One-loop superstring amplitudes

Carlos R. Mafra

(With Oliver Schlotterer, arXiv:1812.{10969,10970,10971})

Supported by a Royal Society University Fellowship

STAG Research Centre and Mathematical Sciences,
University of Southampton, UK

Motivation

- Compute the n-point open superstring correlator at one loop using pure spinors
- Correlator $\mathcal{K}_n(\ell)$ defined by:

$$\mathcal{A}_n = \sum_{\text{top}} C_{\text{top}} \int_{D_{\text{top}}} d\tau\, dz_2\, dz_3 \, \ldots \, dz_n \int d^D\ell \; |\mathcal{I}_n(\ell)| \, \langle \mathcal{K}_n(\ell) \rangle$$

The most important conditions we impose on the correlator are

1. BRST invariant (ie susy and gauge invariant)

$$Q\mathcal{K}_n(\ell) = 0$$

2. monodromy invariant under $a-$ and $b-$cycles

$$D\mathcal{K}_n(\ell) = 0$$

D=10 SYM theory

- Covariant description using single-particle (label i) superfields (Witten'86)

$$A^i_\alpha(x,\theta),\ A^i_m(x,\theta),\ W^\alpha_i(x,\theta),\ F^i_{mn}(x,\theta)$$

- Linearized equations of motion

$$D_\alpha A^i_\beta + D_\beta A^i_\alpha = \gamma^m_{\alpha\beta} A^i_m, \quad D_\alpha A^i_m = (\gamma_m W_i)_\alpha + \partial_m A^i_\alpha$$
$$D_\alpha W^\beta_i = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F^i_{mn}, \quad D_\alpha F^i_{mn} = 2\partial_{[m}(\gamma_{n]} W_i)_\alpha$$

where D_α is covariant derivative in 10D superspace (x^m, θ^α)

Pure spinor amplitude prescription at one loop

- (Berkovits'04)

$$\mathcal{A}_1 = \int_{\text{moduli}} \left\langle (\mu, b)(PCs) V^1(0) \int dz\; U^2 \cdots \int dz\; U^n \right\rangle$$

- vertex operators using single-particle SYM superfields

$$V^1 = \lambda^\alpha A^1_\alpha(x,\theta),$$
$$U^i = \partial\theta^\alpha A^i_\alpha + A^i_m \Pi^m + d_\alpha W^\alpha_i + \frac{1}{2} N^{mn} F^i_{mn}$$

- CFT calculation: zero modes and OPEs
- OPEs among vertices organized using multiparticle superfields with covariant BRST variations (CM, Schlotterer '14)
- BRST charge $Q = \lambda^\alpha D_\alpha$

Multiparticle SYM superfields (CM, Schlotterer '14, '15)

$$K_B \in \{A_\alpha^B,\ A_B^m,\ W_B^\alpha,\ F_B^{mn}\}$$

- Recursive definition of multiparticle superfields inspired by OPE computations and satisfy generalized EOMs

$$W_{12}^\alpha = \frac{1}{4}(\gamma^{mn}W^2)^\alpha F_{mn}^1 + W_2^\alpha(k^2\cdot A^1) - (1\leftrightarrow 2)$$
$$D_\alpha W_{12}^\beta = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}^{12} + (k^1\cdot k^2)(A_\alpha^1 W_2^\beta - A_\alpha^2 W_1^\beta)$$

- They satisfy generalized Jacobi identities

Generalized Jacobi symmetries

- The superfields K_B satisfy generalized Jacobi symmetries

$$
\begin{aligned}
0 &= K_{12} + K_{21}, \\
0 &= K_{123} + K_{231} + K_{312}, \quad \text{(Jacobi identity)} \\
0 &= K_{1234} - K_{1243} + K_{3412} - K_{3421} \\
0 &= K_{A\ell(B)} + K_{B\ell(A)}
\end{aligned}
$$

- $\ell(A)$ is the Dynkin operator (left-to-right nested brackets)
- These are the same symmetries obeyed by nested commutators

$$K_{1234\ldots p} \rightarrow K_{[\ldots[[[1,2],3],4],\ldots,p]} = K_{\ell(P)}$$

- Bern-Carrasco-Johansson numerators are natural in this framework

Kinematic building blocks

- Pure spinor zero-mode saturation rules give rise to kinematic building blocks satisfying covariant BRST variations
- From tree-level we get $V_B = \lambda^\alpha A^B_\alpha$

$$QV_1 = 0$$
$$QV_{123} = (k^1 \cdot k^2)(V_1 V_{23} + V_{13} V_2) + (k^{12} \cdot k^3) V_{12} V_3$$

- At one-loop there are more possibilities V_A, $T_{A,B,C}$, $T^m_{A,B,C,D}$, $T^{mn}_{\dots}$ etc with BRST algebra

$$QT_{1,2,3} = 0$$
$$QT_{12,3,4} = s_{12}(V_1 T_{2,3,4} - V_2 T_{1,3,4})$$
$$QT^m_{1,2,3,4} = k^m_1 V_1 T_{2,3,4} + (1 \leftrightarrow 2,3,4)$$

- They satisfy generalized Jacobi symmetries within each "word" slot

One-loop superstring correlators

A Lie polynomial is an expression written in terms of nested commutators

Ree's theorem

If $\mathcal{Z}_P$ satisfies shuffle symmetries $\mathcal{Z}_{A \sqcup B} = 0$ and t^{p_i} are non-commutative variables then

$$\sum_P \mathcal{Z}_{p_1 p_2 p_3 \dots} t^{p_1} t^{p_2} t^{p_3} \dots$$

is a Lie polynomial

- Example: $\mathcal{Z}_{12}$ satisfies shuffle if it is antisymmetric, so

$$\mathcal{Z}_{12} t^1 t^2 + \mathcal{Z}_{21} t^2 t^1 = \mathcal{Z}_{12}[t^1, t^2]$$

is a Lie polynomial

Lessons from tree-level

- n-point disk correlator can be rewritten in a suggestive way:

$$\mathcal{K}_n^{\rm tree} = \sum_{AB=23\ldots n-2} \big(\mathcal{Z}^{\rm tree}_{1A} V_{1A}\big)\big(\mathcal{Z}^{\rm tree}_{n-1,B} V_{n-1,B}\big) V_n + {\rm perm}(23\ldots n-2)\,.$$

1. Worldsheet functions satisfy shuffle symmetries

$$\mathcal{Z}^{\rm tree}_{123\ldots p} \equiv \frac{1}{z_{12}z_{23}\ldots z_{p-1,p}} \qquad \longrightarrow \qquad \mathcal{Z}^{\rm tree}_{A\shuffle B} = 0$$

2. associated kinematics satisfy generalized Jacobi symmetries

$$V_P \equiv \lambda^\alpha A^P_\alpha \qquad \longrightarrow \qquad V_{A\ell(B)} + V_{B\ell(A)} = 0$$

- This has the same structure of a Lie polynomial!

$$\sum_P \mathcal{Z}^{\rm tree}_P V_P$$

Ansatz for one-loop correlators

Tree-level reinterpretation key to unlock the one-loop correlators

1. Assume Lie-polynomial structure for one-loop correlators:

$$\mathcal{K}_n \to \sum \mathcal{Z}_{A,B,C,D} V_A T_{B,C,D} + \cdots$$

2. kinematic factors $V_A T_{B,C,D}$ satisfying generalized Jacobi symmetries
3. one-loop worldsheet functions $\mathcal{Z}_{A,B,C,\ldots}$ satisfying shuffle symmetries

- Singular behavior of $\mathcal{Z}_{A,B,\ldots}$ as vertices collide is known from OPEs
- Unlike at tree-level, OPEs don't determine the complete functions as regular pieces are not fixed by singularities

The shuffle-symmetry requirement was helpful in fixing the functions

- How do these functions look like?

Shuffle-symmetric functions

- PS zero-mode rules and OPEs imply at low multiplicities

$$\mathcal{Z}_{1,2,3,4} = 1$$
$$\mathcal{Z}_{12,3,4,5} = g^{(1)}_{12}, \qquad \mathcal{Z}^m_{1,2,3,4,5} = \ell^m$$

- Notation: $g^{(n)}_{ij} \equiv g^{(n)}(z_i - z_j, \tau)$
- $g^{(1)}_{12}$ is part of an infinite family of functions (Brown, Levin)

$$g^{(0)}(z,\tau) = 1$$
$$g^{(1)}(z,\tau) = \partial_z \ln \theta_1(z,\tau)$$
$$2g^{(2)}(z,\tau) = (\partial_z \ln \theta_1(z,\tau))^2 + \partial_z^2 \ln \theta_1(z,\tau) - \frac{\theta_1'''(0,\tau)}{3\theta_1'(0,\tau)}$$

- Defined from the Kronecker–Eisenstein series

$$\frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(\alpha,\tau)\theta_1(z,\tau)} \equiv \sum_{n=0}^{\infty} \alpha^{n-1} g^{(n)}(z,\tau)$$

Kronecker-Eisenstein coefficient functions

- $g^{(1)}(z,\tau) = \partial \log \theta_1(z,\tau)$ is the genus-one generalization of tree-level $1/z$ function
- $g^{(n)}(z,\tau)$ for $n \geq 2$ have no singularities on the surface as $z \to 0$
- $g^{(n)}(z,\tau)$ are single-valued around a-cycles
- monodromies around b-cycles given by

$$Dg_{ij}^{(n)} = \Omega_{ij} g_{ij}^{(n-1)}, \quad \Omega_{ij} \equiv \Omega_i - \Omega_j$$

  where $D \equiv -\frac{1}{2\pi i} \sum_{j=1}^{n} \Omega_j \delta_j$ is a monodromy operator and Ω_i are abstract parameters
- $g_{ij}^{(n)}$ satisfy Fay identities, eg

$$g_{12}^{(1)} g_{23}^{(1)} + g_{12}^{(2)} + \mathrm{cyc}(1,2,3) = 0$$

- $\mathcal{Z}_{12,3,4,5}$ is antisymmetric in [12], so it obeys shuffle symmetry
- Casting the 4 and 5-pt correlators in Lie-polynomial form we get

$$\begin{aligned}
\mathcal{K}_4(\ell) &= V_1\,T_{2,3,4}\mathcal{Z}_{1,2,3,4}\\
\mathcal{K}_5(\ell) &= V_1\,T^m_{2,3,4,5}\mathcal{Z}^m_{1,2,3,4,5} + \big[V_{12}\,T_{3,4,5}\mathcal{Z}_{12,3,4,5} + (2 \leftrightarrow 3,4,5)\big]\\
&\quad + \big[V_1\,T_{23,4,5}\mathcal{Z}_{1,23,4,5} + (2,3|2,3,4,5)\big]
\end{aligned}$$

- what about 6 points?

Shuffle-symmetric function for 6-pt correlator

- We need a shuffle-symmetric one-loop counterpart of the tree-level

$$\mathcal{Z}^{\rm tree}_{123} = \frac{1}{z_{12}z_{23}}$$

- However, both

$$g^{(1)}_{12}g^{(1)}_{23} + \frac{1}{2}(g^{(2)}_{12} + g^{(2)}_{23})$$

  and

$$g^{(1)}_{12}g^{(1)}_{23} + g^{(2)}_{12} + g^{(2)}_{23} - g^{(2)}_{13}$$

  satisfy shuffle symmetries in $P = 123$ (using Fay ids)
- Which one to use at six points?
- A new (double-copy) duality comes to the rescue! BRST invariants vs generalized elliptic integrands (CM, Schlotterer '17)

Worldsheet function/BRST-invariants duality

- One can show that (recall $\mathcal{Z}_{12,3,4,5} = g^{(1)}_{12}$)

$$E_{1|23,4,5} = \mathcal{Z}_{1,23,4,5} + \mathcal{Z}_{12,3,4,5} - \mathcal{Z}_{13,2,4,5}$$

  is is single valued, $DE_{1|23,4,5} = 0$
- Seen this combinatorial pattern before: 5-pt BRST invariant

$$C_{1|23,4,5} = M_1 M_{23,4,5} + M_{12} M_{3,4,5} - M_{13} M_{2,4,5}$$

  satisfying $QC_{1|23,4,5} = 0$, where $M_A M_{B,C,D}$ are the non-local versions of $V_A T_{B,C,D}$ (CM, Schlotterer'14)
- Duality: elliptic integrands vs BRST invariants

$$E_{1|23,4,5} \longleftrightarrow C_{1|23,4,5}$$

$$DE_{1|23,4,5} = 0 \longleftrightarrow QC_{1|23,4,5} = 0$$

- Tensorial generalization (CM, Schlotterer '18)

Worldsheet function/BRST invariant duality

- Using the Jacobi theta functions and integration by parts can show

$$k_2^m E^m_{1|2,3,4,5} + \big[s_{23} E_{1|23,4,5} + (3 \leftrightarrow 4,5)\big] = 0$$

- We have seen an identity of identical structure for the BRST invariants:

$$k_2^m C^m_{1|2,3,4,5} + \big[s_{23} C_{1|23,4,5} + (3 \leftrightarrow 4,5)\big] = 0$$

- Similarly, identical symmetry relations hold for the GEIs

$$\begin{aligned} E_{2|34,1,5} &= E_{1|34,2,5} + E_{1|23,4,5} - E_{1|24,3,5} \\ E_{2|13,4,5} &= -E_{1|23,4,5} \\ E^m_{2|1,3,4,5} &= E^m_{1|2,3,4,5} + \big[k_3^m E_{1|23,4,5} + (3 \leftrightarrow 4,5)\big]\ , \end{aligned}$$

- Duality between elliptic functions and BRST invariants!

Bootstrapping worldsheet functions

- This duality can be exploited to derive higher-point worldsheet functions!
- Inspired by the BRST variation written in terms of BRST invariants

$$QM_{123,4,5} = C_{1|23,4,5} - C_{3|12,4,5}$$

assume the following monodromy variation of the 6pt worldsheet function

$$D\mathcal{Z}_{123,4,5,6} = \Omega_1 E_{1|23,4,5,6} - \Omega_3 E_{3|12,4,5,6}$$

where the elliptic functions $E_{1|23,4,5,6}$ are obtained from 5pt functions using the combinatorics of 5pt BRST invariants

Higher-point one-loop correlators

- There is a unique solution:

$$\mathcal{Z}_{123,4,5,6} = g^{(1)}_{12} g^{(1)}_{23} + g^{(2)}_{12} + g^{(2)}_{23} - g^{(2)}_{13}$$

- This is the function we should use in 6pt ansatz!
- Can find the tensorial shuffle-symmetric functions similarly
  - Require the monodromy variations of $\mathcal{Z}^{mn\dots}_{A,B,C,\dots}$ to match the BRST variation of the corresponding Berends-Giele superfield $M_A M^{mn\dots}_{B,C,\dots}$

Examples

- 6 point correlator

$$\begin{aligned}\mathcal{K}_6(\ell) &= \frac{1}{2} V_1\, T^{mn}_{2,3,4,5,6} \mathcal{Z}^{mn}_{1,2,3,4,5,6} \\ &+ \big[V_{12}\, T^{m}_{3,4,5,6} \mathcal{Z}^{m}_{12,3,4,5,6} + (2 \leftrightarrow 3,4,5,6) \big] \\ &+ \big[V_1\, T^{m}_{23,4,5,6} \mathcal{Z}^{m}_{1,23,4,5,6} + (2,3|2,3,4,5,6) \big] \\ &+ \big[V_{123}\, T_{4,5,6} \mathcal{Z}_{123,4,5,6} + V_{132}\, T_{4,5,6} \mathcal{Z}_{132,4,5,6} + (2,3|2,3,4,5,6) \big] \\ &+ \big[(V_{12}\, T_{34,5,6} \mathcal{Z}_{12,34,5,6} + \mathrm{cyc}(2,3,4)) + (2,3,4|2,3,4,5,6) \big] \\ &+ \big[(V_1\, T_{2,34,56} \mathcal{Z}_{1,2,34,56} + \mathrm{cyc}(3,4,5)) + (2 \leftrightarrow 3,4,5,6) \big] \\ &+ \big[V_1\, T_{234,5,6} \mathcal{Z}_{1,234,5,6} + V_1\, T_{243,5,6} \mathcal{Z}_{1,243,5,6} + (2,3,4|2,3,4,5,6) \big]\end{aligned}$$

- Nice combinatorics of Stirling set and cycle numbers:

$$\mathcal{K}_6(\ell) = \sum_{r=0}^{2} \frac{1}{r!} \Big(V_{A_1}\, T^{m_1\ldots m_r}_{A_2,\ldots,A_{r+4}} \mathcal{Z}^{m_1\ldots m_r}_{A_2,\ldots,A_{r+4}} + \big[12\ldots 6|A_1,\ldots,A_{r+4}\big] \Big)$$

6pt anomaly cancellation (Green, Schwarz 84)

- 6pt correlator is not BRST invariant by itself
- However BRST variation is a total derivative on moduli space

$$Q\mathcal{K}_6(\ell) = -\frac{1}{2} V_1 Y_{2,3,4,5,6} \mathcal{Z}^{mm}_{1,2,3,4,5,6} = -2\pi i\, V_1 Y_{2,3,4,5,6} \frac{\partial}{\partial\tau} \log \mathcal{I}_6(\ell) \cong 0$$

where $Y_{2,3,4,5,6}$ is the anomaly kinematic factor (CM, Berkovits 2006)

$$Y_{2,3,4,5,6} \equiv \frac{1}{2}(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)(\lambda\gamma^p W_4)(W_5\gamma_{mnp} W_6)$$

- To show this need identities for τ derivatives of the Kronecker-Eisenstein series, several BRST variations etc
- So anomaly cancels after summing over one-loop topologies for $SO(32)$ (Green, Schwarz 84)

Higher-point one-loop correlators

- This corresponds to a non-refined sector ($d = 0$) and generalizes to a Lie-polynomial structure

$$\mathcal{K}_n^{(0)}(\ell) = \sum_{r=0}^{n-4} \frac{1}{r!}\Big(V_{A_1} T^{m_1\ldots m_r}_{A_2,\ldots,A_{r+4}} \mathcal{Z}^{m_1\ldots m_r}_{A_1,\ldots,A_{r+4}} + \big[12\ldots n|A_1,\ldots,A_{r+4}\big]\Big)$$

- The full correlator includes refined ($d \neq 0$) and anomalous (Y) superfields (omitted entirely from this talk)

$$\mathcal{K}_n(\ell) \equiv \sum_{d=0}^{\lfloor \frac{n-4}{2} \rfloor} (-1)^d \mathcal{K}_n^{(d)}(\ell) + \mathcal{K}_n^Y(\ell)$$

- Leads to BRST-invariant and single-valued 7-pt correlator
- Puzzle at 8-points: modular form of weight four $G_4(\tau)$ remains in the BRST variation
- Probably requires a new class of term that we missed, but the Lie-polynomial structure of the correlator should be the same