Energy-Energy Correlation At Small Angles

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LD, Ian Moult, HuaXing Zhu, 1905.01310

Amplitudes 2019

Trinity College Dublin

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NATIONAL ACCELERATOR LABORATORY

Why a "cross section" at "Amplitudes"?

- EEC is measurable for QCD – measured for decades, in fact
- Much interest in it for N=4 SYM, not just QCD
  Hofman, Maldacena, 0803.1467; Belitsky, Hohenegger, Korchemsky, Sokatchev, Zhiboedov, 1309.0769, 1309.1424, 1311.6800, 1409.2502; Henn, Sokatchev, Yan, Zhiboedov, 0903.05314; Kologlu, Kravchuk, Simmons-Duffin, Zhiboedov, 1905.01311; Korchemsky, 1905.01444
- Among simplest infrared-safe event-shapes, can apply "amplitudes" methods
- Observable χ lives on compact domain, $[0,\pi]$: large logarithms on **both** ends can be resummed. **Discuss $\chi \to 0$ limit here**
- Sum rules constrain it → avoid direct computations
- As $\chi \to 0$, probe jet substructure. Generalize to computable jet substructure variables for LHC, correlate multiple small angles
  Moult, Necib, Thaler, 1609.07483

The EEC

- Energy-energy correlation (EEC) in e^+e^- annihilation:

one of first infrared safe event-shapes in QCD, from over 40 years ago   Basham, Brown, Love, S. Ellis, PRD, PRL 1978

$$\frac{d\Sigma}{d\cos\chi} = \sum_{\text{partons } i,j} \int d\sigma \, \frac{E_i E_j}{Q^2} \delta(\cos\theta_{ij} - \cos\chi)$$

Collinear parton splitting

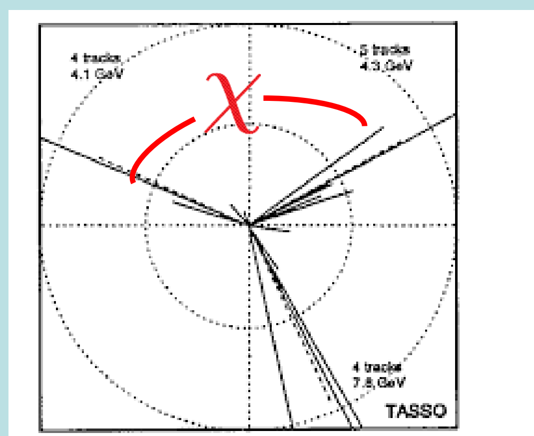
$$E_i \to xE_i + (1-x)E_i$$

preserves observable.

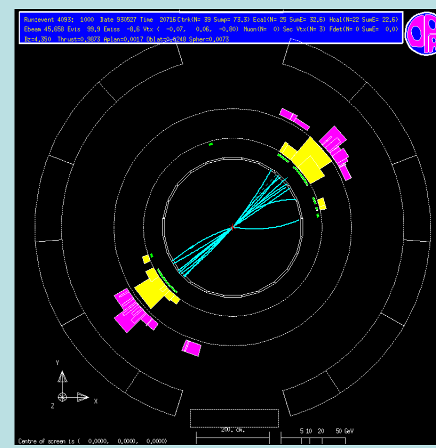
So does soft emission

→ IR safe

Data from wide range of CM energies →

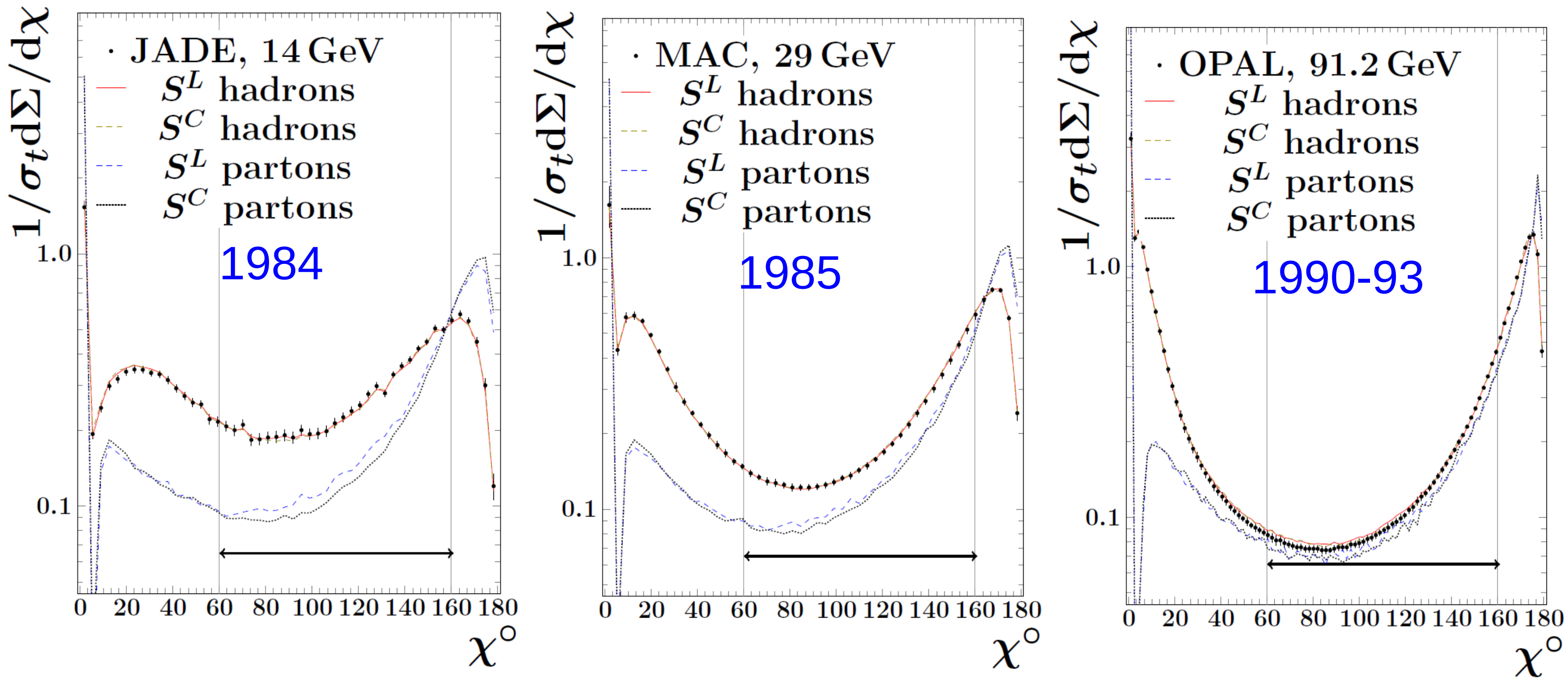


$Q \approx 22$ GeV



$Q = 91$ GeV

Evolution with energy clearly visible

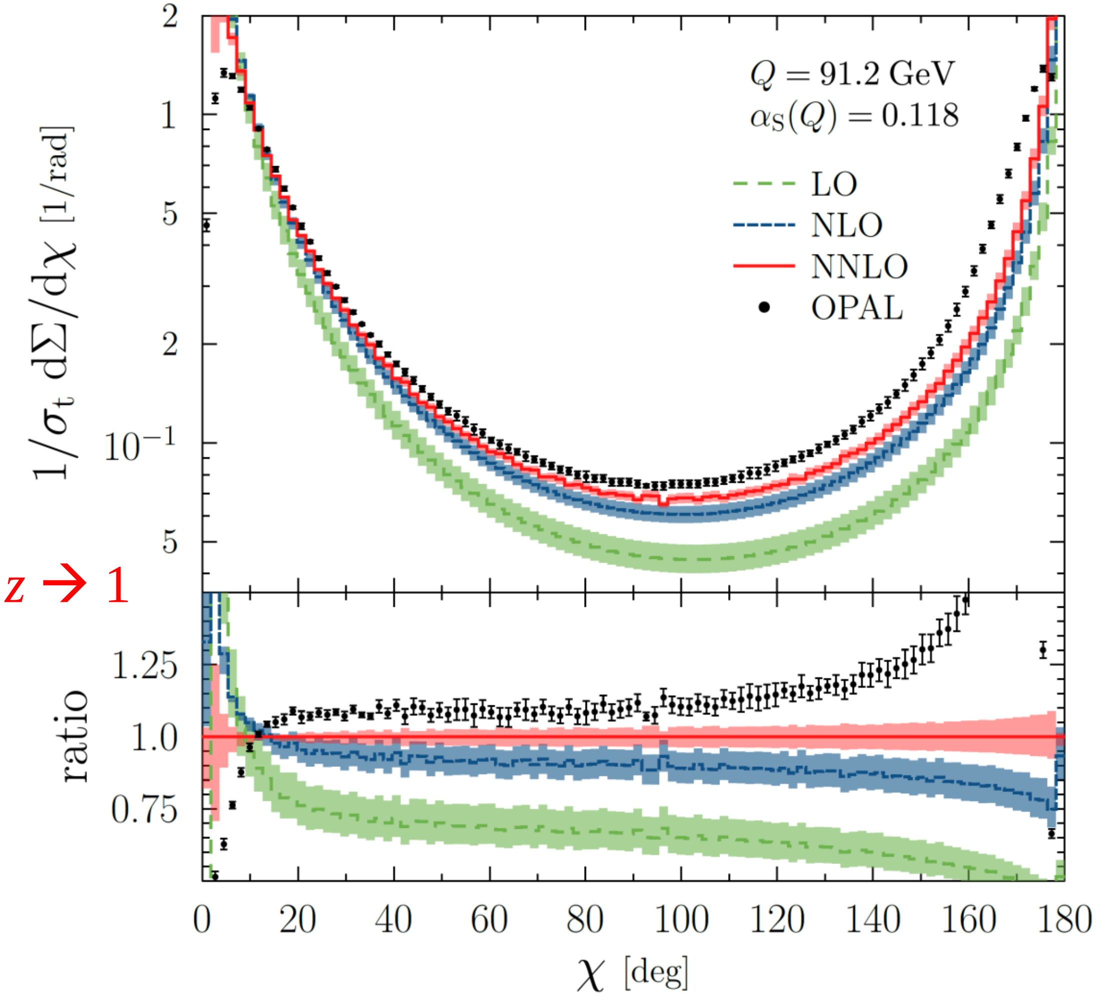


data reviewed recently in Kardos et al, 1804.09146

EEC in QCD at generic angle χ

- Computed at NLO numerically in 1980s and 1990s
  Richards, WJ Stirling, Ellis, 1982, 1983; Ali, Barreiro, 1982, 1984; Schneider, Kramer, Schierholz, 1984; Falck, Kramer, 1989; Kunszt, Nason, Marchesini, Webber, LEP Yellow Book, 1989; Glover, Sutton, 1994; Clay, Ellis, 1995; Kramer, Spiesberger, 1996; Catani, Seymour, 1996 [EVENT2].
- Computed numerically at NNLO only 3 years ago
  Del Duca, Duhr, Kardos, Somogyi, Trocsanyi, 1603.08927
- Computed analytically at NLO in QCD last year
  LD, Luo, Shtabovenko, Yang, Zhu,1801.03129

NNLO QCD vs. LEP data



Tulipant, Kardos, Somogyi, 1708.04093

$z \to 1$

$z \to 0$

$$z = \frac{1}{2}(1 - \cos\chi) \;\in\; [0, 1]$$

Use EEC to measure α_s

Kardos, Kluth, Somogyi, Tulipant, Verbytskyi, 1804.09146

- Clean initial state, but nonperturbative hadronization corrections are large, estimate with Monte Carlo.
- Included NNLO + NNLL resummation as $z \rightarrow 1$
- Competitive result:

$$\begin{aligned}\alpha_s(M_Z) &= 0.11750 \pm 0.00018(exp.) \pm 0.00102(hadr.) \\ &\quad \pm 0.00257(ren.) \pm 0.00078(res.)\end{aligned}$$

- Still room for theory improvement:
- NNLL $z \rightarrow 0$ resummation (this talk)
- NNNLL $z \rightarrow 1$ resummation (soon)
- [Approximate] NNNLO?

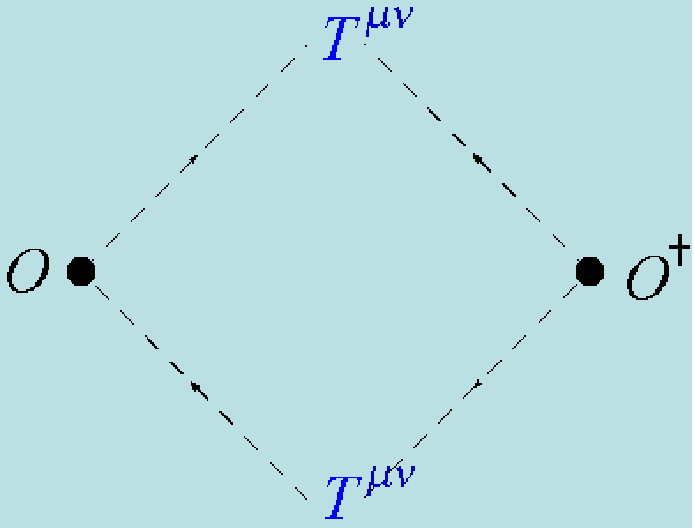
EEC in a CFT

- Energy-momentum tensor is fundamental in any QFT, but also in conformal field theories.
- Alternative methods of study in CFT, especially N=4 SYM:
- Mellin representation of a four-point correlation function → analytic results in N=4 SYM at NLO, NNLO

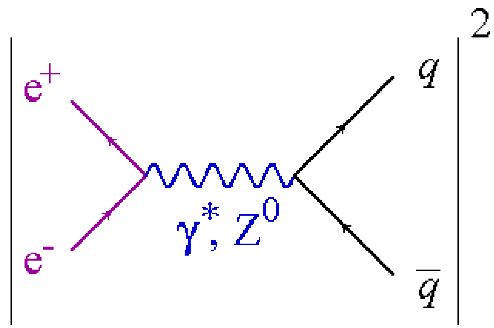
Belitsky et al., 1309.0769, 1309.1424, 1311.6800, 1409.2502;
Henn, Sokatchev, Yan, Zhiboedov, 1903.05314; Korchemsky, 1905.01444

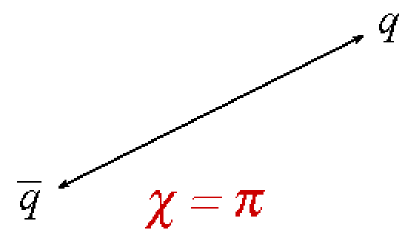
- Using properties of "ANEC" light-ray operators

Kologlu, Kravchuk, Simmons-Duffin, Zhiboedov, 1905.01311

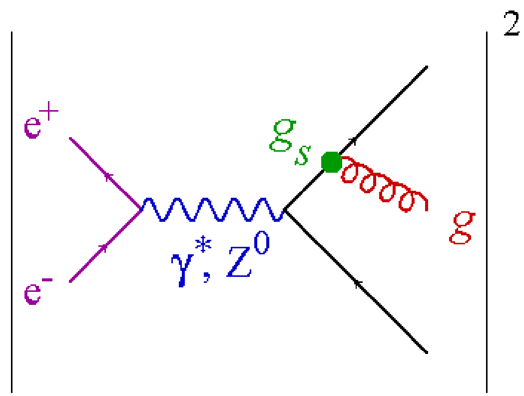


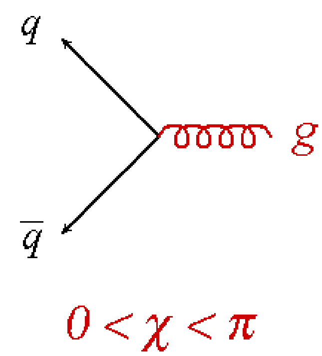
LO EEC for $0 < \chi < \pi$ is $O(\alpha_s)$





$$\alpha_s = \frac{g_s^2}{4\pi}$$

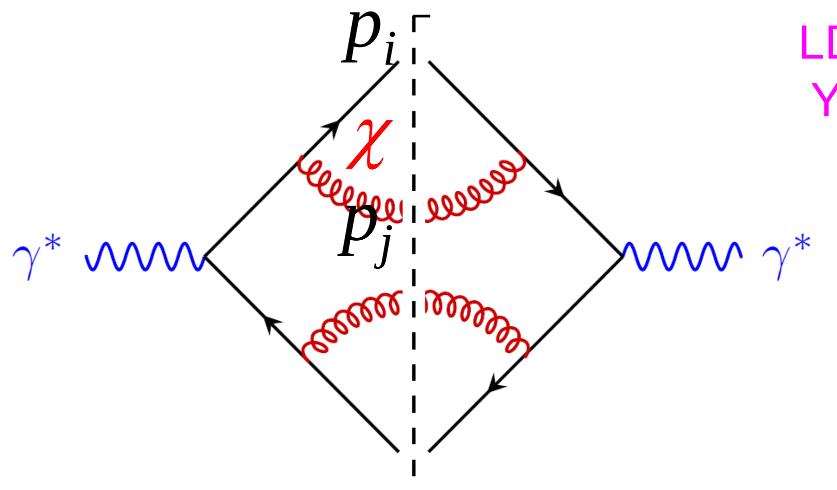




How to compute at NLO in QCD?

LD, Luo, Shtabovenko, Yang, Zhu, 1801.03129

Sample NLO real emission contribution



- Interference method with Feynman diagrams (gasp!)
- Reverse unitarity Anastasiou, Melnikov (2003): All momenta → loop momenta, put cut momenta on shell, impose $\delta(\cos\theta_{ij} - \cos\chi)$
- IBPs/Laporta algorithm Chetyrkin, Tkachov (1981), Laporta (2001)
- Differential equations for master integrals Gehrmann, Remiddi (2000)

can all be solved in terms of polylogarithms.

- Same method works also for Higgs → gg → hadrons Luo, Shtabovenko, Yang, Zhu, 1903.07277

NLO QCD result

$$\frac{1}{\sigma_0}\frac{d\Sigma}{d\cos\chi} = \frac{\alpha_s(\mu)}{2\pi}A(z) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \left(\beta_0 \log\frac{\mu}{Q}A(z) + B(z)\right) + \mathcal{O}(\alpha_s^3)$$

$$z = \frac{1}{2}(1-\cos\chi) \ \in \ [0,1]$$

LO result fits on one line:

Basham, Brown, Love, S. Ellis, 1978

$$A(z) = C_F \frac{3-2z}{4(1-z)z^5}[3z(2-3z) + 2(2z^2-6z+3)\ln(1-z)]$$

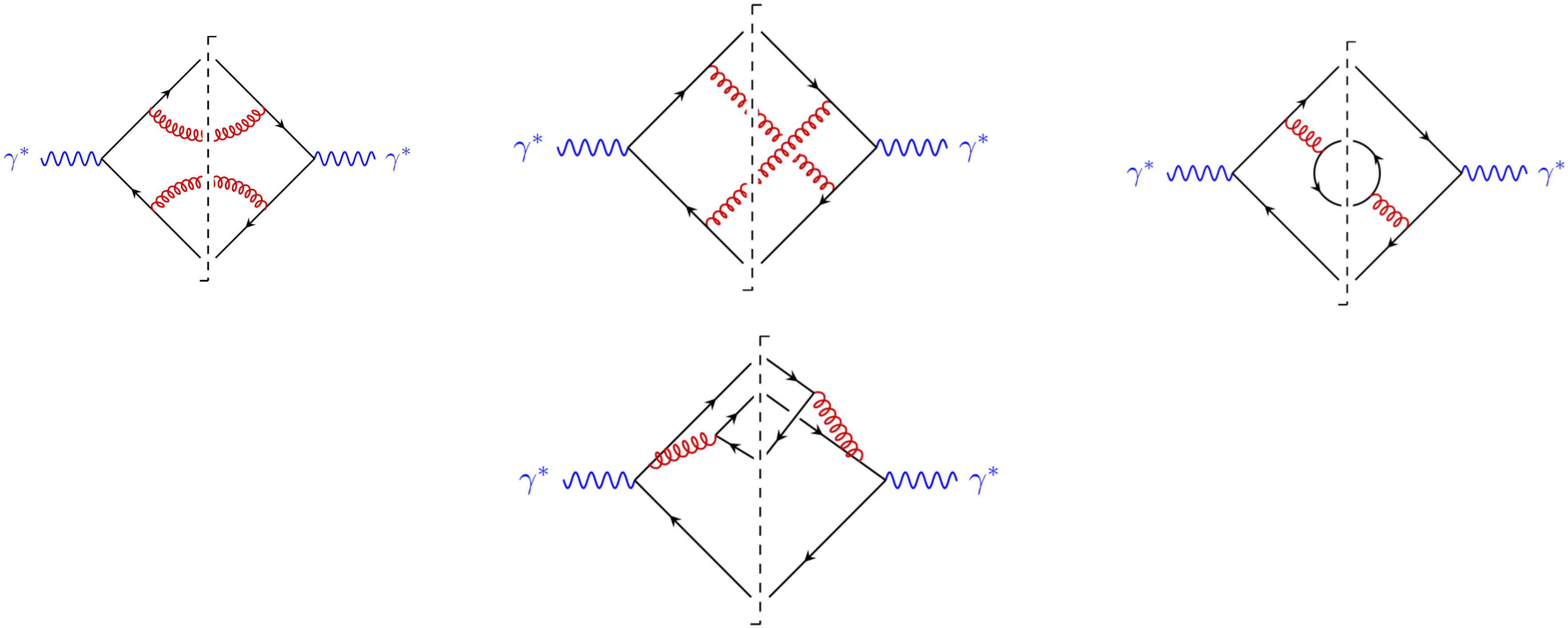
NLO result much lengthier, but expressible in terms of classical polylogarithms:

$$\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t}\mathrm{Li}_{n-1}(t), \quad \mathrm{Li}_1(t) = -\ln(1-t)$$

NNLO result? maybe elliptic polylogs? [based on N=4 Henn et al. 0903.05314]

Color structure of NLO QCD result

$$B(z) = C_F^2 B_{\rm lc}(z) + C_F(C_A - 2C_F) B_{\rm nlc}(z) + C_F N_f T_f B_{N_f}(z)$$



Leading color coefficient fits on one page

$$
\begin{aligned}
B_{\rm lc} = &+ \frac{122400z^7 - 244800z^6 + 157060z^5 - 31000z^4 + 2064z^3 + 72305z^2 - 143577z + 63298}{1440(1-z)z^4} \\
&- \frac{-244800z^9 + 673200z^8 - 667280z^7 + 283140z^6 - 48122z^5 + 2716z^4 - 6201z^3 + 11309z^2 - 9329z + 3007}{720(1-z)z^5} g_1^{(1)} \\
&- \frac{244800z^8 - 550800z^7 + 422480z^6 - 126900z^5 + 13052z^4 - 336z^3 + 17261z^2 - 38295z + 19938}{720(1-z)z^4} g_2^{(1)} \\
&+ \frac{4z^7 + 10z^6 - 17z^5 + 25z^4 - 96z^3 + 296z^2 - 211z + 87}{24(1-z)z^5} g_1^{(2)} \\
&+ \frac{-40800z^8 + 61200z^7 - 28480z^6 + 4040z^5 - 320z^4 - 160z^3 + 1126z^2 - 4726z + 3323}{120z^5} g_2^{(2)} \\
&- \frac{1-11z}{48z^{7/2}} g_3^{(2)} - \frac{120z^6 + 60z^5 + 160z^4 - 2246z^3 + 8812z^2 - 10159z + 4193}{120(1-z)z^5} g_4^{(2)} \\
&- 2\left(85z^4 - 170z^3 + 116z^2 - 31z + 3\right) g_1^{(3)} + \frac{-4z^3 + 18z^2 - 21z + 5}{6(1-z)z^5} g_2^{(3)} + \frac{z^2+1}{12(1-z)} g_3^{(3)} \,,
\end{aligned}
$$

where

$$
g_1^{(1)} = \log(1-z)\,, \qquad g_2^{(1)} = \log(z)\,, \qquad g_1^{(2)} = 2(\mathrm{Li}_2(z) + \zeta_2) + \log^2(1-z)\,,
$$

$$
g_2^{(2)} = \mathrm{Li}_2(1-z) - \mathrm{Li}_2(z)\,, \qquad g_3^{(2)} = -2\,\mathrm{Li}_2\left(-\sqrt{z}\right) + 2\,\mathrm{Li}_2\left(\sqrt{z}\right) + \log\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)\log(z)\,, \qquad g_4^{(2)} = \zeta_2
$$

$$
g_1^{(3)} = -6\left[\mathrm{Li}_3\left(-\frac{z}{1-z}\right) - \zeta_3\right] - \log\left(\frac{z}{1-z}\right)\left(2(\mathrm{Li}_2(z) + \zeta_2) + \log^2(1-z)\right)\,,
$$

$$
g_2^{(3)} = -12\left[\mathrm{Li}_3(z) + \mathrm{Li}_3\left(-\frac{z}{1-z}\right)\right] + 6\,\mathrm{Li}_2(z)\log(1-z) + \log^3(1-z)\,,
$$

$$
g_3^{(3)} = 6\log(1-z)\left(\mathrm{Li}_2(z) - \zeta_2\right) - 12\,\mathrm{Li}_3(z) + \log^3(1-z)\,.
$$

NLO, intra-jet limit, $z \to 0$

LL

$$B(z) = C_F\Big\{\frac{1}{z}\Big[\ln z\Big(-\frac{107C_A}{120}+\frac{25C_F}{32}+\frac{53N_fT_f}{240}\Big)+C_A\Big(-\frac{25\zeta_2}{12}+\frac{\zeta_3}{2}+\frac{17683}{2700}\Big)$$
$$+C_F\Big(\frac{43\zeta_2}{12}-\zeta_3-\frac{8263}{1728}\Big)-\frac{4913N_fT_f}{3600}\Big]$$
$$+\ln z\Big[C_A\Big(\frac{33\zeta_2}{2}-\frac{703439}{25200}\Big)+C_F\Big(\frac{42109}{1200}-21\zeta_2\Big)+N_fT_f\Big(\frac{86501}{12600}-4\zeta_2\Big)\Big]$$
$$+C_A\Big(\frac{213\zeta_2}{5}-\frac{101\zeta_3}{2}-\frac{26986007}{5292000}\Big)+C_F\Big(-\frac{1541\zeta_2}{30}+65\zeta_3+\frac{18563}{2700}\Big)$$
$$+N_fT_f\Big(-\frac{46\zeta_2}{3}+12\zeta_3+\frac{2987627}{330750}\Big)\Big\}+\mathcal{O}(z)$$

NLL

leading power (first two lines)

first subleading power (remaining lines)

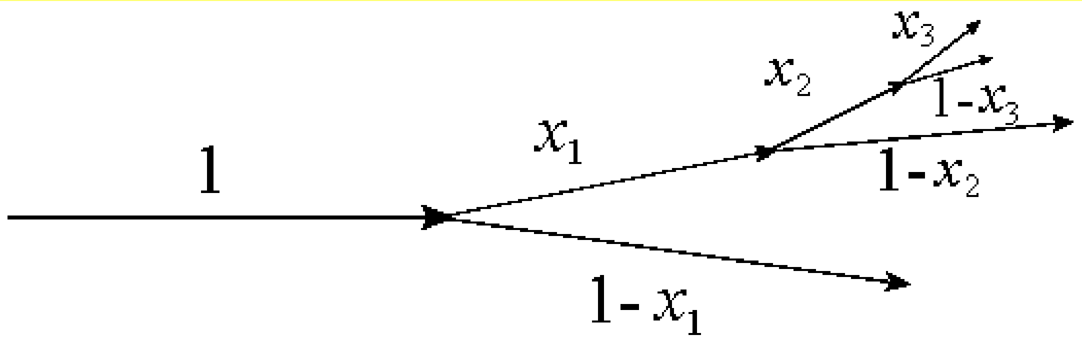
Single log behavior $\ln^L z/z$
characteristic of pure collinear observable

"Jet Calculus" for LL resummation

- Collinear dominated.
- Only a single Mellin moment N=3 of time-like splitting function (twist 2 anomalous dimension)

$$\gamma_{ij}^{(N)} \equiv -\int_0^1 dx\, x^{N-1} P_{ij}(x)$$

Konishi, Ukawa, Veneziano Phys.Lett.1978,1979;
Richards, Stirling, Ellis, NPB229, 317, 1983



Energy weighting →

$$\int_0^1 dx\, x(1-x)\, P_{ij}(x) \;\to\; -\int_0^1 dx\, x^2\, P_{ij}(x) \;\equiv\; \gamma_{ij}^{(N=3)}$$

Momentum sum rule controls x^1 term,
→ can drop it.

$$\int_0^1 dx\, x\, P_{ij}(x) \;\equiv\; -\gamma_{ij}^{(N=2)}$$

LL resummed formula

Richards, Stirling, Ellis, NPB229, 317, 1983

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\cos\chi} = \frac{\alpha_s(\sqrt{z}Q)}{16\pi\ z} \sum_{i,j=q,g} \gamma_{ij}^{(0)} \left[\frac{\alpha_s(\sqrt{z}Q)}{\alpha_s(Q)}\right]_{jq}^{-\gamma^{(0)}/\beta_0}$$

$$\gamma_{ij}^{(0)} = \begin{bmatrix} \frac{25}{6}C_F & -\frac{7}{15}n_f \\ -\frac{7}{6}C_F & \frac{14}{5}C_A + \frac{2}{3}n_f \end{bmatrix} \qquad \beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f$$

One-loop (LO) $N=3$ time-like moments

Beyond LL as $z \to 0$

LD, Moult, Zhu, 1905.01310

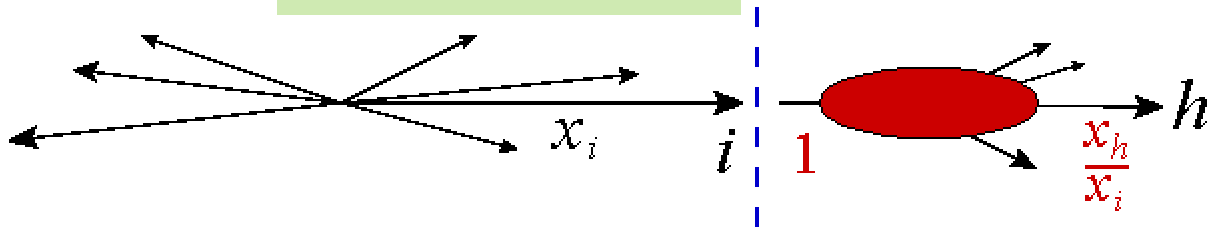
- Factorize on single parton states, similar to production of identified hadrons h with momentum $p_h = x \times Q/2$

$$\frac{d\sigma(e^+e^- \to h + X)}{dx_h} = \sum_{i=q,g} \int_0^1 dx_i \underbrace{\frac{d\sigma(e^+e^- \to i + X)}{dx_i}} \underbrace{D_{i\to h}(x_h/x_i)}$$

…, Mitov, Moch, Vogt, 2006
Moch, Vogt, 0709.3899,
Almasy, Moch, Vogt, 1107.2263

perturbative hard function, computed to NNLO + evolution

nonperturbative fragmentation function

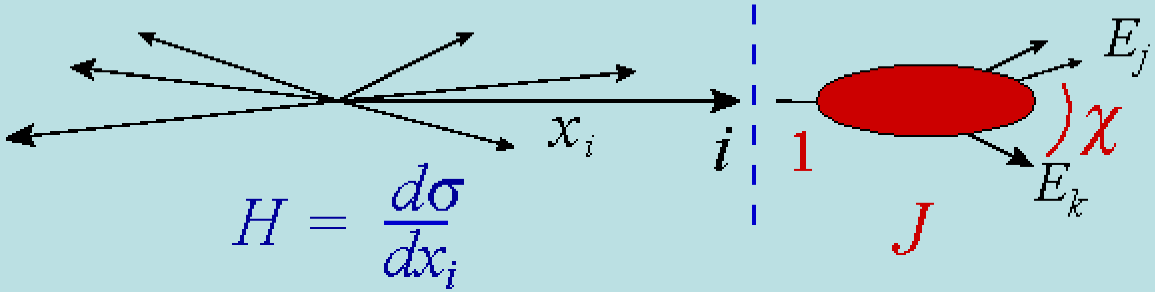


All orders EEC factorization

Cumulant $\Sigma\left(z, \ln\frac{Q^2}{\mu^2}, \alpha_s(\mu)\right) \equiv \frac{1}{\sigma_0}\int_0^z dz' \frac{d\sigma}{dz}\left(z', \ln\frac{Q^2}{\mu^2}, \alpha_s(\mu)\right)$

$$\Sigma\left(z, \ln\frac{Q^2}{\mu^2}, \alpha_s(\mu)\right) = \int_0^1 dx\, x^2 \vec{J}\left(\ln\frac{zx^2Q^2}{\mu^2}, \alpha_s(\mu)\right) \cdot \vec{H}\left(\ln\frac{Q^2}{\mu^2}, \alpha_s(\mu)\right)$$

- Reuses hard function



- Replaces nonperturbative fragmentation function with perturbative jet function J which includes the small angle EEC measurement
- J depends on its only physical scale: $q_T^2 \approx (\chi x Q/2)^2 \approx zx^2Q^2$

Evolution of jet function

- To resum large logs, evolve jet function from its natural scale, $\mu = \sqrt{z}Q$ up to natural scale of hard function, $\mu = Q$
- Hard function evolves with time-like splitting kernel, $P_T(y,\mu)$:

$$\frac{d\vec{H}(x)}{d\ln\mu^2} = -\int_x^1 \frac{dy}{y}\widehat{P}_T(y,\mu)\cdot\vec{H}(x/y)$$

- Σ is RGE invariant, i.e. independent of μ
- Leads to evolution equation for J:

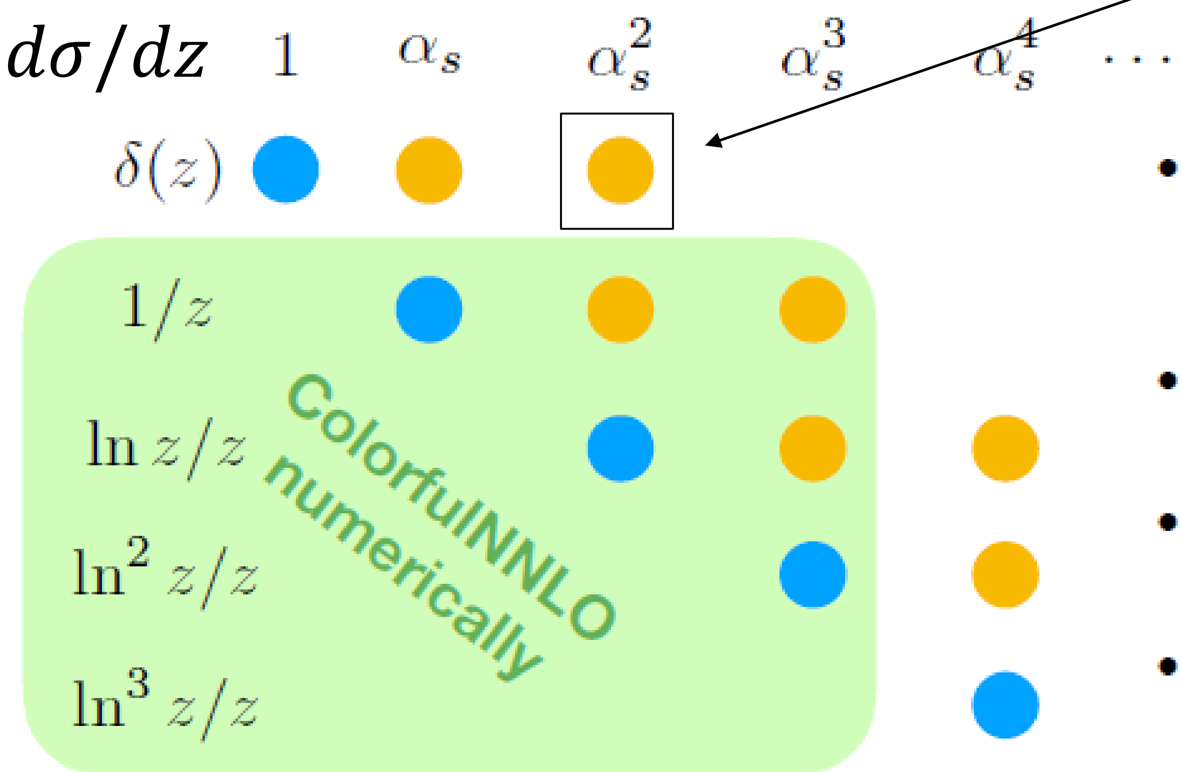
$$\frac{d\vec{J}\left(\ln\frac{zQ^2}{\mu^2},\alpha_s\right)}{d\ln\mu^2} = \int_0^1 dy\, y^2\, \vec{J}\left(\ln\frac{zy^2Q^2}{\mu^2}\right)\cdot\widehat{P}_T(y,\mu)$$

- LL evolution only uses N=3 time-like moments (y^2)
- Beyond LL, need "nearby" moments, $\ln y \leftrightarrow \frac{\partial}{\partial N}$

Counting the order

- LL — Konishi, Ukawa, Veneziano, 1979
- NLL + NNLL — Dixon, Moult, HXZ, 2019, This talk

Get these jet function constants indirectly using sum rules



- To get to NNLL require:
- NNLO splitting kernel
  Moch, Vermaseren, Vogt
- NNLO hard function
  Mitov, Moch, 2006;
  Almasy, Moch, Vogt, 2011
- NNLO jet function
  Very challenging!

Sum rules

- Energy conservation, $Q^2 = (\sum E_i)^2 = \sum E_i E_j$ implies sum rule,

$$\int_0^1 dz \frac{d\sigma}{dz} = \sigma_{\text{tot}}$$

- Momentum conservation involving

$$p_i \cdot p_j = E_i E_j (1 - \cos\chi)$$

→ second sum rule Korchemsky, 1905.01444, Kologlu et al., 1905.01311

$$\int_0^1 dz\, z \frac{d\sigma}{dz} = \int_0^1 dz\, (1-z) \frac{d\sigma}{dz} = \frac{1}{2} \sigma_{\text{tot}}$$

Use sum rule(s) to get $\alpha_s^2\,\delta(z)$

- $\sigma_{\rm tot}$ known, for e^+e^- and Higgs, e.g.
  Herzog, Ruijl, Ueda, Vermaseren, Vogt, 1707.01044
- First sum rule needs both $\delta(z)$ and $\delta(1-z)$ terms
- Second sum rule decouples them, although $\alpha_s^2\,\delta(1-z)$ term also known Zhu, et al. (2019)
- α_s^2 distribution for e^+e^- and Higgs for $0 < z < 1$ known analytically. Integrate it, use PSLQ to get in terms of ζ_n
- $\delta(z)$ coefficients involve sum of H and J.
- Get H from Almasy, Mitov, Moch, Vogt
- → Use two $\delta(z)$ coefficients to fix the two 2-loop J_q, J_g

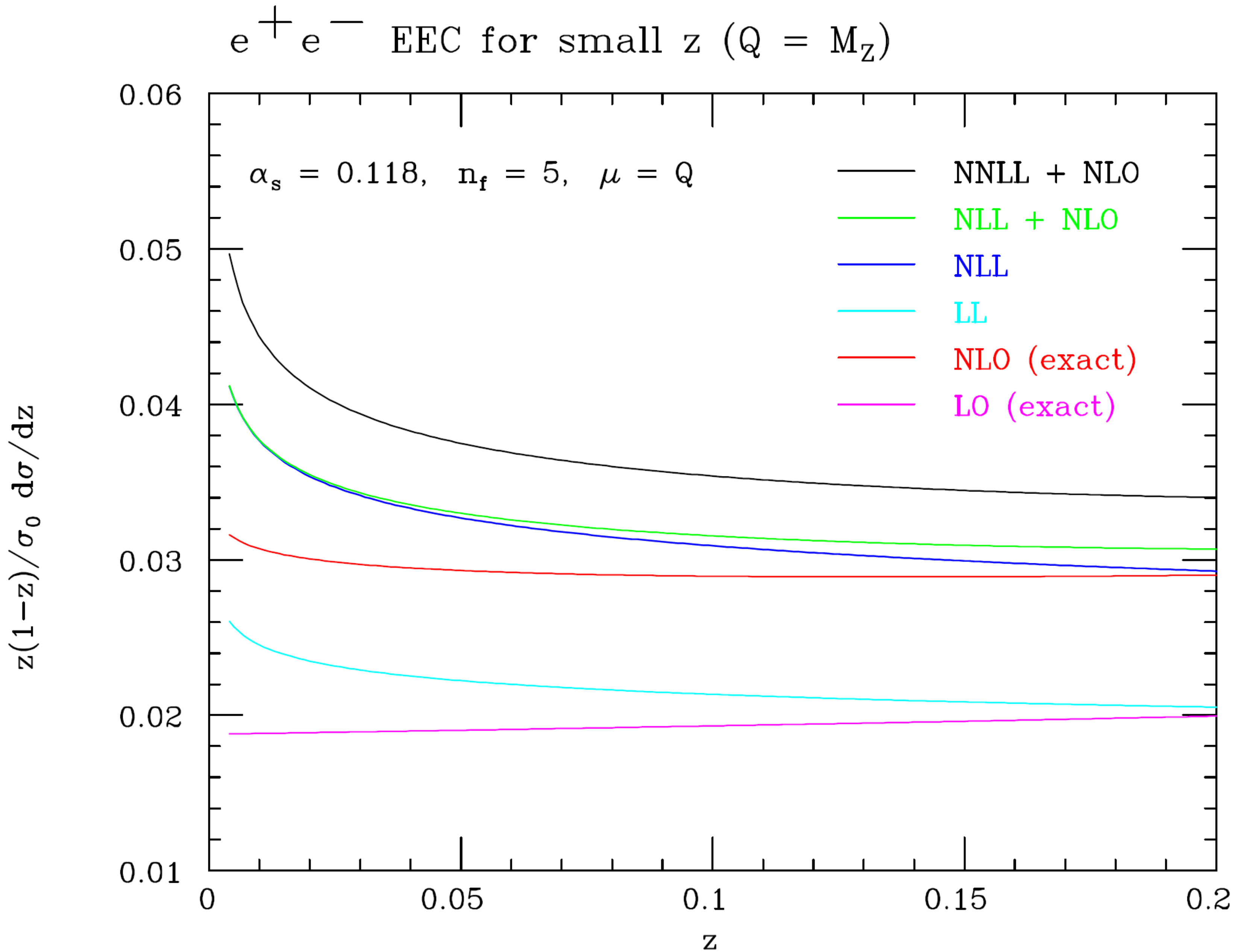
Two loop jet constants in QCD

$$j_2^q = C_F n_f \left(\frac{9}{5}\zeta_2 + \frac{703847}{24000}\right) + C_F C_A \left(-76\zeta_4 + 280\zeta_3 + \frac{1063}{15}\zeta_2 - \frac{164883727}{324000}\right)$$
$$+ C_F^2 \left(152\zeta_4 - 478\zeta_3 - 106\zeta_2 + \frac{3498505}{5184}\right),$$
$$j_2^g = n_f^2 \left(-\frac{8}{15}\zeta_2 + \frac{2344}{1125}\right) + C_F n_f \left(4\zeta_3 + \frac{14}{5}\zeta_2 - \frac{1528667}{108000}\right)$$
$$+ C_A n_f \left(\frac{44}{5}\zeta_3 - \frac{127}{25}\zeta_2 + \frac{68111303}{1620000}\right) + C_A^2 \left(76\zeta_4 - \frac{1054}{5}\zeta_3 - \frac{2159}{75}\zeta_2 + \frac{133639871}{810000}\right)$$

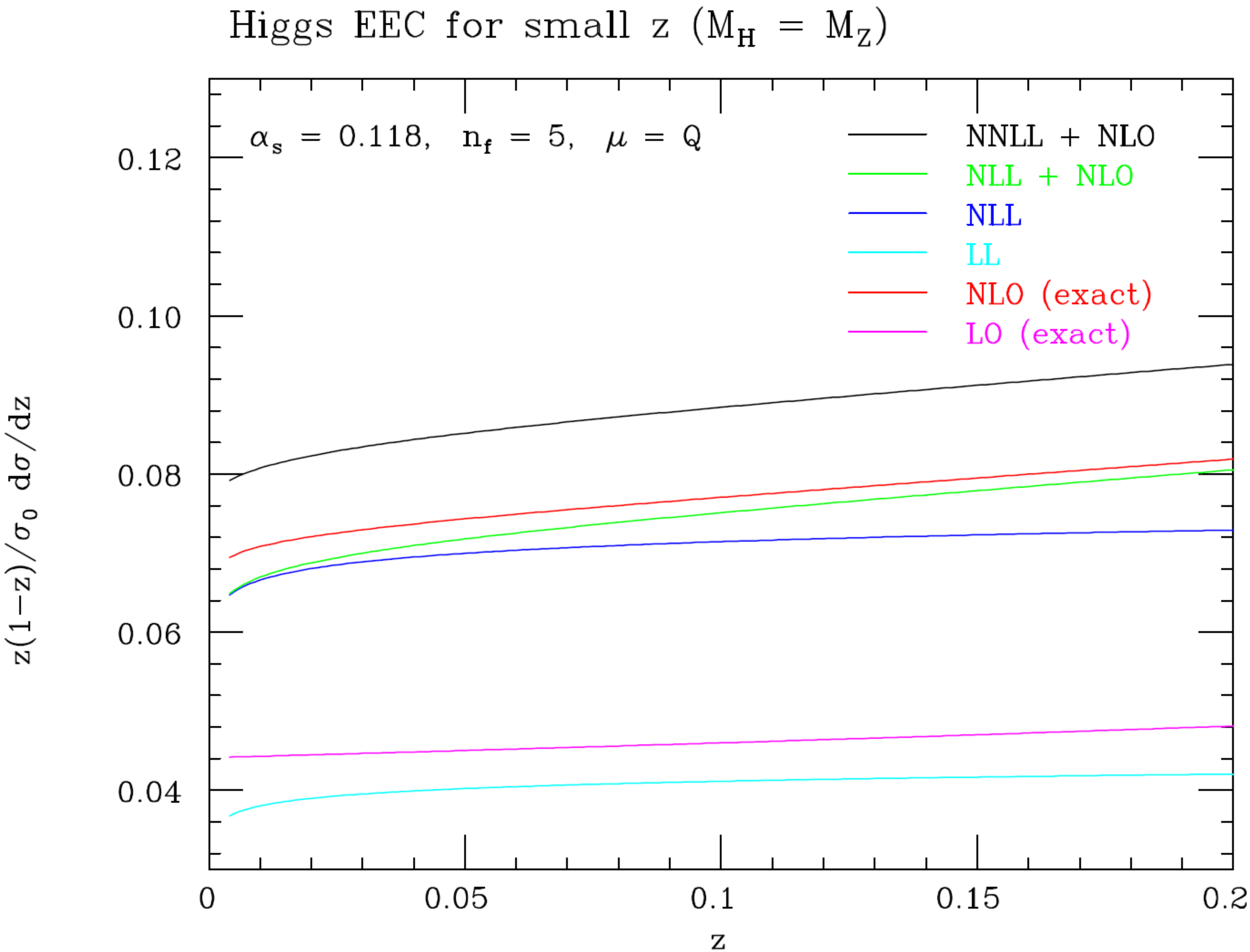
Did not need to compute directly, thanks to sum rule(s)!

Solve jet evolution for QCD

- 2 x 2 equation coupling J_q, J_g
- For $z > 0.004$, solve order by order through 9 loops
- Do for both e^+e^- and Higgs to illustrate different behavior of quark and gluon jets.
- Competition between β function and splitting plays out very differently for quarks and gluons, $C_F = \frac{4}{3}$ versus $C_A = 3$.

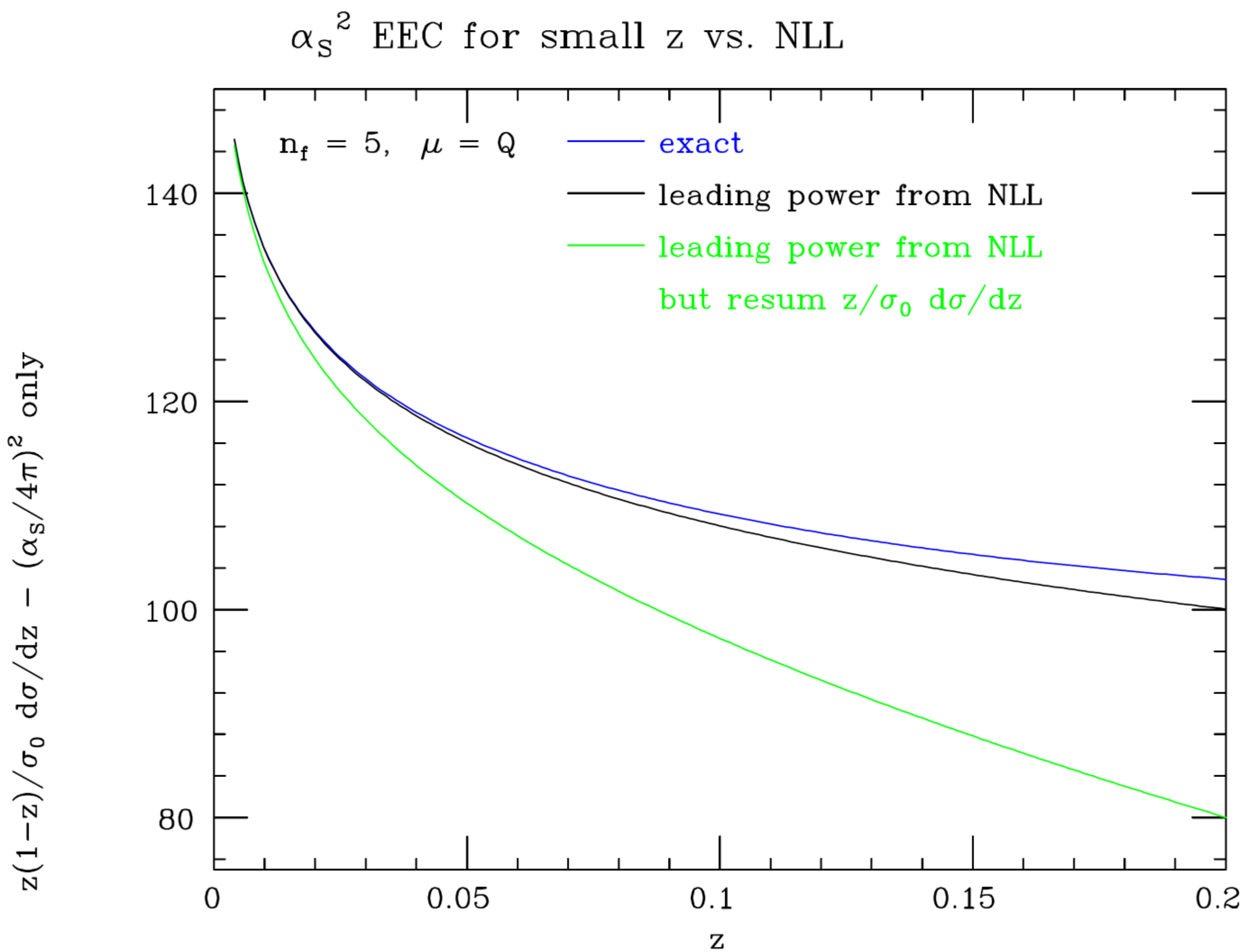


e^+e^- EEC for small z ($Q = M_Z$)
$\alpha_s = 0.118$, $n_f = 5$, $\mu = Q$
NNLL + NLO
NLL + NLO
NLL
LL
NLO (exact)
LO (exact)
0.06
0.05
0.04
0.03
0.02
0.01
0
0.05
0.1
0.15
0.2
z
$z(1-z)/\sigma_0\, d\sigma/dz$



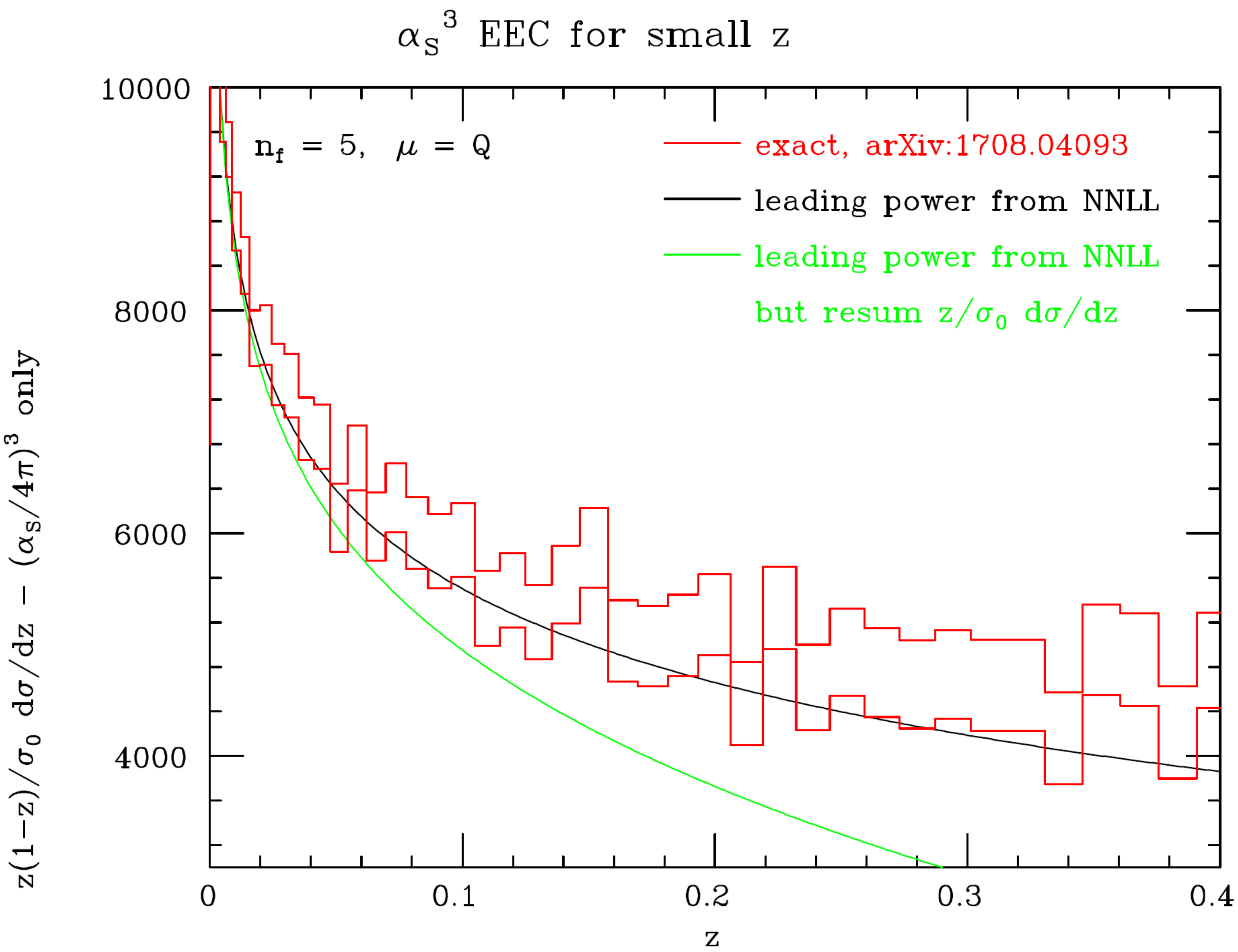
Higgs EEC for small z ($M_H = M_Z$)
$\alpha_s = 0.118, \quad n_f = 5, \quad \mu = Q$
NNLL + NLO
NLL + NLO
NLL
LL
NLO (exact)
LO (exact)
0.12
0.10
0.08
0.06
0.04
$z(1-z)/\sigma_0 \; d\sigma/dz$
0
0.05
0.1
0.15
0.2
z

Comparison with fixed order at α_s^2



Power corrections very small for $z(1\text{-}z)$ normalization

Comparison with fixed order at α_s^3



Thanks to Del Duca, Duhr, Kardos, Somogyi, Trocsanyi, Tulipant for numbers

With $z(1-z)$ normalization, leading power remarkably accurate all the way out to $z = 0.4$

Jet evolution for N=4 SYM

- Scale invariance → solution for N=4 SYM a pure power law:

$$J\left(\frac{zQ^2}{\mu^2},\alpha_s\right) = C_J(\alpha_s)\left(\frac{zQ^2}{\mu^2}\right)^{\gamma_J^{\mathcal{N}=4}(\alpha_s)}$$

- Insert into evolution equation, find that

$$\begin{aligned} 2\gamma_J^{\mathcal{N}=4} &= -2\int_0^1 dy\, y^{2+2\gamma_J^{\mathcal{N}=4}} P_{T,\text{uni.}}(y) \\ &= 2\gamma_T^{\mathcal{N}=4}(N=1+2\gamma_J^{\mathcal{N}=4}) \end{aligned}$$

- Using "time-like space-like reciprocity relation" Drell, Levy, Yan (1969), Gribov, Lipatov (1972), …, Basso, Korchemsky, hep-th/0612247 this is actually the space-like N=3 moment Caron-Huot (N=1 for "universal" N=4 SYM anomalous dimension):

$$\gamma_J^{\mathcal{N}=4} = \gamma_S^{\mathcal{N}=4}(N=1)$$

N=4 SYM result

- Jet function solution leads to:

$$\Sigma(z) = \frac{1}{2} C(\alpha_s)\, z^{\gamma_J^{\mathcal{N}=4}(\alpha_s)}$$

- Space-like anomalous dimension

$$\gamma_J^{\mathcal{N}=4}(\alpha_s) = \gamma_S^{\mathcal{N}=4}(N=1, \alpha_s)$$

extracted through 4 loops from

Kotikov, Lipatov, Rej, Staudacher, Velizhanin, 0704.3586; Bajnok, Janik, Lukowski, 0811.4448

- Sum rule at α_s^2 gives:

$$C(\alpha_s) = 1 - \frac{C_A \alpha_s}{\pi} + \left(\frac{11}{4}\zeta_4 - 3\zeta_2 + 7\right)\left(\frac{C_A \alpha_s}{\pi}\right)^2 + \mathcal{O}(\alpha_s^3)$$

- Result agrees precisely with recent “space-like” analysis
  Kologlu et al., 1905.01311; Korchemsky, 1905.01444

- And with $z \to 0$ limit of NNLO result
  Henn, Sokatchev, Yan, Zhiboedov, 0903.05314

Conclusions

- All order time-like factorization formula for small angle EEC for generic theories.
- Explicitly computed through NNLL, two orders more accurate than previous jet calculus approach.
- Quite different behavior for quarks (e^+e^-) vs. gluons (Higgs)
- Good agreement with ColorfulNNLO results at $O(\alpha_s^3)$
- Time-like/space-like reciprocity of twist 2 anomalous dimensions relates formula to other N=4 SYM approaches

Outlook

- May be able to go to NNNLL in QCD, at least approximately. Also approximate NNNLO?
- Same jet functions also apply to suitable "jet substructure" observables at LHC, could use to discriminate quark/gluon jets at LHC.
- May eventually lead to more precise value of α_s, as well as more precise jet substructure understanding at LHC

Extra Slides

Two loop jet constants in N=1 SYM

$$j_2^q = C_F n_f \left(\frac{9}{5}\zeta_2 + \frac{703847}{24000}\right) + C_F C_A \left(-76\zeta_4 + 280\zeta_3 + \frac{1063}{15}\zeta_2 - \frac{164883727}{324000}\right)$$
$$+ C_F^2 \left(152\zeta_4 - 478\zeta_3 - 106\zeta_2 + \frac{3498505}{5184}\right),$$
$$j_2^g = n_f^2 \left(-\frac{8}{15}\zeta_2 + \frac{2344}{1125}\right) + C_F n_f \left(4\zeta_3 + \frac{14}{5}\zeta_2 - \frac{1528667}{108000}\right)$$
$$+ C_A n_f \left(\frac{44}{5}\zeta_3 - \frac{127}{25}\zeta_2 + \frac{68111303}{1620000}\right) + C_A^2 \left(76\zeta_4 - \frac{1054}{5}\zeta_3 - \frac{2159}{75}\zeta_2 + \frac{133639871}{810000}\right)$$

- In pure N=1 SYM ($C_F \to C_A$, $n_f \to C_A$), collapse to:

$$j_2^{q,\ \mathcal{N}=1} = C_A^2\left(76\,\zeta_4 - 198\,\zeta_3 - \frac{100}{3}\zeta_2 + \frac{78117}{400}\right)$$
$$j_2^{g,\ \mathcal{N}=1} = C_A^2\left(76\,\zeta_4 - 198\,\zeta_3 - \frac{158}{5}\zeta_2 + \frac{263197}{1350}\right)$$

Solving jet evolution for N=1 SYM

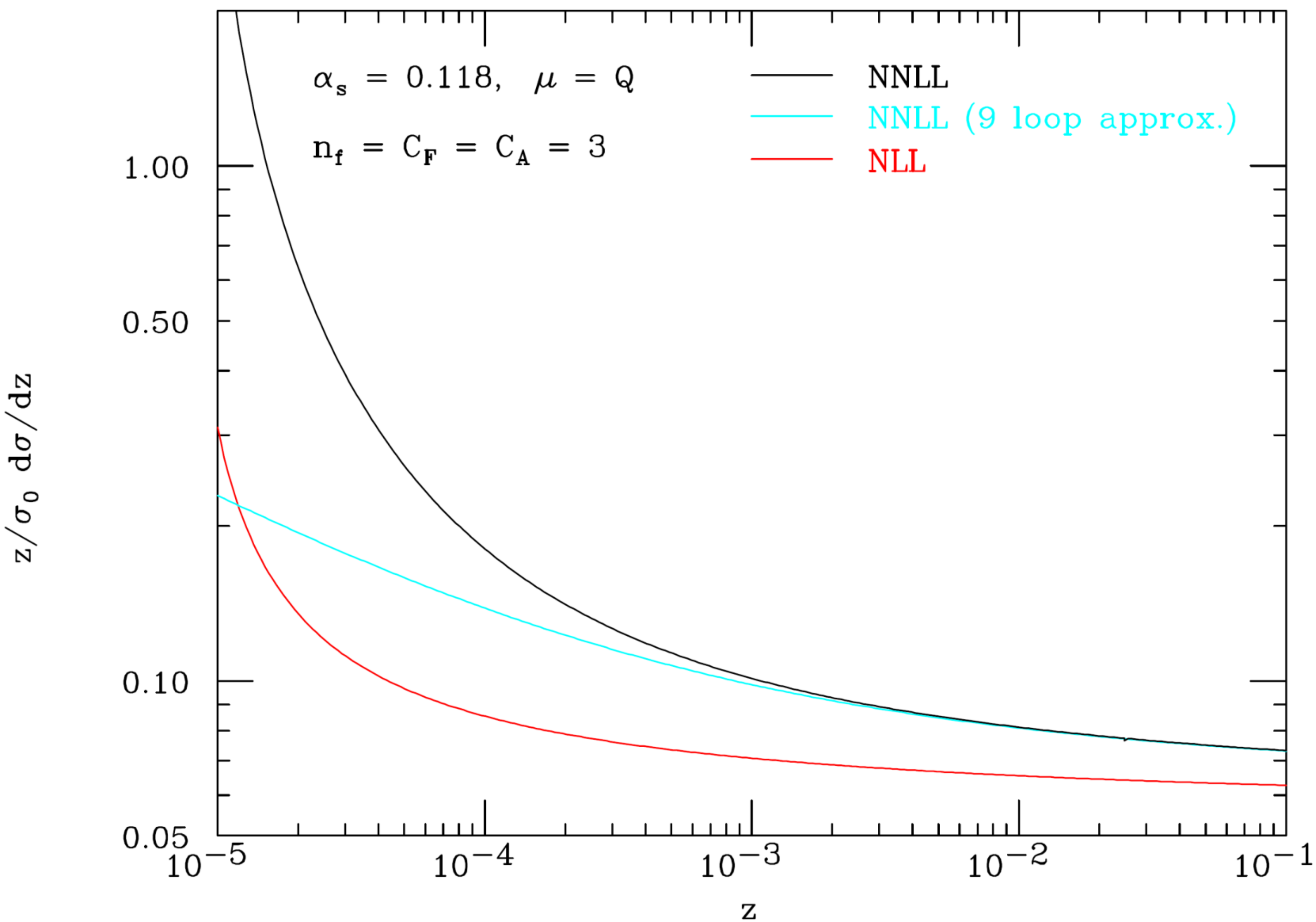
- We can do it exactly at NNLL, because 2 x 2 matrix equation is effectively 1 x 1. Solution is:

$$\Sigma^{\mathcal{N}=1}_{\mathrm{NNLL}}(z) = c_1^S(\alpha_s) + c_2^S(\alpha_s)\ln z + c_3^S(\alpha_s)\frac{\ln z}{1+\beta_0 a_s \ln z}$$
$$+ c_4^S(\alpha_s)\ln[1+\beta_0 a_s \ln z]$$
$$+ c_5^S(\alpha_s)\ln\left(1 - 2C_A a_s \frac{\ln[1+\beta_0 a_s \ln z]}{1+\beta_0 a_s \ln z}\right)$$

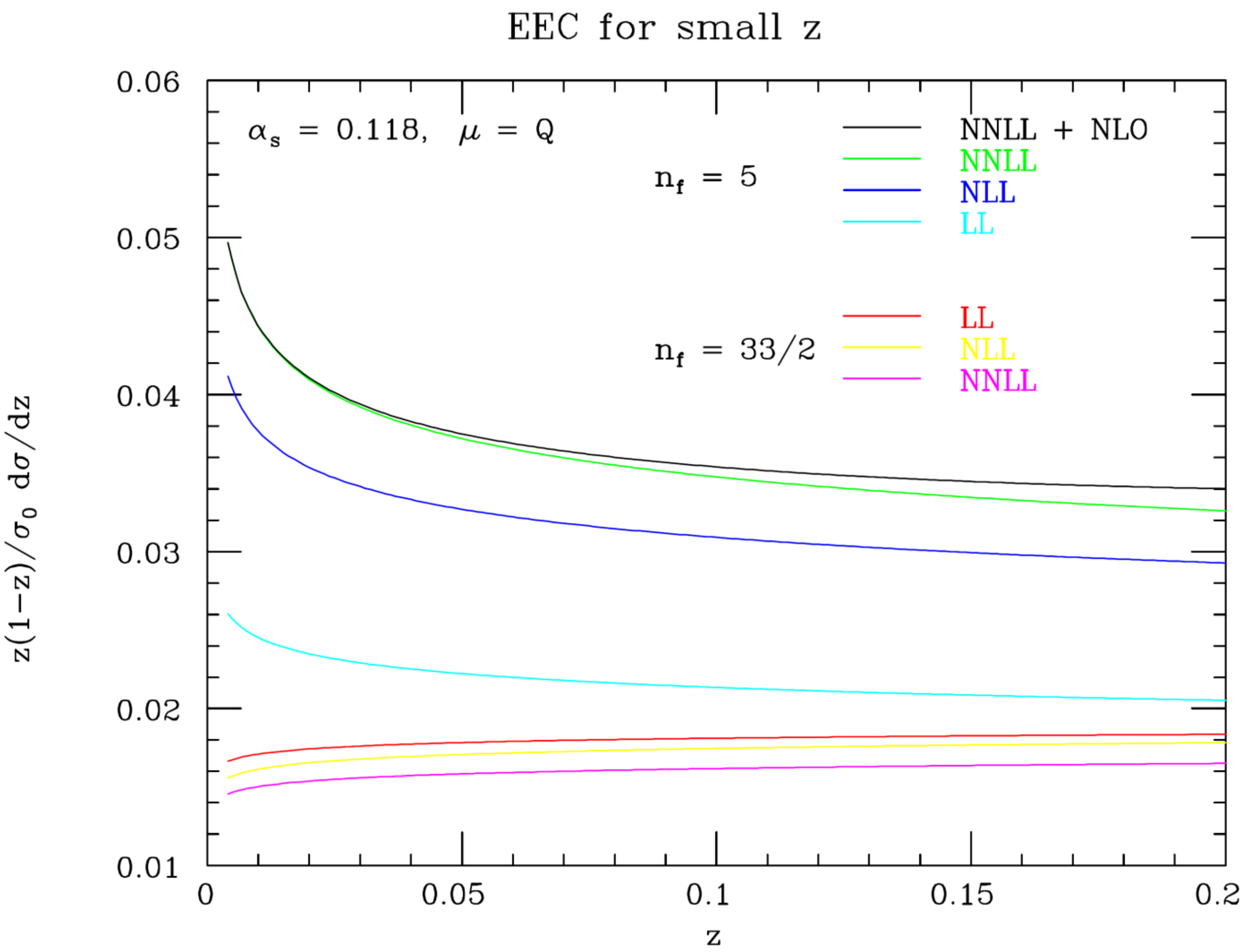
$$c_1^H = \frac{1}{2} + \frac{69}{8}a + a^2\left(22\zeta_4 - 66\zeta_3 - \frac{95}{3}\zeta_2 + \frac{81949}{432}\right),$$
$$c_1^\gamma = \frac{1}{2} + \frac{13}{24}a + a^2\left(22\zeta_4 - 44\zeta_3 + \frac{22}{9}\zeta_2 + \frac{2911}{162}\right),$$

etc., $a = \dfrac{C_A \alpha_s}{4\pi}$

e^+e^- EEC in N=1 SYM for small z ($Q = M_Z$)



Near Banks-Zaks fixed point



Analytic properties of QCD moments

- With analytic formulae, compute the integrals

$$B_N = \int_0^1 dz\ z^N B(z)$$

numerically to high accuracy, for each color coefficient

- Using PSLQ, it is always of the form

$$B_N = r_N^{(4)}\zeta(4)\ +\ r_N^{(3)}\zeta(3)\ +\ r_N^{(2)}\zeta(2)\ +\ r_N^{(0)}$$

where the $r_N^{(w)}$ are rational numbers.

- E.g. $B_3(C_A) = -\frac{207}{2}\zeta(4) + \frac{14902}{35}\zeta(3) - \frac{553}{450}\zeta(2) - \frac{2369041}{5040}$
- Could they be zeta values at higher loop orders too?
- Expression for general N in terms of $\psi(N)$ functions?

Back-to-back limit, $z \to 1$

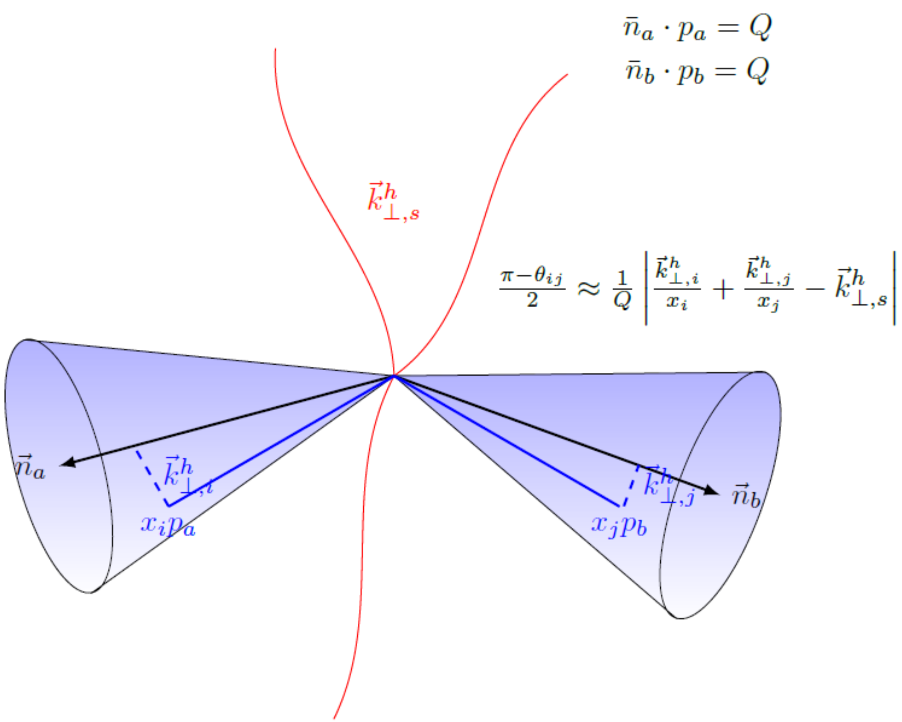
$$
\begin{aligned}
B(z) = C_F \Big\{ \frac{1}{1-z} \Big[& \frac{1}{2} C_F \ln^3(1-z) + \ln^2(1-z) \left(\frac{11 C_A}{12} + \frac{9 C_F}{4} - \frac{N_f T_f}{3} \right) \\
& + \ln(1-z) \left(C_A \left(\frac{\zeta_2}{2} - \frac{35}{72} \right) + C_F \left(\zeta_2 + \frac{17}{4} \right) + \frac{N_f T_f}{18} \right) \\
& + C_A \left(\frac{11\zeta_2}{4} + \frac{3\zeta_3}{2} - \frac{35}{16} \right) + C_F \left(3\zeta_2 - \zeta_3 + \frac{45}{16} \right) + N_f T_f \left(\frac{3}{4} - \zeta_2 \right) \Big] \\
& + \left(\frac{C_A}{2} + C_F \right) \ln^3(1-z) + \ln^2(1-z) \left(\frac{27 C_A}{8} + \frac{13 C_F}{2} - \frac{N_f T_f}{2} \right) \\
& + \ln(1-z) \left[C_A \left(22\zeta_2 - \frac{2011}{72} \right) + C_F (47 - 19\zeta_2) + N_f T_f \left(\frac{361}{36} - 4\zeta_2 \right) \right] \\
& + C_A \left(\frac{6347\zeta_2}{80} - 21\zeta_2 \ln 2 - \frac{137\zeta_3}{4} - \frac{3305}{72} \right) \\
& + C_F \left(-\frac{1727\zeta_2}{20} + 42\zeta_2 \ln 2 + \frac{121\zeta_3}{2} + \frac{3437}{96} \right) \\
& + N_f T_f \left(-\frac{1747\zeta_2}{120} + 12\zeta_3 + \frac{2099}{144} \right) \Big\} + \mathcal{O}(1-z)
\end{aligned}
$$

leading power

first subleading power

- Double log behavior, $\ln^{2L+1}(1-z)/(1-z)$ characteristic of Sudakov suppression from soft/collinear gluon emission. Collins, Soper,…
- Coefficients of leading-power terms agree precisely with NNLL resummation DeFlorian, Grazzini, hep-ph/0407241

$z \rightarrow 1$ (cont.)



Moult, Zhu, 1801.02627

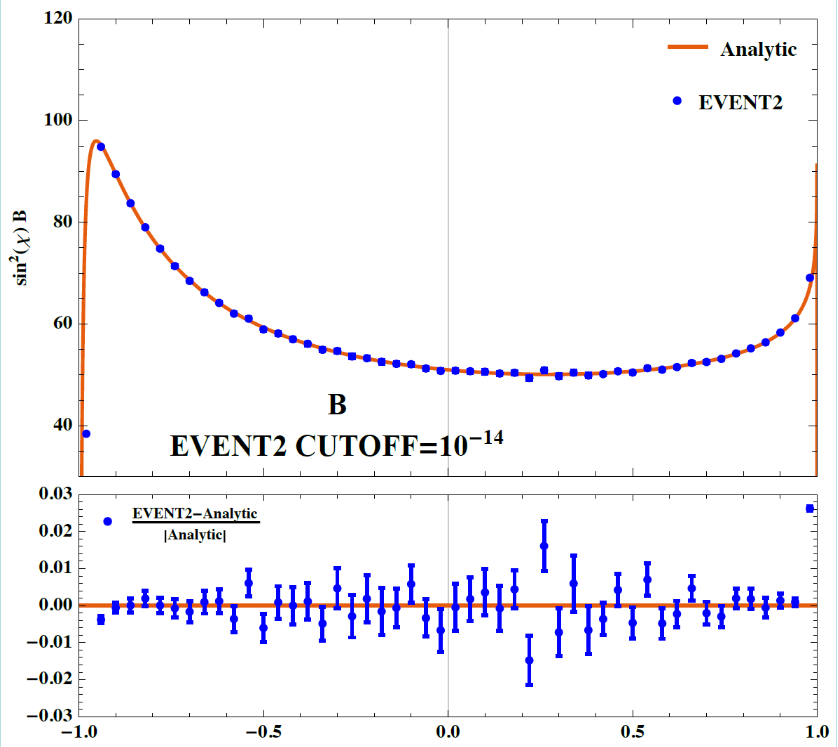
Soft gluons contribute, but only via recoil, by deflecting the hard quark jet

- Factorization theorem recently proved: Relate EEC to back-to-back production of identified hadrons Collins, Soper 1981-1982
- Should allow NNNLL resummation soon

Why analytic?

- Validate accuracy of numerical QCD results.
- Compare with analytic NLO result in N=4 SYM

Belitsky, Hohenegger, Korchemsky, Sokatchev, Zhiboedov, 1309.0769, 1309.1424, 1311.6800



- Study limits as $\chi \to 0,\pi$ to aid resummation of large logarithms there.

Belitsky et al. method for N=4 SYM

- Very different from "QCD method", which uses dimensional regularization; divergences cancel between virtual and real
- Exploit conformal invariance of 4-point function with two "energy flow operators"

$$\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\rangle_q = \int d^4x\, \mathrm{e}^{iq\cdot x} \langle 0|O^\dagger(x)\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)O(0)|0\rangle$$

$$\mathcal{E}(\vec{n}) = \int_{-\infty}^{\infty} d\tau \lim_{r\to\infty} r^2 n^i T_{0i}(t=\tau+r, r\vec{n})$$

- Analytically continue from Euclidean to physical region using double Mellin transform
- No infrared divergences at any step!
- Recently pushed to NNLO (semi-analytic): Henn, Sokatchev, Yan and Zhiboedov, 1903.05314