String amplitudes from genus zero to genus one

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Joint work with Don Zagier (arXiv:1906.12339 [math.NT])

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String amplitudes

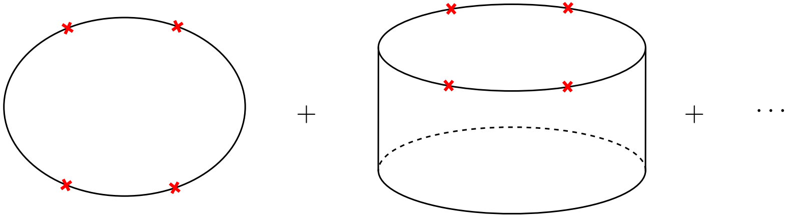

Figure: Four open strings

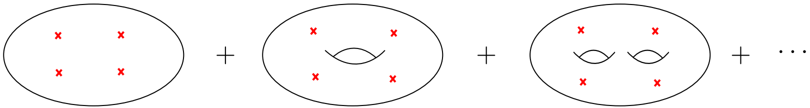

Figure: Four closed strings

Plan of the talk

$$A_{0,4}^{(\mathrm{op})}(s,t) \qquad A_{0,4}^{(\mathrm{cl})}(s,t)$$

$$A_{1,2}^{(\mathrm{op})}(s,\tau) \qquad A_{1,2}^{(\mathrm{cl})}(s,\tau)$$

Plan of the talk

$$A_{0,4}^{(\mathrm{op})}(s,t) \xrightarrow{\color{blue}{\mathrm{sv}}} A_{0,4}^{(\mathrm{cl})}(s,t)$$

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From $A_{0,4}^{(\mathrm{op})}$ to $A_{0,4}^{(\mathrm{cl})}$: the KLT formula

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Open strings: the Veneziano amplitude

$$A_{0,4}^{(\mathrm{op})}(s,t) = \int_{[0,1]} x^{s-1}(1-x)^{t-1}dx = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

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Closed strings: the Virasoro-Shapiro amplitude

$$A_{0,4}^{(\mathrm{cl})}(s,t) = \int_{\mathbb{P}^1_{\mathbb{C}}} |z|^{2s-2}|1-z|^{2t-2}\frac{dzd\overline{z}}{(-2\pi i)} = \frac{\Gamma(s)\Gamma(t)\Gamma(1-s-t)}{\Gamma(1-s)\Gamma(1-t)\Gamma(s+t)}.$$

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$$A_{0,4}^{(\mathrm{cl})}(s,t) = -\frac{s+t}{st}A_{0,4}^{(\mathrm{op})}(s,t)A_{0,4}^{(\mathrm{op})}(-s,-t)^{-1}$$

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$$\implies A^{(\mathrm{cl})}_{0,4}(s,t) = \frac{\sin(\pi s)\sin(\pi t)}{\pi\ \sin(\pi(s+t))}A^{(\mathrm{op})}_{0,4}(s,t)^2.$$

From $A_{0,4}^{(\mathrm{op})}$ to $A_{0,4}^{(\mathrm{cl})}$: the single-valued projection

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Define $\mathrm{sv}\big(\zeta(2k)\big) = 0$ and $\mathrm{sv}\big(\zeta(2k+1)\big) = 2\zeta(2k+1)$
Then $\mathrm{sv}\big(A_{0,4}^{(\mathrm{op})}(s,t)\big) = A_{0,4}^{(\mathrm{cl})}(s,t)$.

Genus one, two points, **closed** strings

Genus one, two points, **closed** strings

For $\tau \in \mathbb{H}$, $\Lambda_\tau := \mathbb{Z}\tau + \mathbb{Z}$ consider the Green's function on $\mathbb{C}/\Lambda_\tau$

$$G(z,\tau) = -\log \left| \frac{\theta_1(z,\tau)}{\eta(\tau)} \right|^2 + \frac{2\pi \, \mathrm{Im}(z)^2}{\mathrm{Im}(\tau)}.$$

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Set $d\mu(z) = \frac{i\,dz\,d\bar{z}}{2\,\mathsf{Im}(\tau)}$ and define

$$A^{(\mathrm{cl})}_{1,2}(s,\tau) := \int_{\mathbb{C}/\Lambda_\tau} e^{s\,G(z,\tau)}\,d\mu(z) = \sum_{l\geq 0}\frac{s^l}{l!}\int_{\mathbb{C}/\Lambda_\tau} G(z,\tau)^l\,d\mu(z),$$

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$D_l(\tau) := \int_{\mathbb{C}/\Lambda_\tau} G(z,\tau)^l d\mu(z)$ are "2-point modular graph functions".

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Examples

$$D_1(\tau) = 0, \qquad D_2(\tau) = E(2,\tau), \qquad D_3(\tau) = E(3,\tau) + \zeta(3).$$

The leading term as $\tau \to i\infty$

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Proposition (Green, Russo, Vanhove; Zagier)

$$A^{(\mathrm{cl})}_{1,2}(s,\tau) = e^{sY/6}\sum_{n=0}^{\infty}\left(\frac{Y^n}{(2n+1)!!} - \sum_{k=1}^{n-1}\frac{(2k-3)!!\gamma_{n,k}}{(-Y)^k}\right)\left(\frac{-s}{2}\right)^n$$
$$+\,O(e^{-Y}) \qquad (Y := 2\pi\mathrm{Im}(\tau) \to +\infty),$$

where the coefficients $\gamma_{n,k}$ are given by the special MZVs

$$\gamma_{n,k} := \sum_{r=0}^{n-k-1}(-2)^{r+2}\binom{n-r+k-3}{2k-2}Z(k+n-r,r) \qquad (0<k<n)\,,$$

$$Z(m,r) := \sum_{\substack{\boldsymbol{r}\in\{1,2\}^j,\, j\geq 0\\ r_1+\cdots+r_j=r}} 2^{\#\{i\,:\,r_i=2\}}\,\zeta(\boldsymbol{r},m) \qquad (m\geq 2, r\geq 0).$$

The numbers $\gamma_{n,k}$

$$\gamma_{2,1} = 4\zeta(3)$$
$$\gamma_{3,1} = 0$$
$$\gamma_{4,1} = 12\zeta(5)$$
$$\gamma_{5,1} = -8\zeta(3)^2$$
$$\gamma_{6,1} = 36\zeta(7)$$

$$\gamma_{3,2} = 4\zeta(5)$$
$$\gamma_{4,2} = 8\zeta(3)^2$$
$$\gamma_{5,2} = 24\zeta(7)$$
$$\gamma_{6,2} = 32\zeta(5)\zeta(3)$$
$$\gamma_{7,2} = \frac{292}{3}\zeta(9) - \frac{64}{3}\zeta(3)^2$$

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Define the generating function

$$\mathcal{G}(X,Y) := \frac{1}{X(X+Y)(Y-X)} + \sum_{n>k>0} \gamma_{n,k}\, X^{n-k-1}\, Y^{2k-2}$$

Genus one vs genus zero

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Theorem (Zagier, Z.)

The numbers $\gamma_{n,k}$ are polynomials in odd zeta values with rational coefficients, explicitly given in terms of the coefficients of $A_{0,4}^{(\mathrm{cl})}$ by

$$\mathcal{G}(X,Y) \;=\; \frac{2\,A_{0,4}^{(\mathrm{cl})}(2X,-X-Y)}{(Y-X)^2}.$$

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Main ingredient of the proof

$$A_{0,4}^{(\mathrm{cl})}(s,t) = \sum_{n\geq 0} \binom{-s-t}{n}^2 \left(\frac{1}{n+s}+\frac{1}{n+t}\right)$$

Genus one, two points, **open** strings

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For $\tau \in i\mathbb{R}^+$ consider the "B-cycle propagator"

$$P_B(z,\tau) = -\log\left|\frac{\theta_1(z,\tau)}{\eta(\tau)}\right| - \frac{i\pi}{\tau}\left(z^2 + \frac{1}{6}\right)$$

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and define in a neighborhood of $s = 0$ ($s \in \mathbb{C}$)

$$A_{1,2}^{(\mathrm{op})}(s,\tau) := \int_{\mathbb{R}\tau/\mathbb{Z}\tau} e^{s\,P_B(z,\tau)}\,\frac{dz}{\tau} = \sum_{l\geq 0}\frac{s^l}{l!}\int_{\mathbb{R}\tau/\mathbb{Z}\tau} P_B(z,\tau)^l\,\frac{dz}{\tau},$$

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The unique holomorphic extension of $B_l(\tau) := \int_{\mathbb{R}\tau/\mathbb{Z}\tau} P_B(z,\tau)^l dz/\tau$ to $\mathbb{H}$ are "2-point B-cycle holomorphic graph functions".

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Examples

$$B_1(\tau) = 0, \qquad B_2(\tau) \sim \widetilde{G}_4(\tau), \qquad B_3(\tau) \sim \widetilde{G}_6(\tau).$$

The leading term as $\tau \to i\infty$

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Proposition (Zagier, Z.)

$$A_{1,2}^{(\mathrm{op})} = e^{sT/6-\zeta(2)s/T} \sum_{n\geq 0} \left(\frac{T^n}{(2n+1)!!} - \sum_{k=1}^{n} \frac{(2k-3)!!\,\eta_{n,k}}{(-T)^k} \right) \left(-\frac{s}{2} \right)^n$$

$$+ O(e^{-T}) \qquad (T := \pi\tau/i \to +\infty),$$

where the coefficients $\eta_{n,k}$ are given for $0 < k \leq n$ by the special MZVs

$$\eta_{n,k} := \sum_{r=-1}^{n-k-1} (-2)^{r+2} \binom{k+n-r-3}{2k-2} \zeta(\underbrace{1,\cdots,1}_{r+1}, k+n-r-1).$$

The numbers $\eta_{n,k}$

$$\eta_{1,1} = -\zeta(2) \qquad \eta_{2,2} = -2\zeta(4)$$
$$\eta_{2,1} = 2\zeta(3) \qquad \eta_{3,2} = 2\zeta(5) - 4\zeta(3)\zeta(2)$$
$$\eta_{3,1} = -9\zeta(4) \qquad \eta_{4,2} = 2\zeta(3)^2 - \frac{29}{2}\zeta(6)$$
$$\eta_{4,1} = 6\zeta(5) + 4\zeta(3)\zeta(2) \qquad \eta_{5,2} = 12\zeta(7) - 14\zeta(3)\zeta(4) - 8\zeta(5)\zeta(2)$$
$$\eta_{5,1} = -2\zeta(3)^2 - \frac{61}{2}\zeta(6) \qquad \eta_{6,2} = 8\zeta(3)\zeta(5) + 8\zeta(3)^2\zeta(2) - \frac{703}{12}\zeta(8)$$

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Fact:

$$\sum_{k,r\geq 1} \zeta(\underbrace{1,\cdots,1}_{r-1},k+1)\, s^k\, t^r = 1 - \frac{\Gamma(1-s)\,\Gamma(1-t)}{\Gamma(1-s-t)}.$$

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Fact: $$\sum_{k,r\geq 1} \zeta(\underbrace{1,\cdots,1}_{r-1},k+1)\, s^k\, t^r = 1 - \frac{\Gamma(1-s)\,\Gamma(1-t)}{\Gamma(1-s-t)}.$$

Corollary: All numbers $\eta_{n,k}$ are polynomials in single zeta values with rational coefficients.

Genus one vs genus zero

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Define the generating function

$$\mathcal{E}(X,Y) \;:=\; \frac{1}{X\,(X+Y)\,(Y-X)} \;+\; \sum_{n\geq k>0} \eta_{n,k}\, X^{n-k-1}\, Y^{2k-2}$$

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Proposition (Zagier, Z.)

The numbers $\eta_{n,k}$ are polynomials in single zeta values with rational coefficients, explicitly given in terms of the coefficients of $A_{0,4}^{(\mathrm{op})}$ by

$$\mathcal{E}(X,Y) = \left(\frac{\sin\big(\pi(X+Y)\big)}{\sin\big(\pi(Y-X)\big)} - 1\right) \frac{A_{0,4}^{(\mathrm{op})}(2X,-X-Y)}{X\,(Y-X)}.$$

Single-valued projection at genus 1

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Compare:

$$\eta_{1,1} = -\zeta(2) \qquad \gamma_{1,1} = 0$$
$$\eta_{2,1} = 2\zeta(3) \qquad \gamma_{2,1} = 4\zeta(3)$$
$$\eta_{3,1} = -9\zeta(4) \qquad \gamma_{3,1} = 0$$
$$\eta_{4,1} = 6\zeta(5) + 4\zeta(3)\zeta(2) \qquad \gamma_{4,1} = 12\zeta(5)$$
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$$\gamma_{7,3} = 80\zeta(7)\zeta(3) + 40\zeta(5)^2$$

Single-valued projection at genus 1

Compare:

$$\eta_{1,1} = -\zeta(2) \qquad \gamma_{1,1} = 0$$
$$\eta_{2,1} = 2\zeta(3) \qquad \gamma_{2,1} = 4\zeta(3)$$
$$\eta_{3,1} = -9\zeta(4) \qquad \gamma_{3,1} = 0$$
$$\eta_{4,1} = 6\zeta(5) + 4\zeta(3)\zeta(2) \qquad \gamma_{4,1} = 12\zeta(5)$$
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Conjecture (Broedel, Schlotterer, Z. - 2018): $\mathrm{sv}(\eta_{n,k}) = \gamma_{n,k}$

Single-valued projection at genus 1

Open strings

$$A^{(\mathrm{op})}_{1,2} = e^{sT/6-\zeta(2)s/T} \sum_{n\geq 0} \left(\frac{T^n}{(2n+1)!!} - \sum_{k=1}^{n} \frac{(2k-3)!!\,\eta_{n,k}}{(-T)^k} \right) \left(-\frac{s}{2} \right)^n$$
$$+ O(e^{-T}) \qquad (T := \pi\tau/i \to +\infty).$$

Closed strings

$$A^{(\mathrm{cl})}_{1,2}(s,\tau) = e^{sY/6} \sum_{n=0}^{\infty} \left(\frac{Y^n}{(2n+1)!!} - \sum_{k=1}^{n-1} \frac{(2k-3)!!\gamma_{n,k}}{(-Y)^k} \right) \left(\frac{-s}{2} \right)^n$$
$$+ O(e^{-Y}) \qquad (Y := 2\pi \mathrm{Im}(\tau) \to +\infty),$$

Single-valued projection at genus 1

$$\zeta(2) \to 0, \qquad \eta_{n,k} \to \gamma_{n,k}, \qquad T \to Y \;\; (\log q \to \log |q|^2)$$

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Recall:

- $\mathcal{G}(X,Y) \;=\; 2\,A^{(\mathrm{cl})}_{0,4}(2X,-X-Y)/(Y-X)^2$

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Theorem (Zagier, Z.)

$\mathrm{sv}\big(\mathcal{E}(X,Y)\big) = \mathcal{G}(X,Y)$ and therefore $\mathrm{sv}(\eta_{n,k}) = \gamma_{n,k}$

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Proof: $\mathrm{sv}\left(\frac{\sin\big(\pi(X+Y)\big)}{\sin\big(\pi(Y-X)\big)} - 1\right) = 2X/(Y-X)$.

KLT analogue at genus 1

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$$\Downarrow$$

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$$\Downarrow$$

$$\mathcal{G}(X,Y) = \frac{2X^2}{\pi}\,\frac{\sin\big(\pi(2X)\big)\,\sin\big(\pi(X+Y)\big)\,\sin\big(\pi(Y-X)\big)}{\Big(\sin\big(\pi(X+Y)\big)+\sin\big(\pi(X-Y)\big)\Big)^2}\,\mathcal{E}(X,Y)^2.$$

THE END

Thanks for your attention!