GPLs for Feynman integrals with unrationalizable algebraic symbol letters

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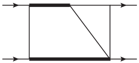
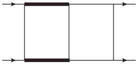
with Matthias Heller and Andreas von Manteuffel

Michigan State University

Outline

Difficulty With Algebraic $\epsilon\,$dln Differential Equations

Bonciani *et. al.*, JHEP **09** (2016) 091

For  (w, z, m^2) and  (w, z, m^2),

have the unrationalizable symbol letter:

$$r = \sqrt{(1 + w^2 z^2)(w + z)^2 + 2w\,z(w - z)^2 + 4w\,z^2(1 + w^2)}$$

The algebraic part of the symbol alphabet reads:

$$\mathcal{L}_A = \{r, -(1 - w)(z - w)(1 - w\,z) + r\,(1 + w),$$
$$-(1 - w)\big(4w\,z + (w + z)(1 + w\,z)\big) - r\,(1 + w),$$
$$r^2 - 2w\,z^2(1 - w)^2 + r\,(w + z)(1 + w\,z),$$
$$r^2(1 - z)^2 + 2z^2(z + w^2)(1 + w^2 z) + r\,(1 - z)(1 + z)\big(2w\,z - (w + z)(1 + w\,z)\big)\}$$

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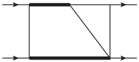
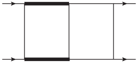
For  (w, z, m^2) and  (w, z, m^2),

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Construction of useful $\epsilon\,$**dln** differential equations not straightforward!

J. M. Henn, Phys. Rev. Lett. **110** (2013) 251601

Factorization Of Algebraic Functions Is Not Canonical

$$-(1-w)(z-w)(1-w\,z)+r\,(1+w)=$$

$$\frac{2w\big(1+z\big)\big(z+w^2\big)\big(2+z-w+w\,z(w+z)+r\big)}{2w^2+z-w+w\,z(w+z)+r}$$

$$-(1-w)\,\big(4w\,z+(w+z)(1+w\,z)\big)-r\,(1+w)=$$

$$\frac{-8\,z\,w^2\big(1+z\big)^3\big(1+w^2\,z\big)\big(2w^2+z-w+w\,z(w+z)+r\big)}{\big(2+z-w+w\,z(w+z)+r\big)\big(-(w+z)(1-w\,z)+r\big)\big(-(z-w)(1+w\,z)+r\big)}$$

$$r^2-2w\,z^2(1-w)^2+r\,(w+z)(1+w\,z)=$$

$$\frac{-z^2\big(2+z-w+w\,z(w+z)+r\big)^2\big(2w^2+z-w+w\,z(w+z)+r\big)^2}{8\big(1+z\big)^2\big(1+w^2\,z\big)^2\big(-(w+z)(1-w\,z)+r\big)^2\big(-(z-w)(1+w\,z)+r\big)^{-2}}$$

$$r^2(1-z)^2+2z^2(z+w^2)(1+w^2z)+r\,(1-z)(1+z)\,\big(2w\,z-(w+z)(1+w\,z)\big)$$

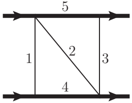
$$=\frac{2z^2\big(1+w^2\,z\big)^2\big(-(w+z)(1-w\,z)+r\big)^2}{\big(-(z-w)(1+w\,z)+r\big)^2}$$

First Non-Trivial Example From Bhabha Scattering

J. M. Henn and V. A. Smirnov, JHEP **1311** (2013) 041;

J. M. Henn, Amplitudes 2017 talk; C. Duhr, Amplitudes 2018 talk

From planar, massive Bhabha electron-positron scattering:



$$(s,t,m^2) = -e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)\left[\prod_{i=1}^{5}\int_0^\infty \mathrm{d}\alpha_i\right]\delta(1-\alpha_5)\,\mathcal{U}^{-1+3\epsilon}\mathcal{F}^{-1-2\epsilon}$$

$$\mathcal{U} = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_5 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_2\alpha_5 + \alpha_3\alpha_4 + \alpha_4\alpha_5$$

$$\mathcal{F} = -s\,\alpha_2\alpha_4\alpha_5 - t\,\alpha_1\alpha_2\alpha_3 + m^2\,(\alpha_4+\alpha_5)\,\mathcal{U} - m^2\big(\alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_3\alpha_5 + \alpha_1\alpha_4\alpha_5 + \alpha_2\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_3\alpha_4\alpha_5\big)$$

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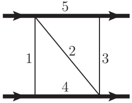

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$$\mathcal{F} = -s\,\alpha_2\alpha_4\alpha_5 - t\,\alpha_1\alpha_2\alpha_3 + m^2\,(\alpha_4+\alpha_5)\,\mathcal{U} - m^2\big(\alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_3\alpha_5 + \alpha_1\alpha_4\alpha_5 + \alpha_2\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_3\alpha_4\alpha_5\big)$$

No evaluation in terms of standard multiple polylogarithms known!

What Polynomial Reduction Has To Offer

F. Brown, Commun. Math. Phys. **287** (2009) 925; arXiv:0910.0114; E. Panzer, arXiv:1506.07243
implemented in `HyperInt`, E. Panzer, Comput. Phys. Commun. **188** (2015) 148

- For Feynman parameter integrals, can conceptualize polynomial reduction as an efficient "simulation" of each direct integration step (in terms of GPLs) of each possible integration order.

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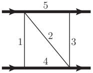
$\Longrightarrow$ 

not *linearly reducible*, can integrate out α_1 or α_3 only

Polynomial Reduction Is Still Useful In The Absence Of Linear Reducibility

Can easily find a good variable change by studying the irreducible polynomials which could appear after integrating out α_1:

$$\mathrm{L}_{\alpha_1} = \Big\{1+\alpha_2+\alpha_3, m^2 - t\,\alpha_2\,\alpha_3, \alpha_2\,(1+\alpha_3) + (1+\alpha_2+\alpha_3)\,\alpha_4,$$

$$m^2\alpha_2 + \left(m^2 + \left(2\,m^2 - s\right)\alpha_2\right)\alpha_4 + m^2\,(1+\alpha_2+\alpha_3)\,\alpha_4^2,$$

$$\alpha_2^2\left(m^2 + t\,\alpha_3\,(1+\alpha_3)\right) + \left(2\,m^2 - s + t\,\alpha_3\right)(1+\alpha_2+\alpha_3)\,\alpha_2\,\alpha_4$$

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$$+\, m^2\,(1+\alpha_2+\alpha_3)^2\,\alpha_4^2\Big\}$$

$\Longrightarrow$ the first variable change for $P_{7,11}$, $\alpha_4 = \frac{x_4\,\alpha_2}{1+\alpha_2+\alpha_3}$, looks promising

E. Panzer, arXiv:1506.07243

Polynomial Reduction Is Still Useful In The Absence Of Linear Reducibility

In contrast to the $P_{7,11}$ analysis, all other polynomials are still fine:

$$\mathrm{L}_{\alpha_1} = \Big\{ 1 + \alpha_2 + \alpha_3, m^2 - t\,\alpha_2\,\alpha_3, \alpha_2\,(1 + \alpha_3 + x_4)\,,$$
$$\frac{\alpha_2}{1 + \alpha_2 + \alpha_3}\Big(m^2\,(1 + \alpha_2 + \alpha_3) + \big(m^2 + \big(2\,m^2 - s\big)\alpha_2\big)x_4 + m^2\alpha_2\,x_4^2\Big),$$
$$\alpha_2{}^2\,\big(m^2 + t\,\alpha_3\,(1 + \alpha_3) + \big(2\,m^2 - s + t\,\alpha_3\big)\,x_4 + m^2 x_4^2\big)\Big\}$$

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$\Longrightarrow$ GPLs with rational weights and arguments after α_2 integration

How To Proceed For The Final Two Integrations?

Rerunning the polynomial reduction in `HyperInt`, we get stuck again:

$$\mathrm{L}_{\alpha_1\alpha_2} = \Big\{1+\alpha_3, 1+\alpha_3+x_4, m^2+t\,\alpha_3+t\,\alpha_3^2, m^2+\left(2\,m^2-s\right)x_4+m^2x_4^2,$$
$$m^2\left(1+x_4\right)-s+\left(m^2(2+x_4)-s\right)\alpha_3, m^2+\left(2\,m^2-s\right)x_4+m^2x_4^2$$
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How to proceed without leaving the space of GPLs?

- $m^2+t\,\alpha_3+t\,\alpha_3^2$ and $m^2+\left(2\,m^2-s\right)x_4+m^2x_4^2$ no issue, dealt with via kinematic variable changes.
- $m^2+\left(2\,m^2-s\right)x_4+m^2x_4^2+t(1+x_4)\alpha_3+t\,\alpha_3^2$ requires a non-trivial change of variables which maps the domain of the final integration parameter onto $[0,1]$.

Kinematic Variable Changes Eliminate Some Roots

S. Caron-Huot and J. M. Henn, JHEP **1406** (2014) 114

$$s = -\frac{4\,m^2(v_2 - v_1)^2}{(1-v_1)(1+v_1)(1-v_2)(1+v_2)} \qquad \text{and} \qquad t = -\frac{m^2(v_2-v_1)^2}{v_1 v_2}$$

get rid of two square roots which would otherwise arise:

$$m^2 + t\,\alpha_3 + t\,\alpha_3^2 = \frac{m^2\left(v_1 - (v_2-v_1)\alpha_3\right)\left(v_2 + (v_2-v_1)\alpha_3\right)}{v_1 v_2}$$

and

$$m^2 + \left(2\,m^2 - s\right)x_4 + m^2 x_4^2 = \frac{m^2\left((1+v_1)(1-v_2) + (1-v_1)(1+v_2)x_4\right)\left((1-v_1)(1+v_2) + (1+v_1)(1-v_2)x_4\right)}{(1-v_1)(1+v_1)(1-v_2)(1+v_2)}$$

Variable Change Via Parametrization By Lines

E. Panzer, JHEP **1403** (2014) 071; M. Besier *et. al.*, Commun. Num. Theor. Phys. **13** (2019) 253

For technical reasons, set $\alpha_3 = \frac{x_3}{1-x_3}$. The x_3 integration leads to

$$\sqrt{1 + \frac{2(4\,m^2 - 2\,s - t)}{4\,m^2 - t} x_4 + x_4^2}\,.$$

This obstruction is removed by finding a rational parametrization of:

$$1 + \frac{2(4\,m^2 - 2\,s - t)}{4\,m^2 - t} x_4 + x_4^2 = \rho^2.$$

Parametrizing the algebraic variety by lines, $\rho = y_4\left(x_4 - x_4^{(0)}\right) + \rho^{(0)}$, through the rational point $\left(x_4^{(0)}, \rho^{(0)}\right) = \left(-\frac{2(4\,m^2-2\,s-t)}{4\,m^2-t}, 1\right)$ leads to:

$$x_4 = \frac{2\,y_4\left(1 + \frac{4\,m^2-2\,s-t}{4\,m^2-t} y_4\right)}{(1-y_4)(1+y_4)}$$

Weight-Four Result For Bhabha Integral

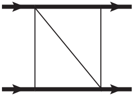
From these considerations, I was able to directly evaluate at $\mathcal{O}\left(\epsilon^0\right)$ in terms of GPLs with the root $\sqrt{p_1(v_1,v_2)p_2(v_1,v_2)p_3(v_1,v_2)p_4(v_1,v_2)}$

$$
\begin{aligned}
p_1(v_1,v_2) &= 1 + v_1 - v_2(1 - v_1) \\
p_2(v_1,v_2) &= 1 + v_2 - v_1(1 - v_2) \\
p_3(v_1,v_2) &= v_1(1 - v_1) + v_2(1 - v_2) + v_1 v_2(2 - v_1 - v_2) \\
p_4(v_1,v_2) &= v_1(1 + v_1) + v_2(1 + v_2) - v_1 v_2(2 + v_1 + v_2)
\end{aligned}
$$

in the weights of the most complicated GPLs. From `GiNaC`:

C. Bauer *et. al.*, J. Symb. Comput. **33** (2002) 1;

J. Vollinga and S. Weinzierl, Comput. Phys. Commun. **167** (2005) 177



$(-1/36, -8/35, 2) \approx -6.31755008947575333016 9497\ldots + \mathcal{O}(\epsilon)$

confirmed with `FIESTA 4`, A. Smirnov, Comput. Phys. Commun. **204** (2016) 189

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- Learn about elliptic multiple polylogarithm reduction identities by comparing the solution for the Bhabha integral of this talk with the solution given last year in terms of eMPLs.