,p_1,\cdots,p_n) \\
& \qquad = \frac{1}{2f^2}\sum_k (T^i)_{ab}(T^i)_{a_k c}\frac{p_k\cdot(p-q)}{p_k\cdot(p+q)} M^{a_1\cdots c\cdots a_n}(p_1,\cdots,p_n)\ .
\end{aligned}
$$

Exceptional scalar EFTs

- Can be studied using Ward identities
- Subleading single soft limits, Berends-Giele recursion relations e.g. for DBI:

$$
\begin{aligned}
& M(p_1, \cdots, p_n, \tau q) \\
= \quad & \tau^2 q_\mu q_\nu \sum_{k=1}^{[n/2]} \sum_{I} \frac{(2k-2)!}{2^{2k-2}[(k-1)!]^2} \left(-F^{-4}\right)^k \\
& \times \sum_{\sigma \in S_{2k+1}} \left[q^\mu_{I_{\sigma(1)}} \; q^\nu_{I_{\sigma(2)}} \prod_{m=1}^{k-1} q_{I_{\sigma(2m+1)}} \cdot q_{I_{\sigma(2m+2)}} \right] \prod_{i=1}^{2k+1} J(\{p_{I^i_j}\}).
\end{aligned}
$$

Extended theory of sGal

CHY gives

$$M_n^{\text{sGal}} = \tau^3 \sum_{a=2}^{n-2} \sum_{\substack{c=2\\c\neq a}}^{n-1} \sum_{\substack{d=1\\d\neq a}}^{n-2} s_{an}s_{cn}s_{dn} M_{n-1}^{\text{sGal}\oplus\text{NLSM}^2\oplus\phi^3}(a,c,1|n-1,d,a)$$
$$+\mathcal{O}(\tau^4).$$

Result from the Ward identity:

$$M_n^{\text{sGal}} = \tau^3 \left\{ \sum_I \frac{2}{\alpha^2} p_{n,I^1} p_{n,I^2} p_{n,I^3} \prod_{i=1}^{3} J(\{p_{I^i_j}\}) \right.$$
$$\left. - \sum_I \frac{1}{6\alpha^4} \sum_{\sigma\in S_5} p_{n,I^{\sigma(1)}} p_{n,I^{\sigma(2)}} p_{n,I^{\sigma(3)}} \left[q^2_{I^{\sigma(4)}} q^2_{I^{\sigma(5)}} - p^2_{I^{\sigma(4)},I^{\sigma(5)}} \right] \prod_{i=1}^{5} J(\{p_{I^i_j}\}) \right\}$$
$$+\mathcal{O}(\tau^4).$$

Identification of Feynman vertices

- The total number of flavor indices of a vertex must be even.
- The allowed vertices with 2 flavor indices are $\Sigma^2\pi^2$ and $\tilde{\Sigma}^2\pi^2$, the Feynman rules of which are the same as the one for the π^4 vertex.
- The allowed vertices with 4 flavor indices are $\phi^2\pi^2$ and $\Sigma\tilde{\Sigma}\phi\pi$, the Feynman rules of which are the same as the one for the π^4 vertex.
- The allowed vertices with 6 flavor indices are ϕ^3, $\phi^3\pi^2$, $\Sigma\tilde{\Sigma}\phi^2\pi$, and $\Sigma^2\tilde{\Sigma}^2\phi$.

Fixing the Feynman vertices

- The Ward identity can fix ϕ^3 and $\phi^3\pi^2$, but not $\Sigma\tilde{\Sigma}\phi^2\pi$ or $\Sigma^2\tilde{\Sigma}^2\phi$
- 8 free parameters left after the constraint of the Ward identity is applied
- All vertices fixed after requiring the amplitudes of the extended theory to satisfy Adler's zero for $\Sigma/\tilde{\Sigma}$, and enhanced Adler's zero for π