Four-loop quark form factor with quartic fundamental colour factor

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in collaboration with Roman Lee, Alexander Smirnov and Matthias Steinhauser

[R.N. Lee, A. Smirnov, V.S. & M. Steinhauser'19]

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The quark-anti-quark-photon form factor with massless quarks which is obtained from the corresponding vertex function Γ_q^μ via

$$F_q(q^2) \;=\; -\frac{1}{4(1-\epsilon)q^2}\mathrm{Tr}\left(\not{q}_2\,\Gamma_q^\mu\,\not{q}_1\,\gamma_\mu\right)\;,$$

where $q = q_1 + q_2$, and q_1 (q_2) is the incoming quark (anti-quark) momentum.

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The analytic evaluation of the contribution with the colour factor $(d_F^{abcd})^2$ which for a SU(N_c) group is given by

$$\frac{(d_F^{abcd})^2}{N_A} \;\; = \;\; \frac{N_c^4 - 6N_c^2 + 18}{96N_c^2}$$

$$F_q \quad = \quad 1 + \sum_{n \geq 1} \left(\frac{\alpha_s^0}{4\pi} \right)^n \left(\frac{\mu^2}{-q^2 - i0} \right)^{n\epsilon} F_q^{(n)} \,,$$

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The cusp and collinear anomalous dimensions $\gamma_{\rm cusp}$ and γ_q are extracted from the pole part of $\log(F_q)$ after renormalization of α_s The corresponding n-loop coefficients are defined by

$$\gamma_x \;=\; \sum_{n\geq 0} \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^n \gamma_x^n\,,$$

with $x = $ cusp or $x = q$.

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Three-loop results
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Analytic results for the three-loop master integrals up to weight 8
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motivated by a future four-loop calculation.

The photon-quark form factor in the large-N_c limit.
[J. Henn, A. Smirnov, V.S. & M. Steinhauser'16;
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The n_f^2 and $n_{q\gamma} n_f$ contributions to the quark and gluon QCD form factors
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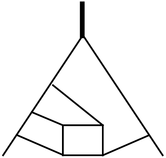
- IBP reduction to master integrals using `FIRE` combined with `LiteRed`.
- Evaluation of the master integrals with differential equations using a canonical basis.



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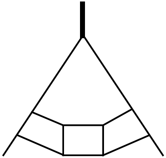

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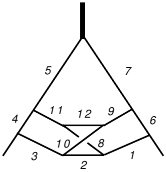
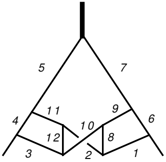
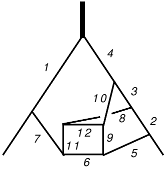


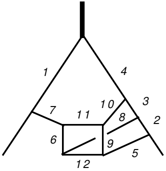
df2-2



df2-3



df2-5



df2-6

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non-planar family	# 1-scale MIs	# 2-scale MIs	number of integrals	size of tables (MB) (1-scale)
df2-2	71	337	14156	98
df2-3	45	244	15278	50
df2-5	41	92	11620	23
df2-6	35	78	11531	18

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Complexity can be defined as the deviation of a given integral $G_{a_1,a_2,\ldots,a_{18}}$ considered as a function of 18 integer indices a_i from the corner point of the corresponding sector, i.e. with indices 1 and 0.
In our calculation, we had in the top sector complexity up to 5 for 1-scale integrals and complexity up to 3 for 2-scale integrals.

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Then the missing reduction became feasible ;)

Some practical recipes

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- Choose numerators as propagators, i.e. as squares of some momenta.
- Choose loop momenta in such a way that the total 'length' of the propagators and numerators will be minimal.

Differential equations as a method to evaluate Feynman integrals [A.V. Kotikov'91]

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$$f'(\epsilon, x) = \epsilon\, A(x)\, f(x, \epsilon)$$

where $\varepsilon = (4 - d)/2$ and f is a vector of master integrals.

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$$f'(\epsilon, x) = \epsilon\, A(x)\, f(x, \epsilon)$$

where $\varepsilon = (4-d)/2$ and f is a vector of master integrals. In our case, $x = q_2^2/q^2$ and

$$A(x) = \sum_{k=0,1} \frac{a_k}{x - x^{(k)}}$$

with $x^{(0)} = 0, x^{(1)} = 1$.

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We choose the point $x = 1$ in order to fix the boundary conditions, where our integrals are expressed in terms of 28 master propagator integrals
[P.A. Baikov & K.G. Chetyrkin'10; R.N. Lee, A. Smirnov & V.S.'11]

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The differential equations are then used to transport the information to the point $x = 0$.

We solve our differential equations asymptotically near the point $x = 0$, where terms with $x^{-k\epsilon}, k = 0, 1, \ldots, 8$ are present, and fix these solutions by matching them to our solution at general x using HPL [D. Maitre'05].

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From the analytic results for the naive part we obtain analytical results for the required one-scale master integrals after changing back to the primary basis.

In this calculation, we proceeded by constructing an *associator* which is a matrix that transforms the vector composed of terms of asymptotic expansion near $x = 1$ into the vector composed of terms of asymptotic expansion near $x = 0$.

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We had to expand the associator up to ϵ^9 (weight 9) for df2-2 and df2-3 since the property of uniform weight is destroyed when mapping the two-scale master integrals to one-scale master integrals in the limit $x \to 0$. In the final result for the form factor all weight-nine constants drop out.

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Checks: by `FIESTA` [A. Smirnov] and by comparison with some partial numerical results [R.H. Boels, T. Huber & G. Yang'11].

$$
\begin{aligned}
G^{(\mathrm{df2\text{-}2})}_{111111111111} = &\\
&+\frac{1}{\epsilon^8}\left[\frac{1}{144}\right]+\frac{1}{\epsilon^7}\left[\frac{73}{576}\right]+\frac{1}{\epsilon^6}\left[\frac{331}{1152}-\frac{7\pi^2}{216}\right]+\frac{1}{\epsilon^5}\left[-\frac{311\zeta_3}{216}-\frac{245\pi^2}{576}-\frac{1765}{1152}\right]\\
&+\frac{1}{\epsilon^4}\left[-\frac{1103\zeta_3}{54}-\frac{37\pi^4}{1440}-\frac{917\pi^2}{1728}+\frac{2297}{576}\right]+\frac{1}{\epsilon^3}\left[\frac{4021\pi^2\zeta_3}{648}-\frac{42053\zeta_3}{1728}-\frac{22667\zeta_5}{360}\right.\\
&\left.-\frac{31327\pi^4}{51840}+\frac{2615\pi^2}{864}-\frac{59}{36}\right]+\frac{1}{\epsilon^2}\left[\frac{10784\zeta_3^2}{81}+\frac{13595\pi^2\zeta_3}{216}+\frac{293837\zeta_3}{1728}-\frac{268139\zeta_5}{360}\right.\\
&\left.-\frac{4901\pi^6}{38880}-\frac{40973\pi^4}{103680}-\frac{347\pi^2}{96}-\frac{21161}{288}\right]+\frac{1}{\epsilon}\left[\frac{1960259\zeta_3^2}{1296}+\frac{1037\pi^4\zeta_3}{160}+\frac{117521\pi^2\zeta_3}{1296}\right.\\
&-\frac{490831\zeta_3}{864}+\frac{508661\pi^2\zeta_5}{2160}-\frac{2028557\zeta_5}{2880}-\frac{10749139\zeta_7}{4032}-\frac{3561371\pi^6}{2177280}+\frac{110171\pi^4}{34560}\\
&\left.-\frac{20797\pi^2}{432}+\frac{222407}{288}\right]-\frac{4937s_{8a}}{6}-\frac{582209\pi^2\zeta_3^2}{1944}+\frac{8605981\zeta_3^2}{5184}+\frac{2064401\zeta_5\zeta_3}{270}\\
&+\frac{3543269\pi^4\zeta_3}{77760}-\frac{876841\pi^2\zeta_3}{1296}+\frac{325039\zeta_3}{216}+\frac{87229\pi^2\zeta_5}{48}+\frac{2528065\zeta_5}{576}-\frac{8894555\zeta_7}{504}\\
&-\frac{17509\pi^8}{1088640}+\frac{579329\pi^6}{2177280}-\frac{547763\pi^4}{51840}+\frac{126427\pi^2}{216}-\frac{1754951}{288}+\mathcal{O}(\epsilon)
\end{aligned}
$$

Our results:

$$
\begin{aligned}
F_q^{(n)}\Big|_{(d_F^{abcd})^2} \quad = \quad & n_f \frac{(d_F^{abcd})^2}{N_F} \Bigg\{ \frac{1}{\epsilon^2} \left[\frac{40\zeta_5}{3} + \frac{8\zeta_3}{3} - \frac{4\pi^2}{3} \right] + \frac{1}{\epsilon} \Bigg[- \frac{148\pi^6}{8505} - \frac{152\zeta_3^2}{3} - \frac{8\pi^2\zeta_3}{3} \\
& + \frac{2720\zeta_5}{9} + \frac{10\pi^4}{27} + \frac{664\zeta_3}{9} - \frac{284\pi^2}{9} + 48 \Bigg] - 1240\zeta_7 - \frac{988\pi^4\zeta_3}{135} \\
& + \frac{496\pi^2\zeta_5}{9} + \frac{10405\pi^6}{10206} + \frac{680\zeta_3^2}{9} + \frac{95098\zeta_5}{27} + \frac{46\pi^2\zeta_3}{9} + \frac{1888\pi^4}{405} \\
& - \frac{13414\zeta_3}{27} - \frac{10783\pi^2}{27} + \frac{3190}{3} \Bigg\} ,
\end{aligned}
$$

where $N_F = N_c = 3$.

The cusp and collinear anomalous dimensions

$$
\begin{aligned}
C_F\gamma^3_{\rm cusp}\Big|_{(d_F^{abcd})^2} &= n_f\frac{(d_F^{abcd})^2}{N_F}\left(-\frac{1280}{3}\zeta_5-\frac{256}{3}\zeta_3+\frac{128}{3}\pi^2\right)\\
&\approx n_f\frac{(d_F^{abcd})^2}{N_F}\left(-123.894910\ldots\right),\\
\gamma^3_q\Big|_{(d_F^{abcd})^2} &= n_f\frac{(d_F^{abcd})^2}{N_F}\left(-\frac{592\pi^6}{8505}-\frac{608\zeta_3^2}{3}+\frac{10880\zeta_5}{9}-\frac{32\pi^2\zeta_3}{3}\right.\\
&\qquad\left.+\frac{40\pi^4}{27}+\frac{2656\zeta_3}{9}-\frac{1136\pi^2}{9}+192\right).
\end{aligned}
$$

The results for γ_q^3 and the finite part of the form factor are new.

Agreement with known four-loop partial results
[S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren & A. Vogt'17,18]
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Agreement of analytic n_f term of the light-like QCD cusp anomalous dimension
[J.M. Henn, T. Peraro, M. Stahlhofen & P. Wasser,19]
with our results.

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to be continued