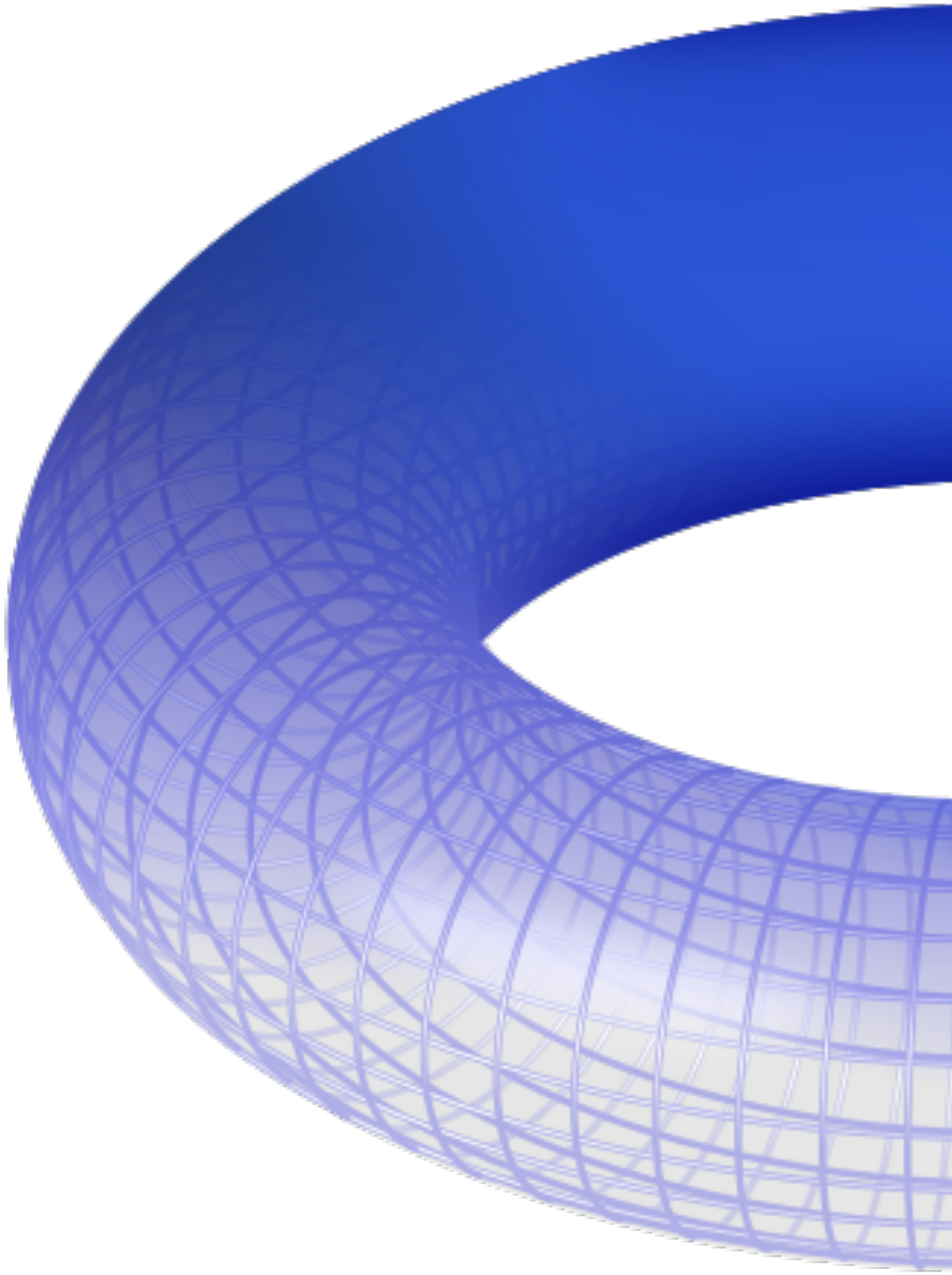
FEYNMAN INTEGRALS AND HIGHER GENUS SURFACES

In collaboration with:

J. Brödel, F. Dulat, C. Duhr, B. Penante



Amplitudes 2019

Dublin 01/07/2019

Lorenzo Tancredi - CERN TH

INTRODUCTION

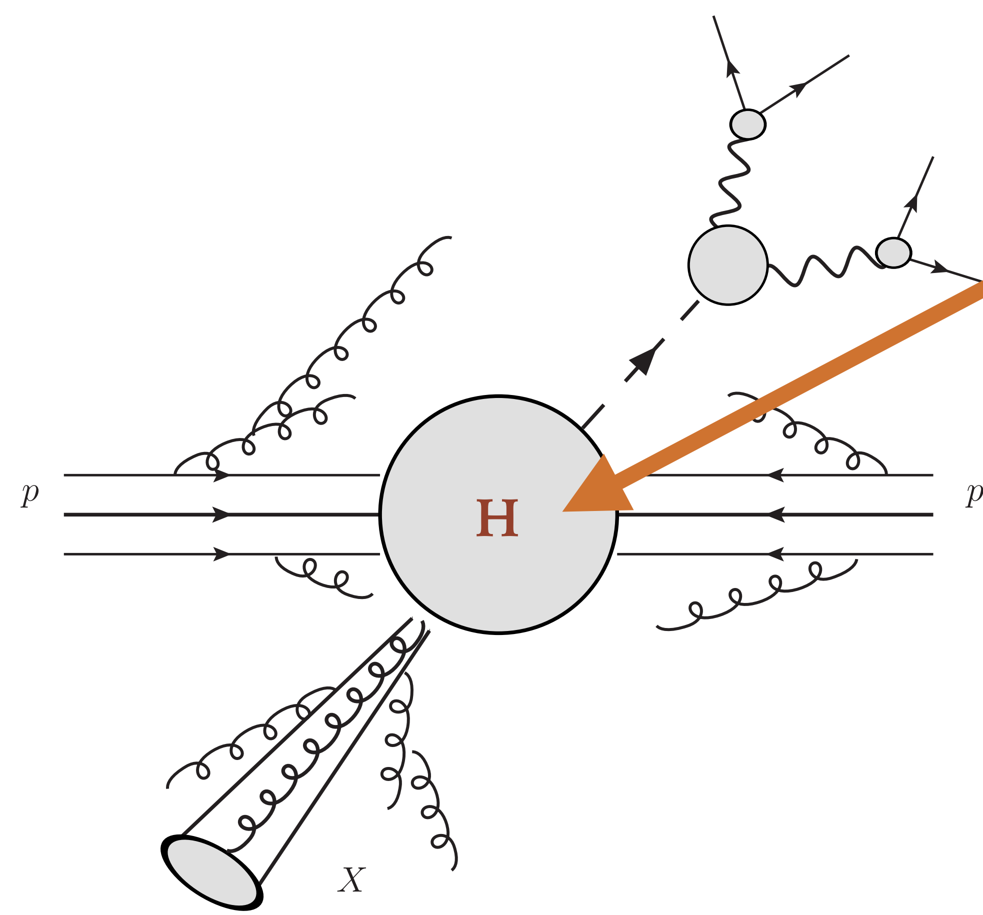
Precision physics @ the LHC means having a lot of ***complex ingredients*** under control

$$pp \to HX \to l_1\bar{l}_1 + l_2\bar{l}_2 + X$$

Factorisation of long and short range physics

Non perturbative corrections

$\mathcal{O}\left(\frac{\Lambda_{QCD}}{Q}\right) \sim$ few percent?



Hard scattering process

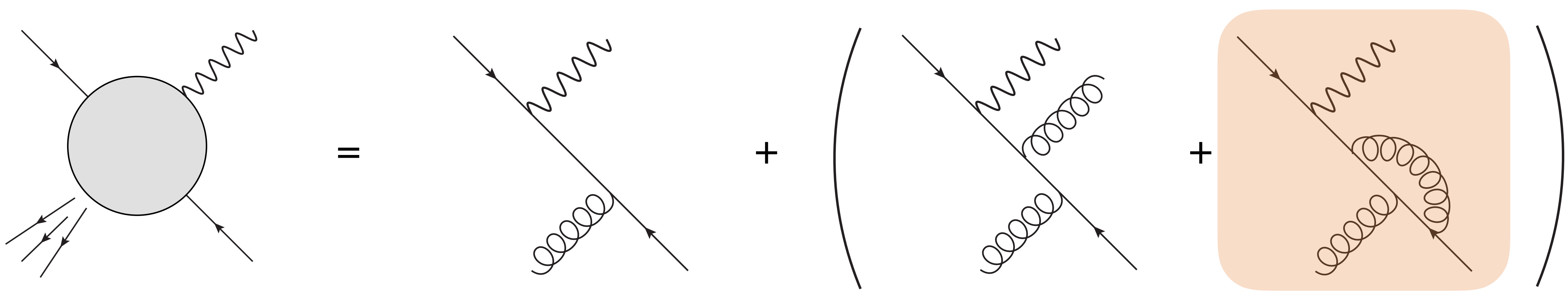
Aim to $\sim$ **few % precision**

Precise determination of **parton content of proton**

PDFs Currently known at level $\sim$ few % for LHC

INTRODUCTION

Hard process requires the calculation of **scattering amplitudes**



+ ....

complicated **set of integrals** to compute

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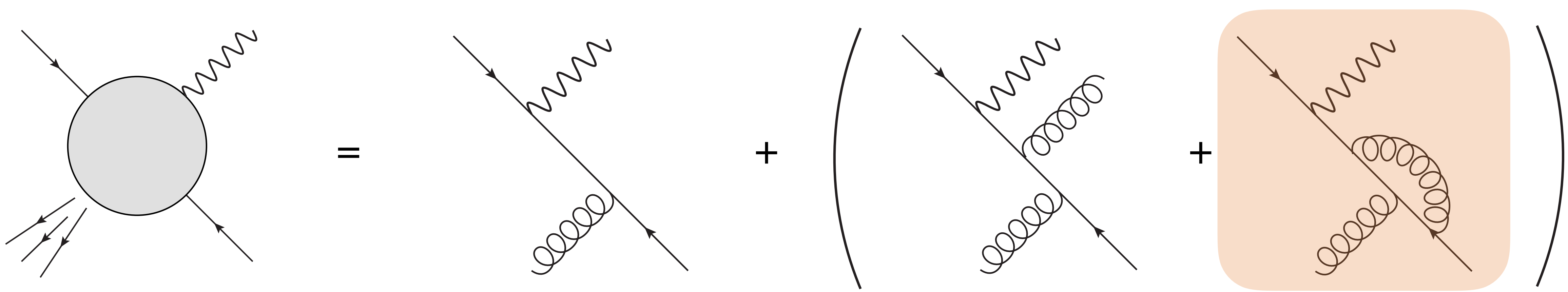


+ ....

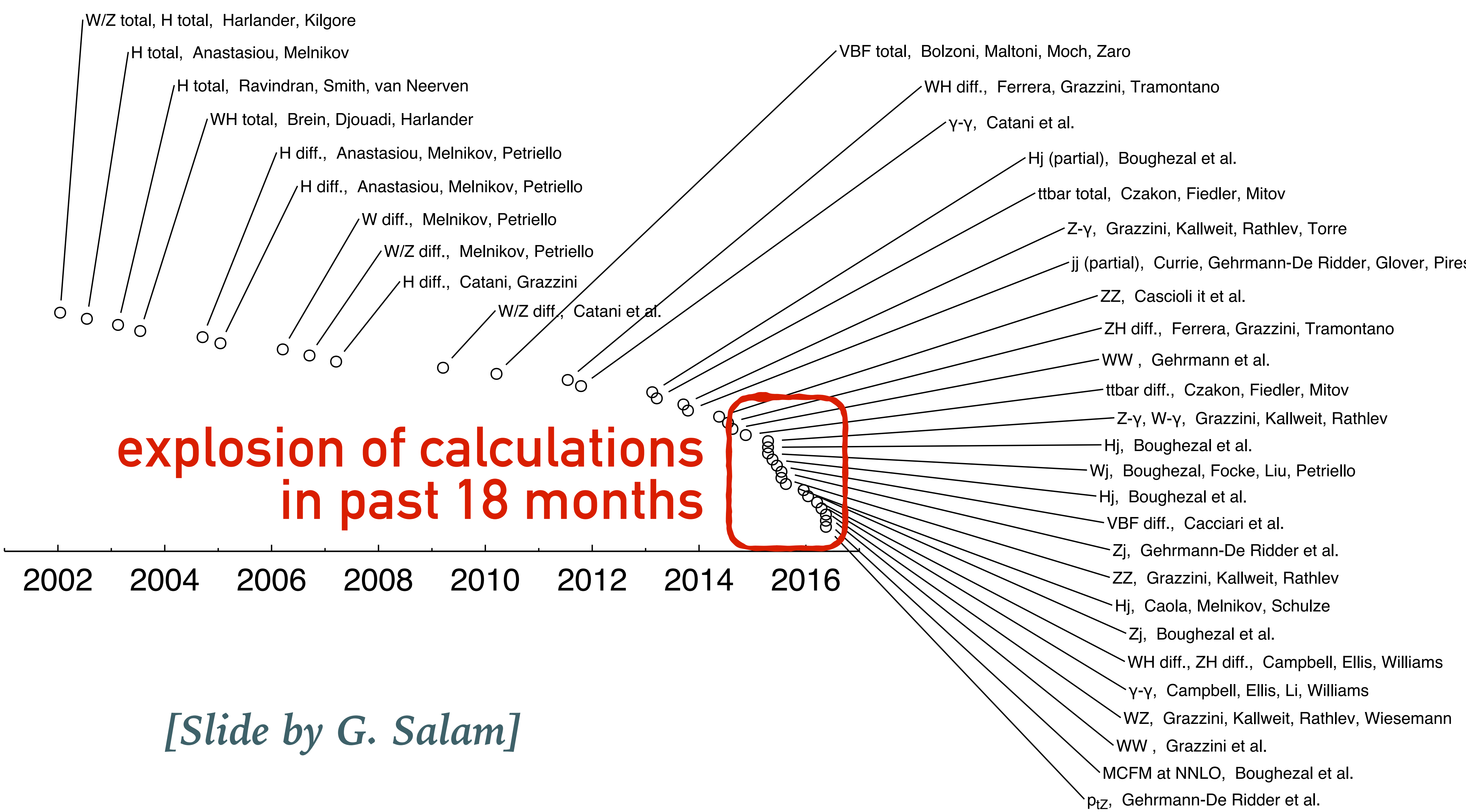
complicated **set of integrals** to compute

Thanks to cross-talk between different communities *(particle physics, amplitudes, string theory, mathematics…)* we have witnessed **impressive developments** in our understanding of ***multi-loop scattering amplitudes***

WHAT WE CAN DO

LHC phenomenology had a lot to gain from these developments…

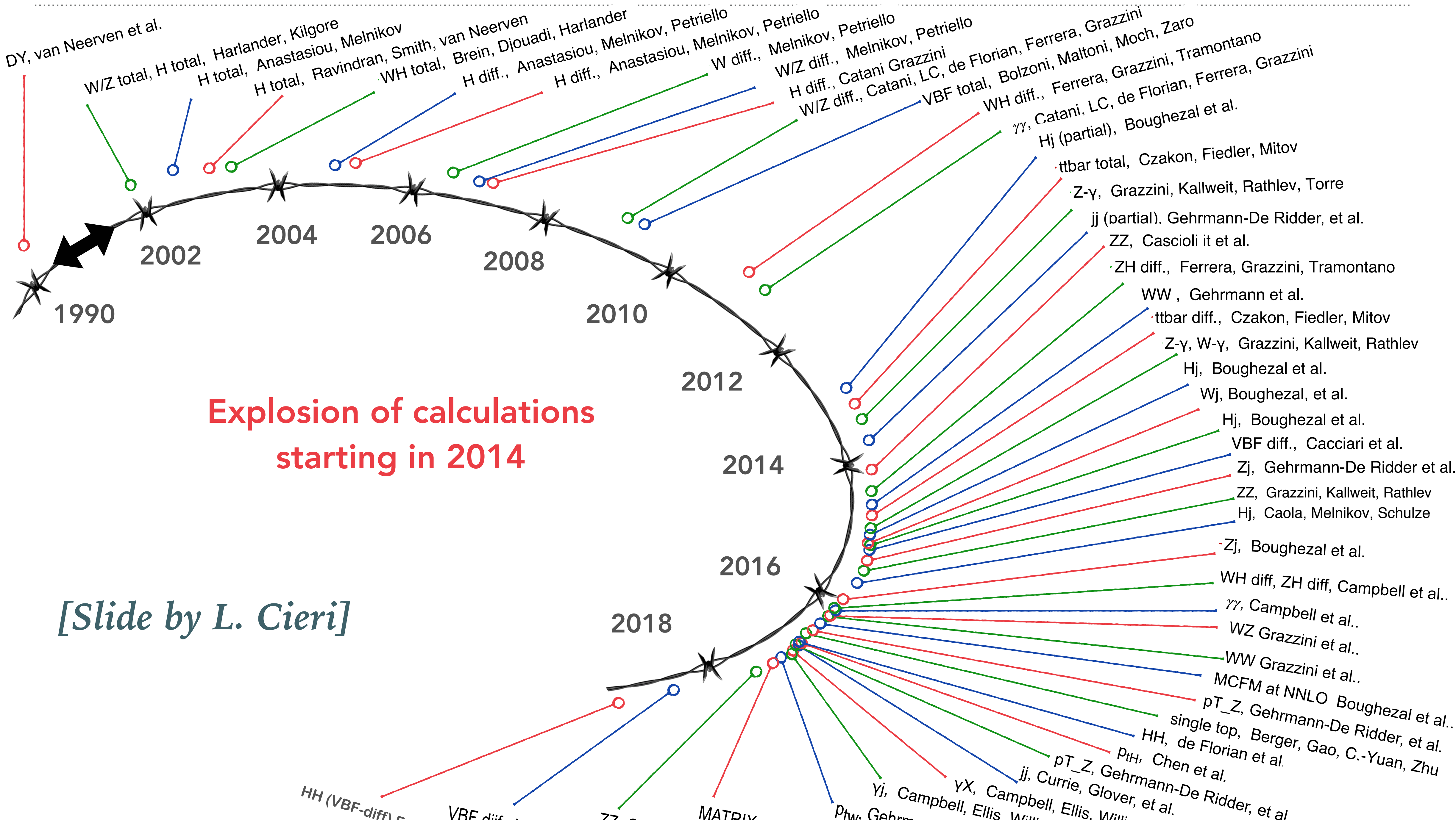
NNLO calculations: the status around the beginning of 2017



[Slide by G. Salam]

BUT THINGS ARE GETTING DIFFICULT AGAIN…

NNLO HADRON-COLLIDER CALCULATIONS VS. TIME



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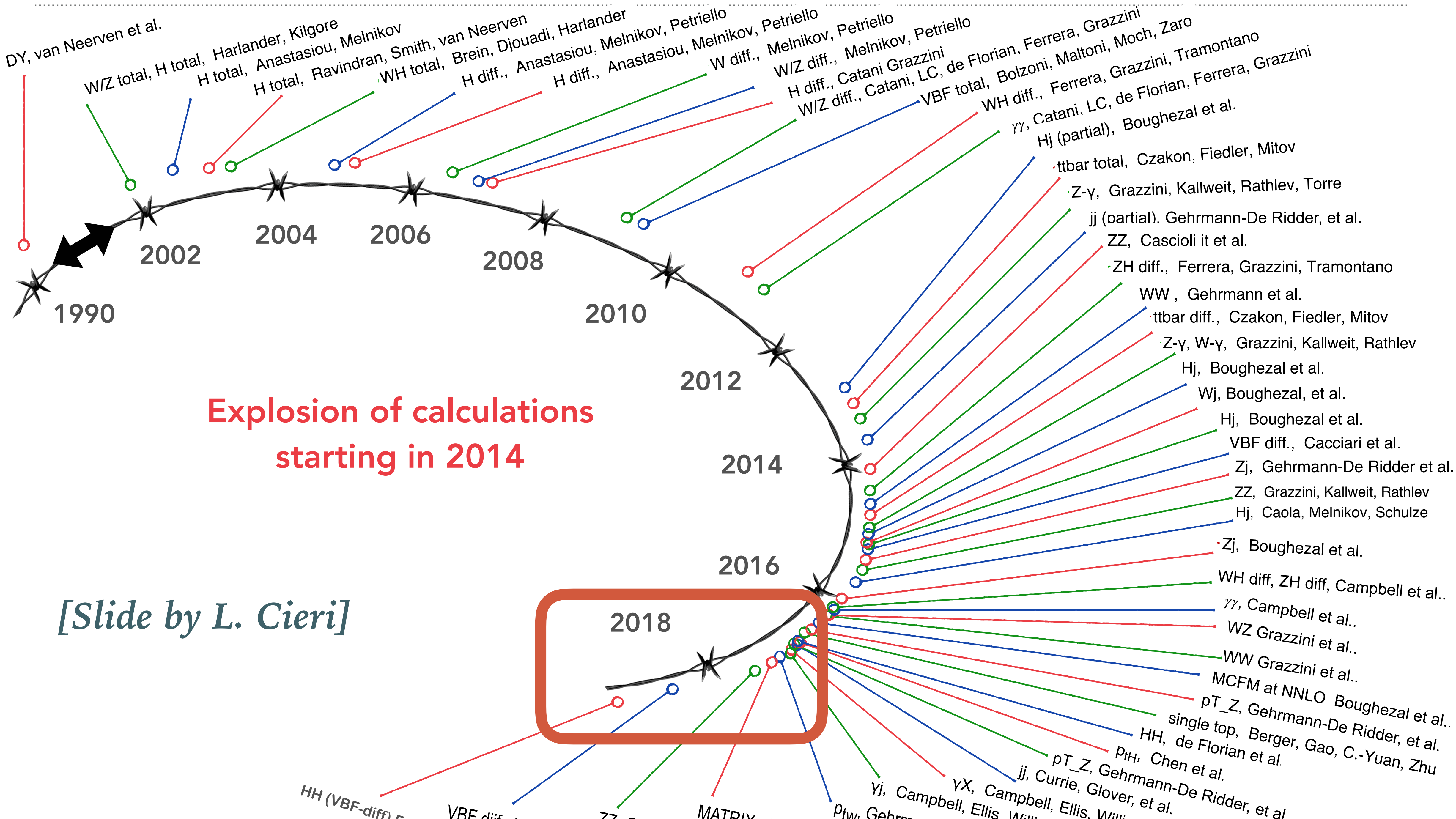
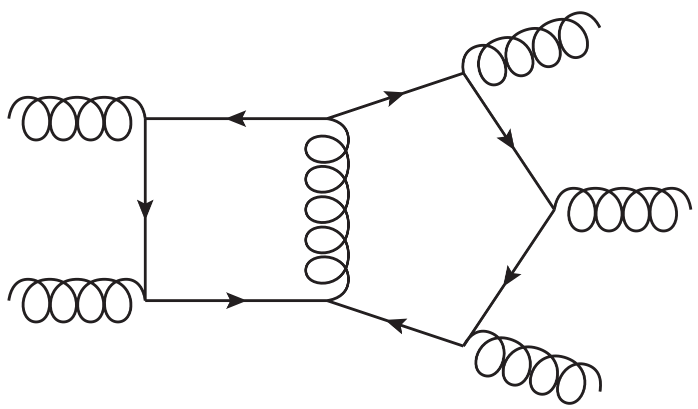
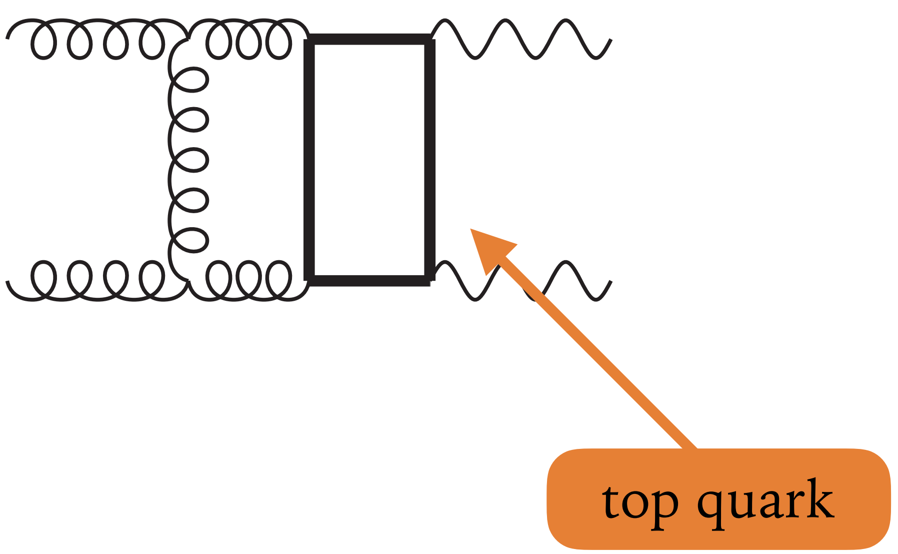


WHAT WE CANNOT DO (YET…)

There is a lot that we can do, but we are not quite there yet…

Properly modelling LHC processes with high precision requires ***more external particles*** and ***massive internal states***





What are we fighting against?

Algebraic complexity

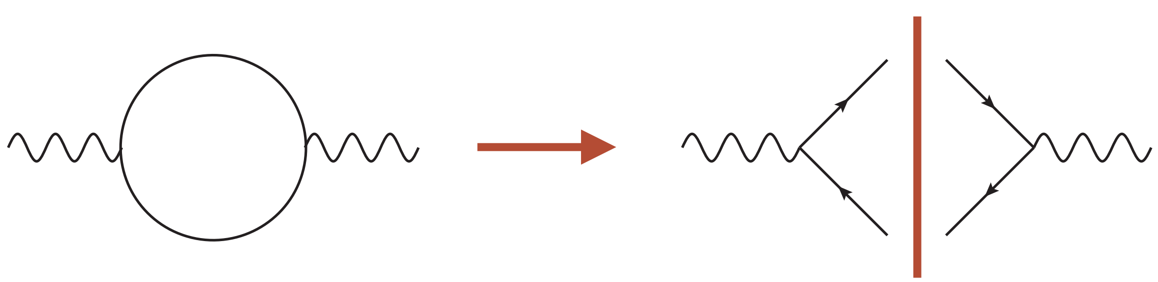
Analytical complexity

New mathematical insight needed to tame these processes!

FEYNMAN INTEGRALS AND THE S-MATRIX

Feynman integrals are source of **analytic structure** of the scattering amplitudes

Multivalued functions. **Branch-cut structure dictated by unitarity!**



$$\sim \int_{4m^2}^{\infty} \frac{ds'}{s'-s-i\epsilon} \frac{1}{\sqrt{s'(s'-4m^2)}}$$

$$\sim \frac{1}{\sqrt{s(s-4m^2)}} \ln\left(\frac{\sqrt{s-4m^2}+\sqrt{s}}{\sqrt{s-4m^2}-\sqrt{s}}\right)$$

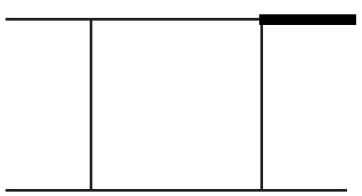
<u>Multivalued functions</u> $\longrightarrow$ ***Riemann surfaces*** and *algebraic geometry*

EXPOSING THE ANALYTIC STRUCTURE

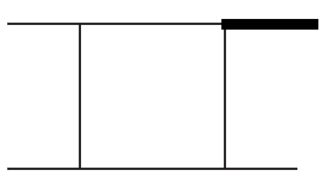
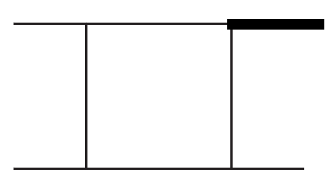
Master Integrals fulfil **differential equations** in kinematical invariants

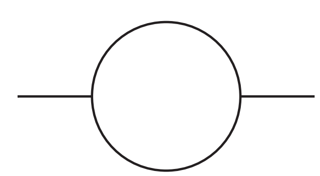
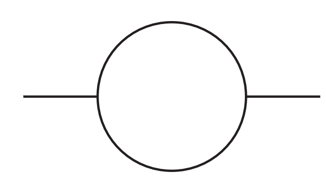
Kotikov, 1990
Bern, Dixon, Kosower, 1993
Remiddi, 1999; Gehrmann, Remiddi 2000

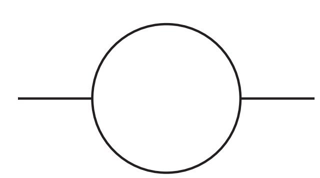
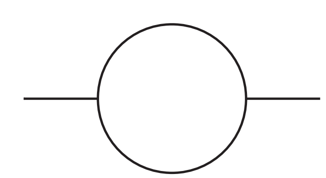
Differential equations trivialise all integrations but one:



$$= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p_2)^2 (k-p_2-p_3)^2 (k-p_1-p_2-p_3)^2}$$

$$s \frac{\partial}{\partial s}$$  $$= \quad \epsilon$$ 

$$+ \frac{1-2\epsilon}{s+t} \left[\frac{1}{s+t+u} \right.$$  $$- \frac{1}{u}$$  $$\left. \vphantom{\frac{1}{u}} \right]$$

$$+ \frac{1-2\epsilon}{s+u} \left[\frac{1}{s+t+u} \right.$$  $$- \frac{1}{t}$$  $$\left. \vphantom{\frac{1}{t}} \right]$$

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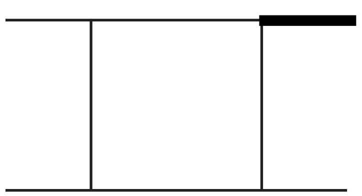

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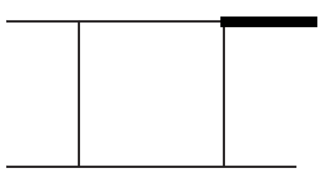

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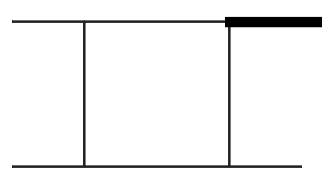

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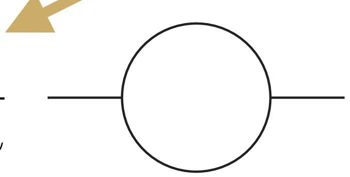

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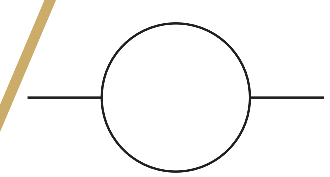

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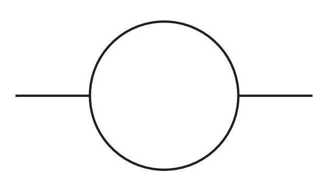

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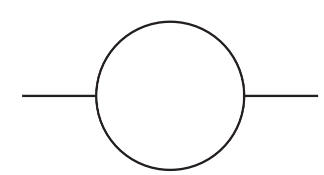


$$\left. \right]$$

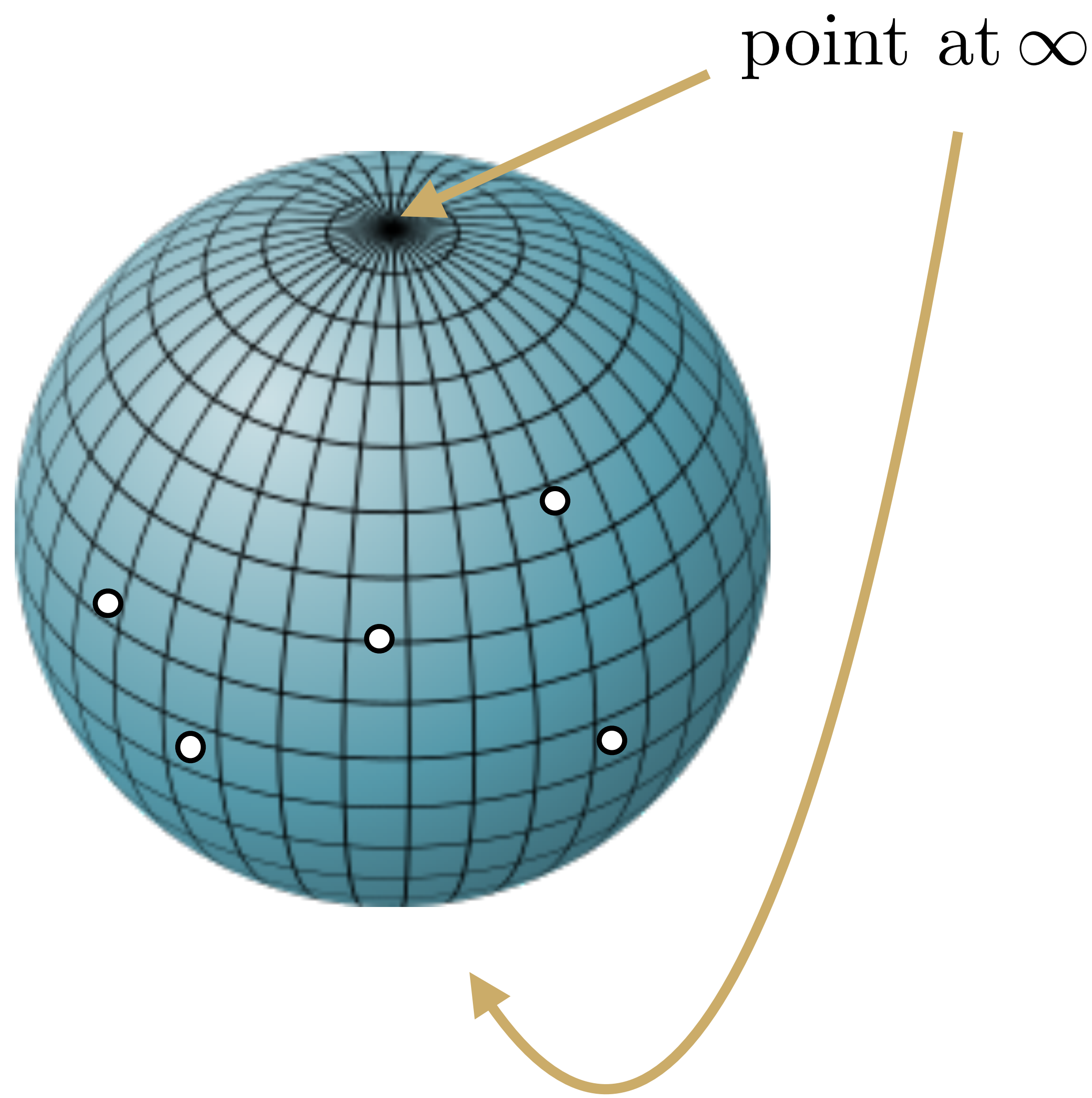
Iterated integrals over rational functions…

A GEOMETRICAL POINT OF VIEW

Dealing with complex functions, the natural concept: **Riemann surface**

$$R(z) = \frac{P(z)}{Q(z)}$$

$$\begin{cases} P(z) & = a_n z^n + \ldots + 1 \\ Q(z) & = b_m z^m + \ldots + 1 \end{cases}$$



A rational function has no **branch cuts**

But it has **poles**

Riemann sphere: *All meromorphic functions on the RS are rational functions*

MULTIPLE POLYLOGARITHMS (MPLS)

Given any rational function R(x), by **factorizing poles** and **partial fractioning** I get

$$\int dx\, R(x) = \int dx\, \frac{p(x)}{q(x)} \sim \left\{ \int dx\, x^n \quad , \quad \int \frac{dx}{(x-c)^k} \right\}$$

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$$\int \frac{dx}{x-c} = \log\left(x-c\right)$$

simple pole

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simple pole

$$\oint_{\gamma_c} \frac{dx}{x-c} = 2\pi i \quad \rightarrow$$

Residue non zero => **multivalued function**

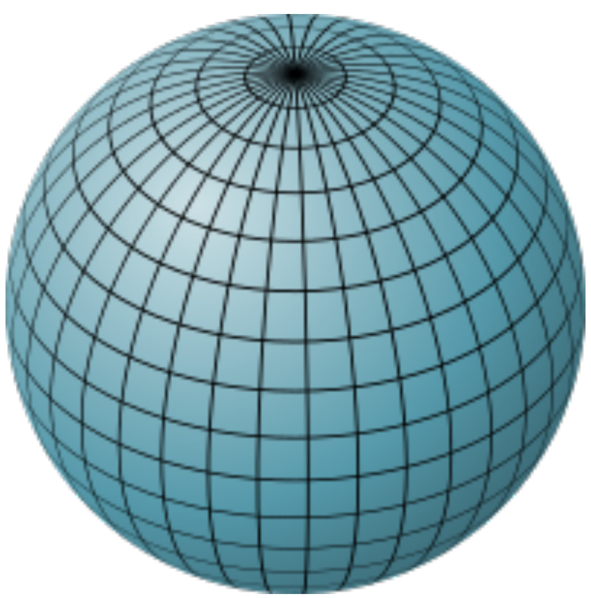
MPLS: INTEGRATING ON THE RIEMANN SPHERE

Integrals of **rational functions** with ***simple poles*** (logarithmic singularities!) on the ***Riemann Sphere***

$$G(c_1, c_2, ..., c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n, t_1)$$

$$= \int_0^x \boxed{\frac{dt_1}{t_1 - c_1}} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} \dots \int_0^{t_{n-1}} \frac{dt_n}{t_n - \boxed{c_n}}$$

We integrate **rational functions**



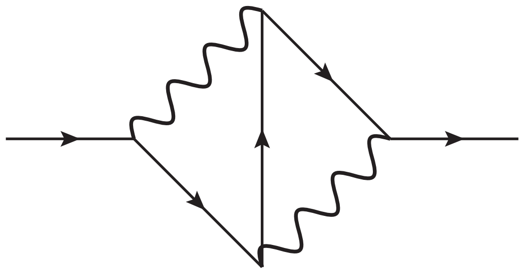
The singularities are generically **complex numbers**!

BEYOND GENUS ZERO

THE ELLIPTIC WORLD

At **two loops**, MPLs with their beautiful properties are not enough.

Electron self-energy in QED @ 2 loops *(computation attempted in 1961 by A. Sabry!)*



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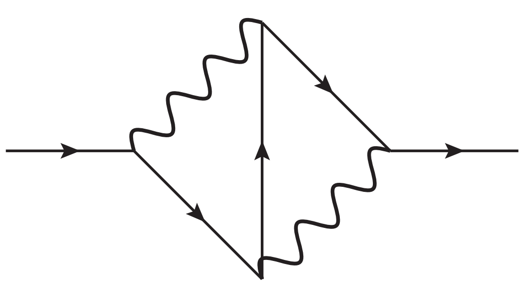
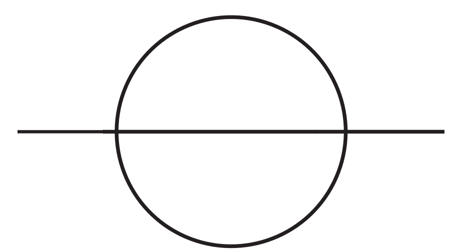
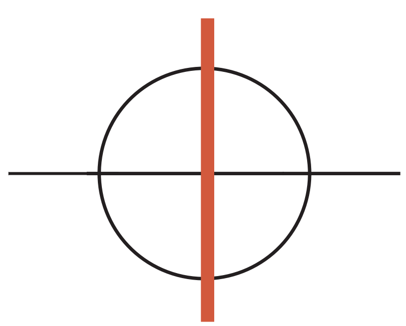




The sunrise integral



$$= \frac{1}{\sqrt{(3m-\sqrt{s})(\sqrt{s}+m)^3}}\,\mathrm{K}\left(\frac{16m^3\sqrt{s}}{(3m-\sqrt{s})(\sqrt{s}+m)^3}\right)$$

Leading singularity -> Elliptic Integral of the first kind:

$$\mathrm{K}(x) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-x\,z^2)}}$$

ELLIPTIC CURVES AND TORII

$$\mathrm{K}(x) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-x\,z^2)}}$$

Consider the function

$$y(z) = \sqrt{(1-z^2)(k^2-z^2)} \quad \text{with} \quad k^2 = \frac{1}{x} > 1$$

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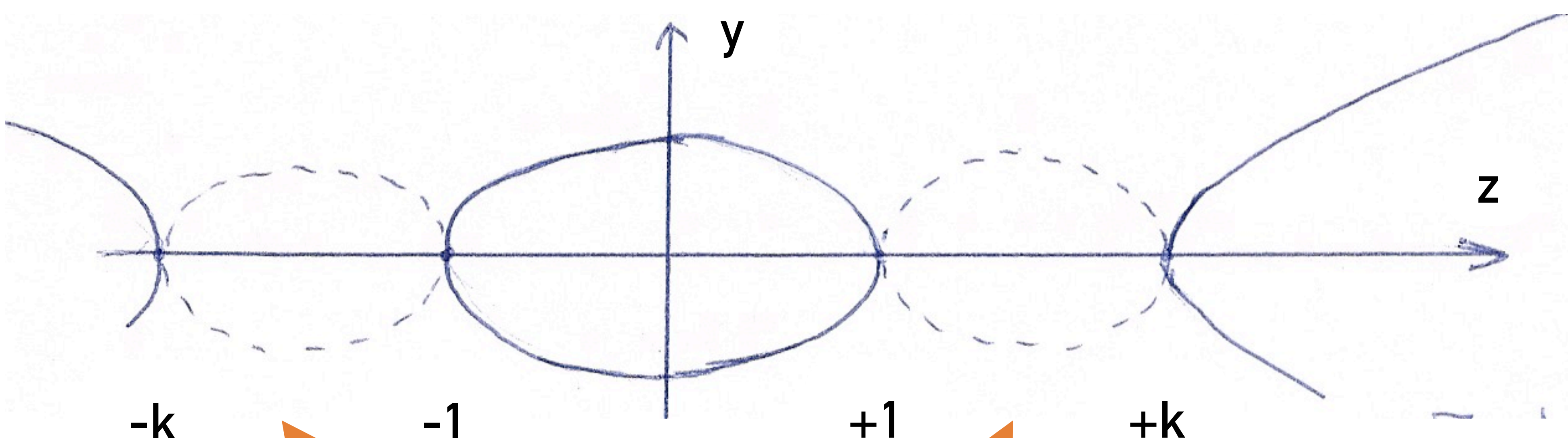
This algebraic equation defines geometrically ***an elliptic curve***

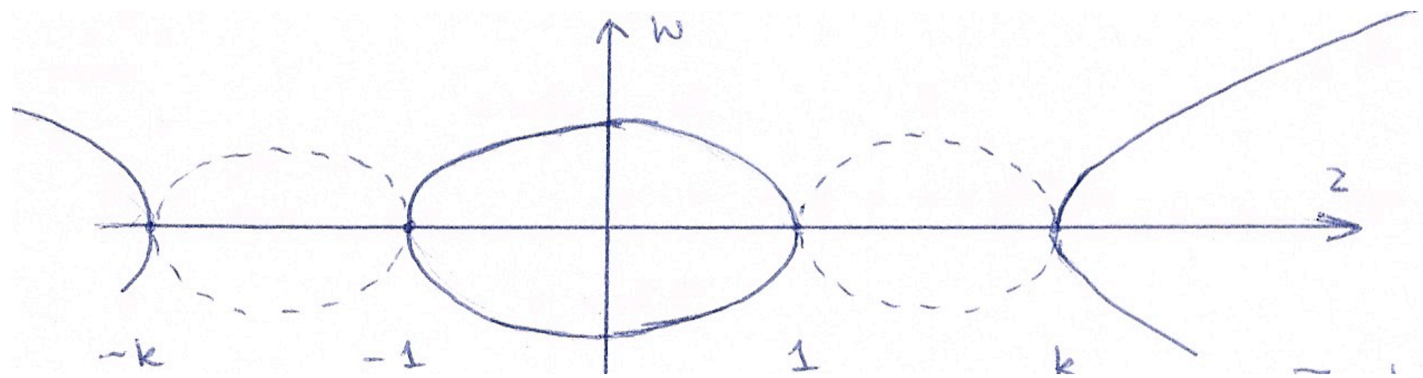


here y becomes imaginary

CHANGING THE GEOMETRY: FROM GENUS 0 TO GENUS 1

They describe by a **new geometry** which has intrinsically **2 degrees of freedom**: a **Torus**!

$$y(z) = \sqrt{(1 - z^2)(k^2 - z^2)}$$



[Drawings by C. Teleman, Riemann Surfaces]

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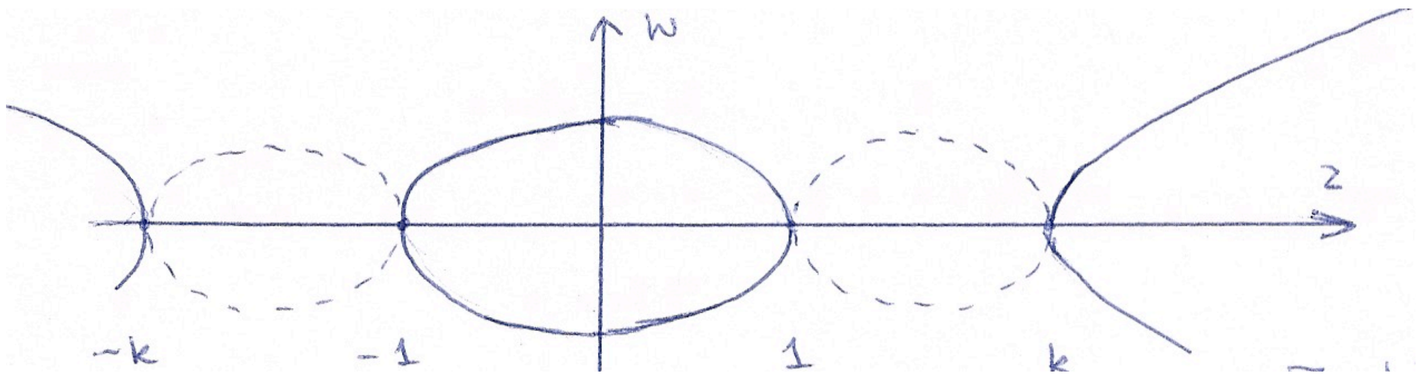

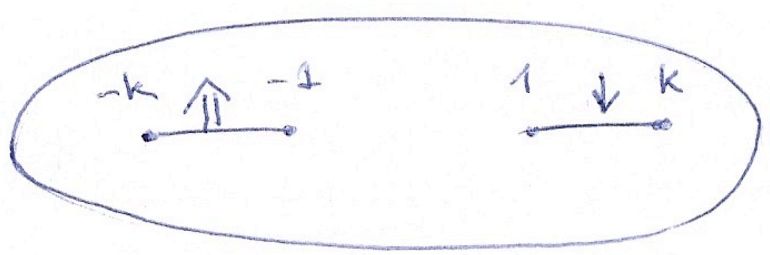

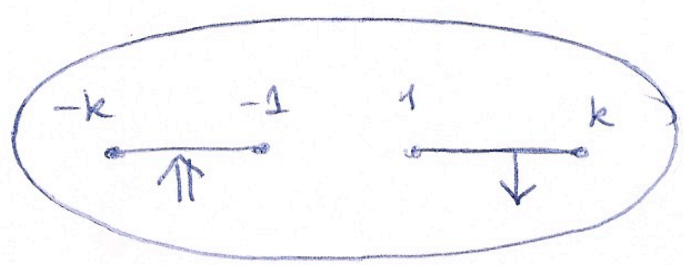

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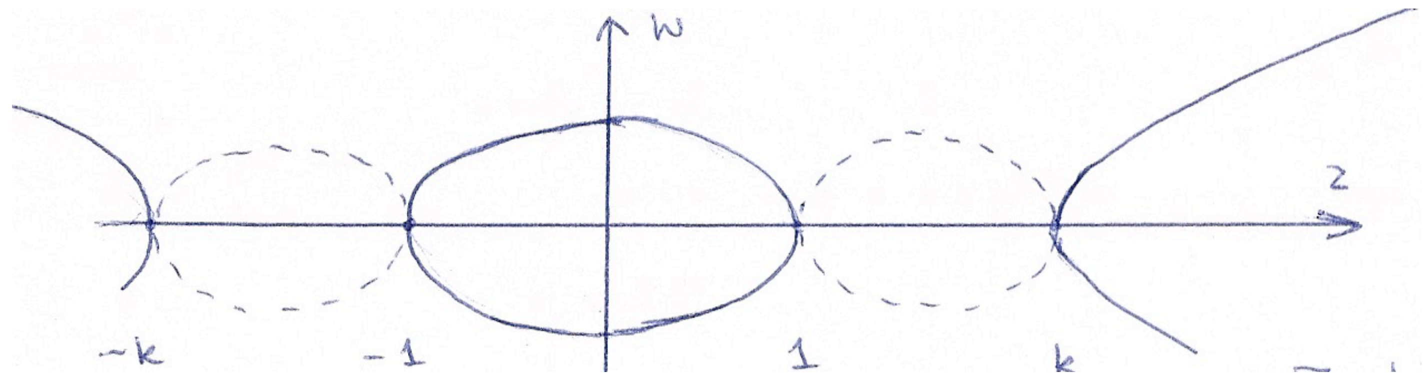

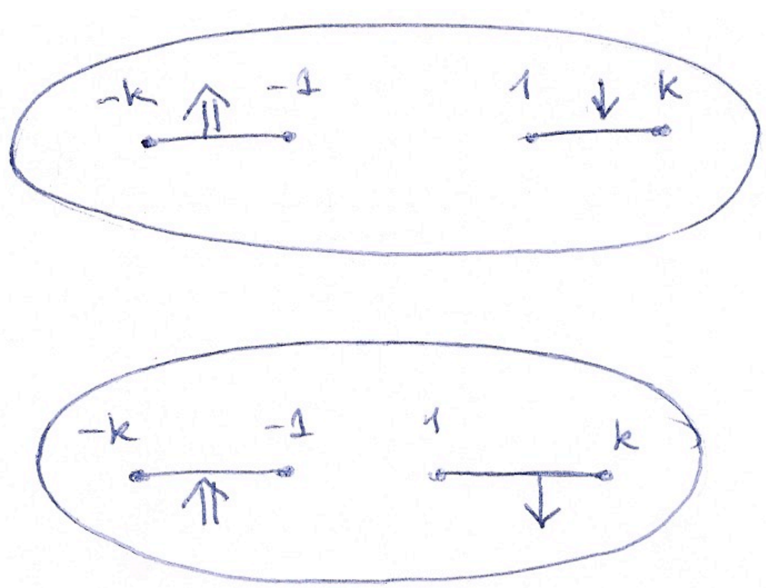

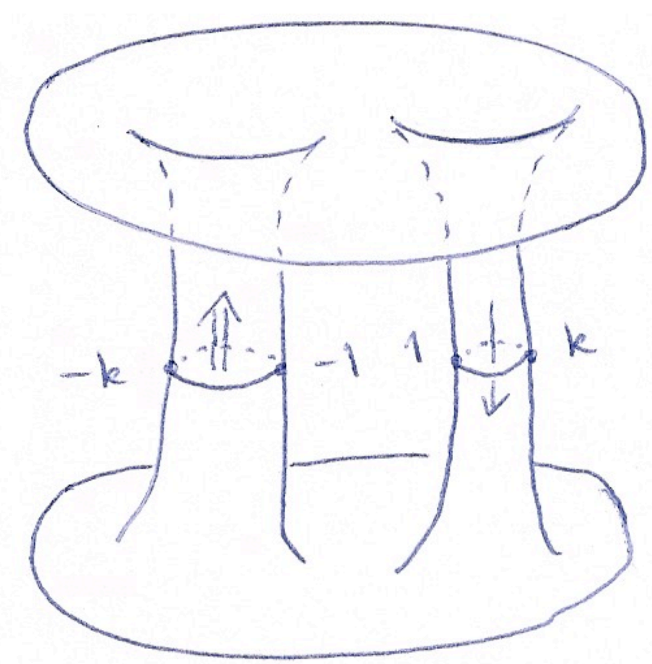

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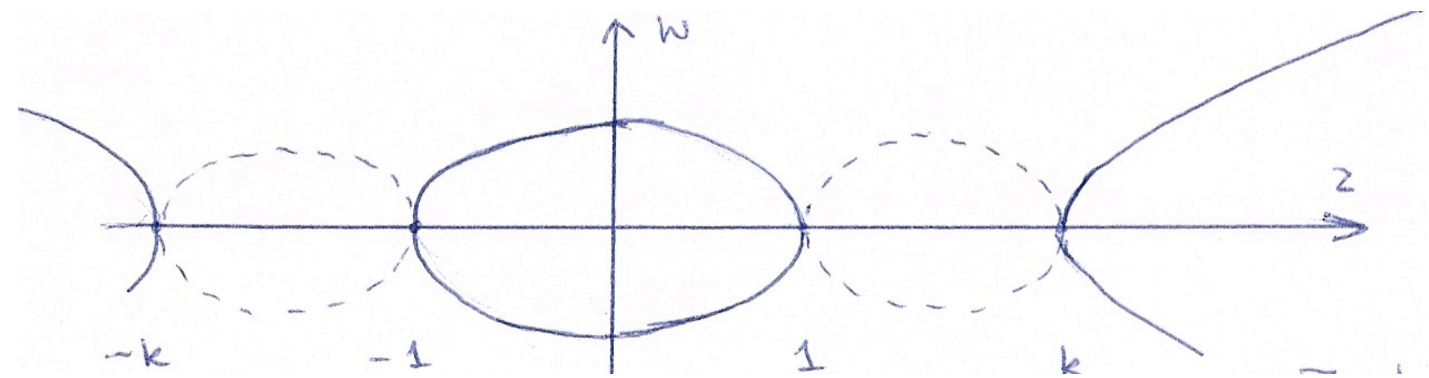

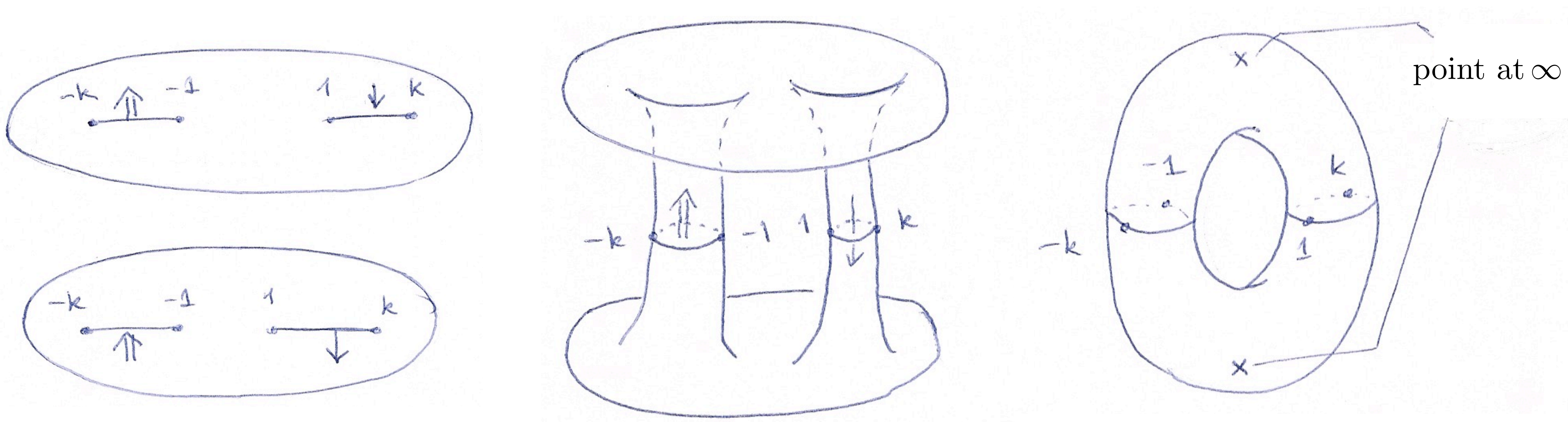

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A NEW GEOMETRY

Some definitions. Take a completely general elliptic curve:

$$y^2 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)$$

We define the two periods as

$$\omega_1 = 2c_4 \int_{a_2}^{a_3} \frac{dx}{y} = 2\,\mathrm{K}(\lambda)\,, \quad \omega_2 = 2c_4 \int_{a_1}^{a_2} \frac{dx}{y} = 2\,i\,\mathrm{K}(1-\lambda)\,,$$

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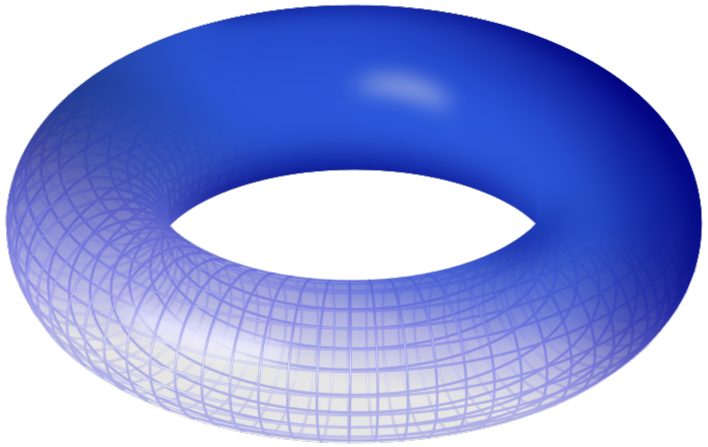

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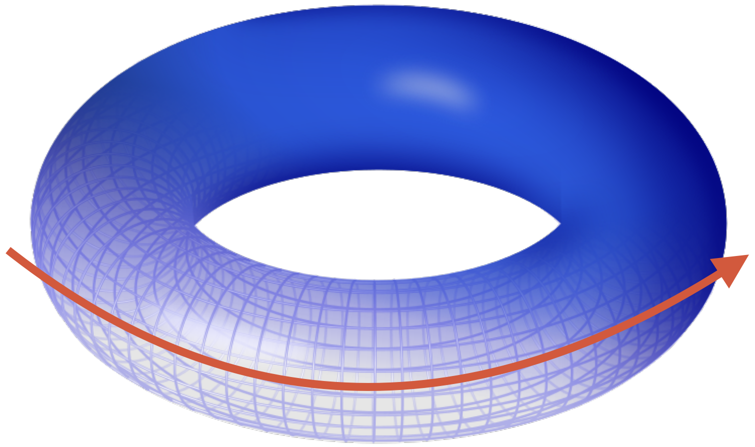

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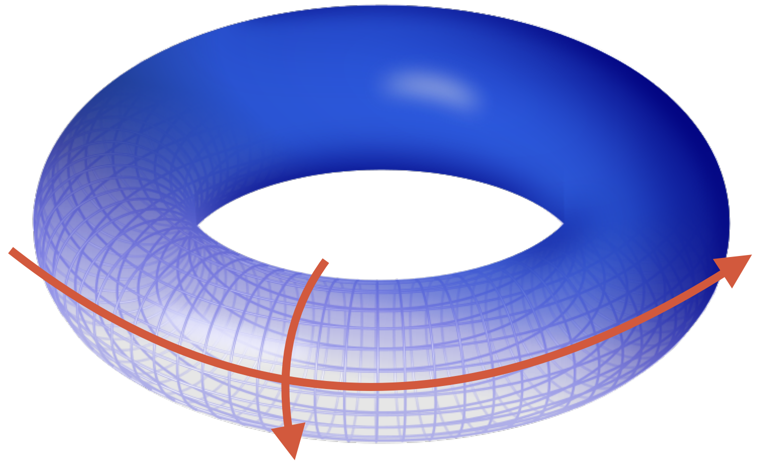

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$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

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$$\boxed{\begin{aligned} \Psi_0(0, x, \vec{a}) &= \frac{c_4}{\omega_1\, y} \\ \Psi_1(c, x, \vec{a}) &= \frac{1}{x-c}\,, \\ \Psi_{-1}(c, x, \vec{a}) &= \frac{y_c}{y(x-c)} + Z_4(c, \vec{a})\,\frac{c_4}{y}\,, \\ \Psi_1(\infty, x, \vec{a}) &= -Z_4(x, \vec{a})\,\frac{c_4}{y}\,, \\ \Psi_{-1}(\infty, x, \vec{a}) &= \frac{x}{y} - \frac{1}{y}\left[a_1 + 2c_4\, G_*(\vec{a})\right]\,, \end{aligned}}$$

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\end{aligned}}$$

$$Z_4(x, \vec{a}) \equiv \int_{a_1}^x dx'\, \Phi_4(x', \vec{a})$$

$$\widetilde{\Phi}_4(x, \vec{a}) \equiv \frac{1}{c_4\, y}\left(x^2 - \frac{s_1}{2}\, x + \frac{s_2}{6}\right)$$

$$\Phi_4(x, \vec{a}) \equiv \widetilde{\Phi}_4(x, \vec{a}) + 4c_4\, \frac{\eta_1}{\omega_1}\, \frac{1}{y}$$

$$z_* = \frac{c_4}{\omega_1} \int_{a_1}^{-\infty} \frac{dx'}{y} \equiv \frac{1}{2} - \frac{\mathrm{F}(\sqrt{\alpha}|\lambda)}{2\,\mathrm{K}(\lambda)}$$

$$G_*(\vec{a}) = \left(\frac{2\eta_1}{\omega_1} - \frac{\lambda}{3} + \frac{2}{3}\right) \mathrm{F}(\sqrt{\alpha}|\lambda) - \mathrm{E}(\sqrt{\alpha}|\lambda) + \sqrt{\frac{\alpha(\alpha\lambda - 1)}{\alpha - 1}}$$

ELLIPTIC MULTIPLE POLYLOGARITHMS

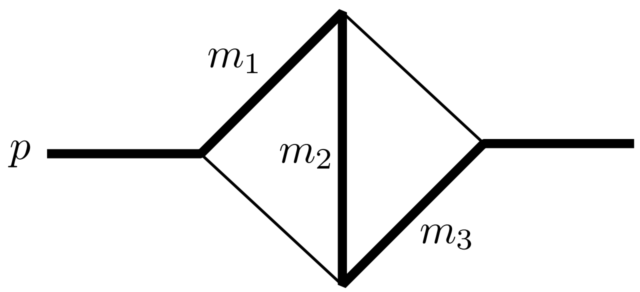
$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

Iterated integrals of rational functions on the elliptic curve

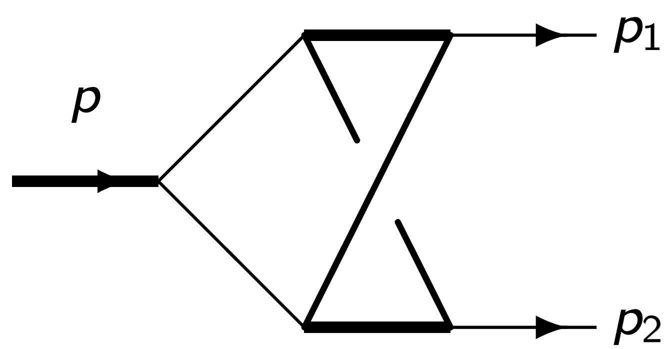
- Only single poles, similarly to multiple polylogarithms (log-singularities)
- Definite parity (invariant as y-> -y, results do not depend on choice of branch cuts)
- Manifestly contain MPLs
- Require an infinite tower of kernels. *Only a small number shows up in physical applications!*

Essentially equivalent to the elliptic polylogarithms introduced in math by ***Brown and Levin*** and studied in ***string theory*** as iterated integrals on the Torus [Broedel, Mafra, Matthes, Schlotterer '14]

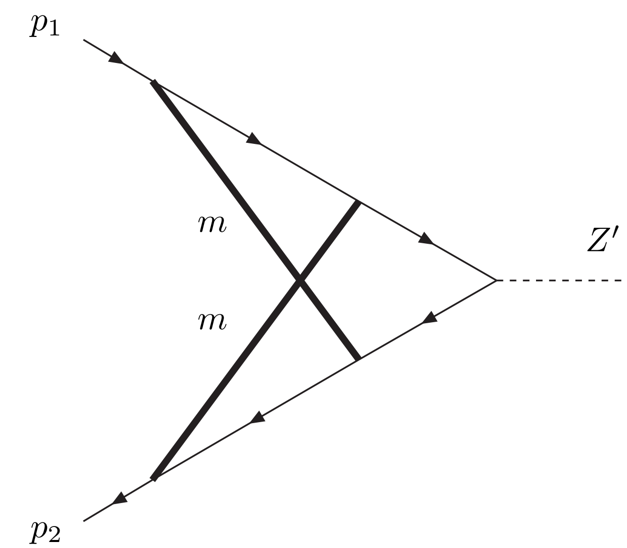
THE ELLIPTIC WORLD



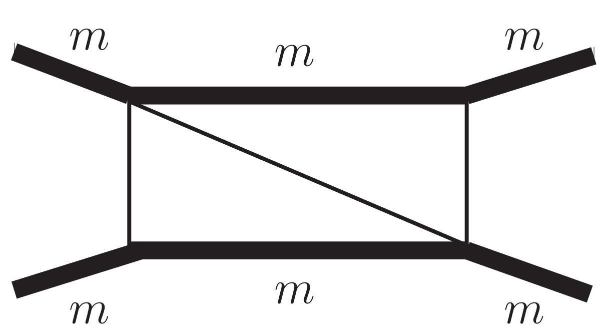
Kite integral (self-energies…)



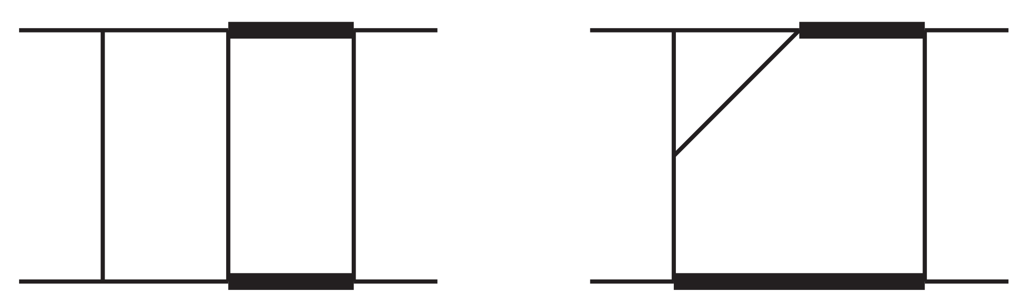
QCD with top quarks



EW form factor



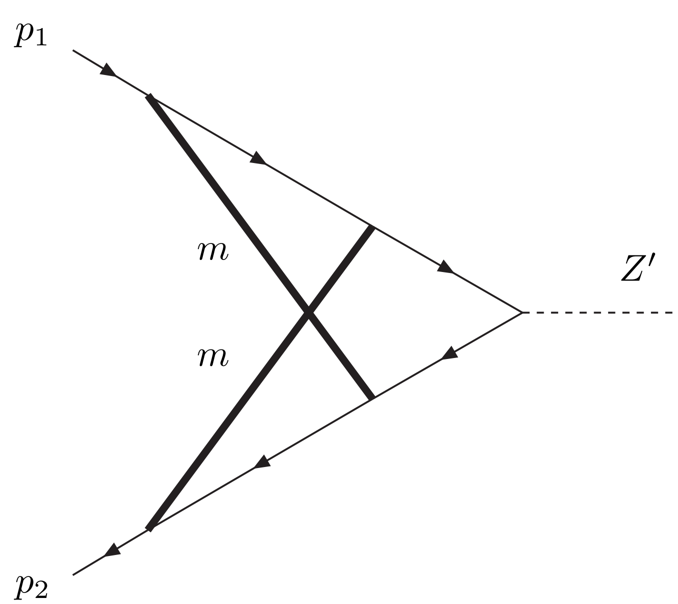
Bhabha scattering in QED



QCD-EW Drell-Yan…

In all these cases, there is always an **elliptic curve** lurking somewhere

EW FORM FACTOR: A VERY NON-TRIVIAL EXAMPLE



$$J_{a_1,\ldots,a_7} = \int \frac{d^D k_1 d^D k_2}{(i\pi)^D} \frac{D_7^{-a_7}}{\prod_{i=1}^6 D_i^{a_i}}$$

$$\begin{aligned} D_1 &= k_1^2 - m^2, & D_3 &= (k_1 - p_1)^2, & D_5 &= (k_1 - k_2 - p_1)^2, \\ D_2 &= k_2^2 - m^2, & D_4 &= (k_2 - p_2)^2, & D_6 &= (k_2 - k_1 - p_2)^2, \end{aligned}$$

$$D_7 = k_1 \cdot p_2$$

Top-sector can be reduced to three master integrals

$$N_1 = J_{1,1,1,1,1,1,0}\,, \quad N_2 = J_{2,1,1,1,1,1,0}\,, \quad N_3 = J_{1,1,1,1,1,1,-1}$$

[Aglietti, Bonciani, Remiddi ’07; Broedel, Duhr, Dulat, Penante, Tancredi ’19]

DIFFERENTIAL EQUATIONS - COUPLED! -

Three master integrals depend on one dimensionless ratio: $x = -\frac{s}{m^2}$

The integrals are **FINITE** in d=4 and satisfy **three coupled diff. eqs.:**

$$\frac{d}{dx}\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} -\frac{2}{x} & \frac{2}{x} & 0 \\ \frac{x+4}{x(x+1)(x-8)} & \frac{7-2x}{(x+1)(x-8)} & 0 \\ -\frac{1}{6x} & \frac{1+x}{3x} & -\frac{1}{x} \end{pmatrix}\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} + \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

Coupled homogeneous system is **2x2**!

Contribution from simpler integrals: ***non-homogeneous part, simpler!***

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Solution given by Maximal Cut of the graphs! [Primo, Tancredi '16,'17]

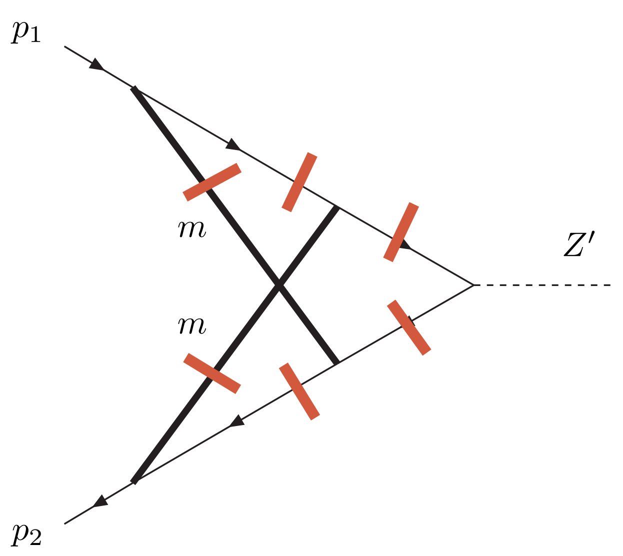
THE HOMOGENEOUS SOLUTIONS (AND THE ELLIPTIC CURVE)

Indeed, if I define: $q^2 = (p_1 + p_2)^2 = -m^2/a$

$$d_\pm = \frac{1}{2}\left(1 - \sqrt{1 - 4a(1+2a) \pm 8\sqrt{a^3(a+1)}}\right)$$

And call $\mathrm{MaxCut}(N_j) = \Omega_j^{\mathrm{ew}}$

$$\rightarrow \quad \Omega_1^{\mathrm{ew}} =$$



THE HOMOGENEOUS SOLUTIONS (AND THE ELLIPTIC CURVE)

Then the **max-cut** of the three master integrals read:

$$\Omega_1^{(\mathrm{ew})} = -\frac{2a^2}{m^4\, d_{+-}} \boxed{\omega_1} \qquad \Omega_3^{(\mathrm{ew})} = \frac{(2-a)a}{3m^2 d_{+-}} \boxed{\omega_1}$$

$$\Omega_2^{(\mathrm{ew})} = \frac{a^2(4a(3a+2)-1)}{2m^4(8a-1)(a+1)\, d_{+-}} \boxed{\omega_1} + 3\frac{a^2\, d_{+-}}{m^4(1+a)(8a-1)} \boxed{\eta_1}$$

Where

$\omega_1, \omega_2, \eta_1, \eta_2$ **Periods** and **quasi-periods** of the elliptic curve defined by:

$$y^2 = (x-d_-)(x-d_+)(x-1+d_-)(x-1+d_+)$$

EW 3-POINT FUNCTION: SOLUTION!

All three master integrals can be computed by ***direct integration over their Feynman parameter representation.*** First master reads:

$$N_1 =$$

$$= a \int \frac{\prod_{i=2}^{6} dx_i}{\left[\left(x_2 x_3 + x_5\right) x_6 - x_4 \left(x_2 + x_3 + x_4 - 1\right) x_5 + a \left(x_3 + x_4 - 1\right) \left(x_2^2 + \left(2x_4 - 1\right) x_2 + \left(x_4 - 1\right) x_4 - x_5 - x_6\right)\right]^2}$$

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We find $N_1 = \Omega_1^{(ew)} \left[2 Q_{-1}(a) + Q_1(a)\right]$

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We find $N_1 = \Omega_1^{(ew)} \left[2Q_{-1}(a) + Q_1(a)\right]$ with:

$$\begin{aligned} Q_{1-}(a) = {} & \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & r_- & 0 \end{smallmatrix}; 1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & r_- & 1 \end{smallmatrix}; 1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & r_- & 0 \end{smallmatrix}; 1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & r_- & 1 \end{smallmatrix}; 1\right) \\ & + 2\left[\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_- & 0 \end{smallmatrix}; 1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_- & 1 \end{smallmatrix}; 1\right)\right] - 4i\pi\left[\mathcal{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & r_- & 0 \end{smallmatrix}; 1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & r_- & 1 \end{smallmatrix}; 1\right)\right] \\ & - \log(a)\left[2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & r_- \end{smallmatrix}; 1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 0 & r_- \end{smallmatrix}; 1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 1 & r_- \end{smallmatrix}; 1\right) - 4i\pi\mathcal{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & r_- \end{smallmatrix}; 1\right)\right] \\ & + \left(r_- \to 1 - r_-\right) \ , \end{aligned}$$

EW 3-POINT FUNCTION: SOLUTION!

$$
\begin{aligned}
Q_1(a) \;=\; & -4\Big[\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & \infty & 0 & -a\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & \infty & 1 & 1+a\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & \infty & 0 & 1\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & \infty & 1 & 0\end{smallmatrix};1\big)\Big]\\
& -3\Big[\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & \infty & 1-a & 1\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & \infty & 1-a & -a\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & \infty & a & 0\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & \infty & a & 1+a\end{smallmatrix};1\big)\\
& +\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & 0 & a & 0\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & 0 & a & 1+a\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & 1 & 1-a & 1\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & 1 & 1-a & -a\end{smallmatrix};1\big)\Big]\\
& +\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & 0 & 1+a & 1\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & 1 & -a & 0\end{smallmatrix};1\big)-2\Big[\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & 0 & 0 & -a\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & 0 & 1 & 1+a\end{smallmatrix};1\big)\\
& +\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & 0 & -a & 0\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & 1 & 0 & -a\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & 1 & 1 & a+1\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & 1 & 1+a & 1\end{smallmatrix};1\big)\Big]\\
& +3\big(\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & \infty & -a & 0\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & \infty & 1+a & 1\end{smallmatrix};1\big)\big)+4i\pi\Big[\mathcal{E}_4\big(\begin{smallmatrix}0 & 0 & 1 & 1\\ 0 & 0 & 1-a & 1\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & 0 & 1 & 1\\ 0 & 0 & 1-a & -a\end{smallmatrix};1\big)\\
& +\mathcal{E}_4\big(\begin{smallmatrix}0 & 0 & 1 & 1\\ 0 & 0 & -a & 0\end{smallmatrix};1\big)+2\mathcal{E}_4\big(\begin{smallmatrix}0 & 0 & 1 & 1\\ 0 & 0 & 0 & -a\end{smallmatrix};1\big)+2\mathcal{E}_4\big(\begin{smallmatrix}0 & 0 & 1 & 1\\ 0 & 0 & 1 & 1+a\end{smallmatrix};1\big)+2\mathcal{E}_4\big(\begin{smallmatrix}0 & 0 & 1 & 1\\ 0 & 0 & a & 0\end{smallmatrix};1\big)\\
& +2\mathcal{E}_4\big(\begin{smallmatrix}0 & 0 & 1 & 1\\ 0 & 0 & a & 1+a\end{smallmatrix};1\big)\Big]+\log\left(\frac{a+1}{a}\right)\Big[-2\big(\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1\\ 0 & 0 & 1\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1\\ 0 & 1 & 1\end{smallmatrix};1\big)\big)\\
& +8i\pi\big(\mathcal{E}_4\big(\begin{smallmatrix}0 & 0 & 1\\ 0 & 0 & a\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & 0 & 1\\ 0 & 0 & 1\end{smallmatrix};1\big)\big)-4\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1\\ 0 & \infty & 1\end{smallmatrix};1\big)-3\big(\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1\\ 0 & \infty & a\end{smallmatrix};1\big)+\mathcal{E}_4\big(\begin{smallmatrix}0 & -1 & 1\\ 0 & 0 & a\end{smallmatrix};1\big)\big)\Big]\\
& +\frac{1}{6}\mathcal{E}_4\big(\begin{smallmatrix}0\\ 0\end{smallmatrix};1\big)\left(-6G(-1,0,-1,a)+6G(0,0,-1,a)-\pi^2\log\left(\frac{a+1}{a}\right)-12\zeta_3\right)\;.
\end{aligned}
$$

Pure function of “elliptic” **transcendental weight = 3**

EW 3-POINT FUNCTION: SOLUTION!

Similarly for the other two master integrals we find by direct calculation

$$N_2 = \Omega_2^{(ew)}\widetilde{N}_1 + H_2^{(ew)}\widetilde{N}_2$$

$$N_3 = \Omega_3^{(ew)}\widetilde{N}_1 + X_3^{(ew)}\widetilde{N}_3$$

Where

$$H_2^{(\text{ew})} = \frac{a^2 d_{+-}}{6m^4(1+a)(1-8a)}\frac{1}{\omega_1} \qquad X_3^{(\text{ew})} = \frac{a}{72\,m^2}$$

And the $\widetilde{N}_j$ are ***pure*** linear combinations of eMPLs !

EW 3-POINT FUNCTION: SOLUTION!

This suggests to perform the basis change

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} \Omega_1^{(\mathrm{ew})} & 0 & 0 \\ \Omega_2^{(\mathrm{ew})} & H_2^{(\mathrm{ew})} & 0 \\ \Omega_3^{(\mathrm{ew})} & 0 & X_3^{(\mathrm{ew})} \end{pmatrix} \begin{pmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \end{pmatrix}$$

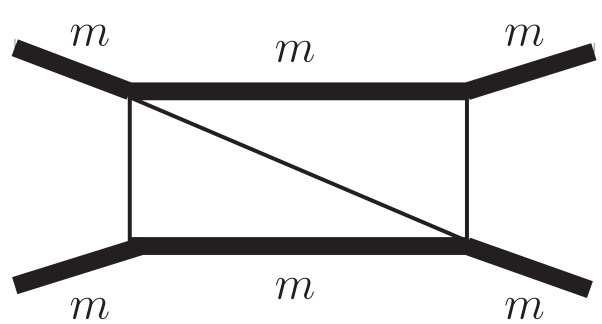
Define a new basis of master integrals of ***uniform transcendental weight***

- Rational/Algebraic functions in pre-factors substituted by ***periods of corresponding algebraic surface*** (in this case elliptic curve!)
- Overall structure remains the same as with MPLs

4-POINT FUNCTIONS: BHABHA SCATTERING

Electron-positron scattering in QED @ 2 loops

$\widetilde{B}(x,y) \quad \sim$



$$\frac{-s}{m^2} = \frac{(1-x)^2}{x} \quad \text{and} \quad \frac{-t}{m^2} = \frac{(1-y)^2}{y},$$

Differential equation in canonical form [Henn, Smirnov '13]

$$d\widetilde{B}(x,y)$$
$$= g_1\, d\log \frac{1-Q}{1+Q} + g_2\, d\log \frac{(1+x)+(1-x)Q}{(1+x)-(1-x)Q} + g_3\, d\log \frac{(1+y)+(1-y)Q}{(1+y)-(1-y)Q}$$

With non-rationalisable square root! $\quad Q = \sqrt{\frac{(x+y)(1+xy)}{x+y-4xy+x^2y+xy^2}}\,.$

4-POINT FUNCTIONS: BHABHA SCATTERING

Q(x,y) turns out to describe an elliptically fibered K3 surface

Integrating in x first and then in y, we can write ***compact result in terms of eMPLs***

$$
\begin{aligned}
\widetilde{B}(x,y) =\;& 16\log\frac{-t}{m^2}\Big[\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1\\ 0 & 1+1/y & 1\end{smallmatrix};\bar{x},\vec{a}\big)-\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1\\ 0 & 1+y & 1\end{smallmatrix};\bar{x},\vec{a}\big)\\
&+\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1\\ \infty & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big)+\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1\\ 1 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big)+\zeta_2\,\mathcal{E}_4\big(\begin{smallmatrix}-1\\ \infty\end{smallmatrix};\bar{x},\vec{a}\big)+\zeta_2\,\mathcal{E}_4\big(\begin{smallmatrix}-1\\ 1\end{smallmatrix};\bar{x},\vec{a}\big)\Big]\\
&-8\left(8\zeta_2+4\mathrm{Li}_2(y)+\log^2 y\right)\Big[\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1\\ 0 & 1+1/y\end{smallmatrix};\bar{x},\vec{a}\big)+\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1\\ 0 & 1+y\end{smallmatrix};\bar{x},\vec{a}\big)-\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1\\ 0 & 1\end{smallmatrix};\bar{x},\vec{a}\big)\Big]\\
&-32\,\zeta_2\,\Big[\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1\\ \infty & 1\end{smallmatrix};\bar{x},\vec{a}\big)-\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1\\ 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big)\Big]+16\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 0 & 1+1/y & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big)\\
&-32\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 0 & 1+1/y & 2 & 1\end{smallmatrix};\bar{x},\vec{a}\big)-16\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 0 & 1+y & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big)+32\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 0 & 1+y & 2 & 1\end{smallmatrix};\bar{x},\vec{a}\big)\\
&+16\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ \infty & 0 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big)-24\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ \infty & 1 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big)-32\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ \infty & 1 & 2 & 1\end{smallmatrix};\bar{x},\vec{a}\big)\\
&+16\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big)+40\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big)-32\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 1 & 1 & 2 & 1\end{smallmatrix};\bar{x},\vec{a}\big)\\
&+\frac{4}{3}\left(12\mathrm{Li}_3(y)+24\zeta_2\log y+\log^3 y\right)\Big[\mathcal{E}_4\big(\begin{smallmatrix}-1\\ \infty\end{smallmatrix};\bar{x},\vec{a}\big)+\mathcal{E}_4\big(\begin{smallmatrix}-1\\ 1\end{smallmatrix};\bar{x},\vec{a}\big)\Big]\\
&+64\zeta_4-32\zeta_2\mathrm{Li}_2(y)+16\mathrm{Li}_4(y)+8\zeta_2\log^2 y+\frac{1}{3}\log^4 y\,.
\end{aligned}
\tag{5.8}
$$

Pure functions weight 4 !

[Broedel, Dulat, Duhr, Penante, Tancredi: to appear]

CONCLUSIONS AND SUMMARY

We are developing a new framework to compute Feynman integrals beyond MPLs which is

- Robust and apparently quite general in the elliptic case (more calculations underway, four point functions for bhabha and drell-yan, etc…)
- Its construction, inspired mainly by work of Brown and Levin, suggests a way to generalise it further to more complicated geometries
- Direct connection to iterated integrals of modular forms (Eisenstein series) and other classes of functions
- It is allowing us to make important steps towards the generalisation of the concept of pure functions/pure Feynman integrals to the elliptic case!

THANK YOU!