Modular Forms and $U(1)$-violating amplitudes in type IIB superstring theory

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Introduction

- We are interested in string corrections to Einstein–Hilbert action

$$\mathcal{L}_{\mathrm{EFT}} \sim R + \frac{a_2^{(0)}}{\Lambda^2} R^2 + \frac{a_3^{(0)}}{\Lambda^4} R^3 + \frac{a_4^{(0)}}{\Lambda^6} R^4 + \frac{a_4^{(2)}}{\Lambda^8} d^2 R^4 + \cdots$$

- Effective action: write down higher-dimensional operators consistent with the symmetries.
- It is easy to say, but hard in practice ······
- On-shell amplitude is a powerful tool to implement the symmetries, and to impose unitarity contraints [Andrew's talk].

Introduction

- The symmetries of 10D type IIB superstring theory: maximal supersymmetry, $SL(2,\mathbb{Z})$ duality symmetry.

- The α'-expansion generates effective action

$$\mathcal{L}^{\rm IIB}_{\rm EFT} \sim R + \alpha' F_2^{(0)}(\tau) \cancel{R^2} + \alpha'^2 F_3^{(0)}(\tau) \cancel{R^3} + \alpha'^3 F_4^{(0)}(\tau) R^4 + \alpha'^5 F_4^{(2)}(\tau) d^4 R^4 + \alpha'^6 F_4^{(3)}(\tau) d^6 R^4 + \cdots\cdots$$

- Coefficients $F(\tau)$ are modular forms of $\tau = C^{(0)} + ie^{-\phi}$, admit expansion in string coupling: $e^{\phi_0} = g_s = 1/\tau_2^0$.

- Good understanding on coefficients of R^4, d^4R^4, d^6R^4 [Green, Gutperle + Vanhove][Green, Sethi]......[Basu][Pioline][Berkovits, Vafa].....[Gomez, Mafra, Schlotterer]

Introduction

- We will derive known results from a different viewpoint: Using constraints from superamplitudes.
- More importantly, extension for general BPS interactions (order $\leq P^{14}$)

$$\mathcal{L}^{(p)}_{n\,i} \sim F^{(p)}_{w\,i}(\tau)\, d^{2p}_{(i)}\, \mathcal{P}^{(w)}_n(\{\Phi\})\,,$$

where i denotes a possible degeneracy, coefficient $F^{(p)}_{w\,i}$ is a weight-w modular function of module τ, which compensates the $U(1)$-weight of the fields in $\mathcal{P}^{(w)}_n(\{\Phi\})$.

Non-holomorphic modular forms

- Non-holomorphic modular forms are functions of τ:

$$F^{(p)}_{w,w'}(\tau) \to (c\tau + d)^{w} (c\bar{\tau} + d)^{w'} F^{(p)}_{w,w'}(\tau)$$

under $SL(2,\mathbb{Z})$ transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}\,.$$

- Covariant derivatives

$$\mathcal{D}_w F^{(p)}_{w,w'}(\tau) := \left(i\tau_2\partial_\tau + \frac{w}{2}\right) F^{(p)}_{w,w'}(\tau) := F^{(p)}_{w+1,w'-1}(\tau)\,,$$

$$\bar{\mathcal{D}}_{w'} F^{(p)}_{w,w'}(\tau) := \left(-i\tau_2\partial_{\bar{\tau}} + \frac{w'}{2}\right) F^{(p)}_{w,w'}(\tau) := F^{(p)}_{w-1,w'+1}(\tau)\,.$$

- We will consider $w' = -w$, $F^{(p)}_{w}(\tau) := F^{(p)}_{w,-w}(\tau)$ transforms with a phase.

Examples: Eisenstein series

- Examples: non-holomorphic Eisenstein series

$$E_w(s,\tau) = \sum_{(m,n)\neq(0,0)} \left(\frac{m+n\bar{\tau}}{m+n\tau}\right)^w \frac{\tau_2^s}{|m+n\tau|^{2s}}$$

It has weight $(w,-w)$.

- Satisfy Homogenous Laplace equation,

$$\begin{aligned}\Delta_-^{(w)} E_w(s,\tau) &:= 4\mathcal{D}_{w-1}\bar{\mathcal{D}}_{-w} E_w(s,\tau) \\ &= (s-w)(s+w-1)\, E_w(s,\tau)\end{aligned}$$

or

$$\begin{aligned}\Delta_+^{(w)} E_w(s,\tau) &:= 4\bar{\mathcal{D}}_{-w-1}\mathcal{D}_{w} E_w(s,\tau) \\ &= (s+w)(s-w-1)\, E_w(s,\tau)\end{aligned}$$

$d^{2p}R^4$ terms with $p = 0, 2, 3$

- R^4 and d^4R^4 [Green, Gutperle + Vanhove][Green, Sethi][Yin, Wang]

$$E_0(\tfrac{3}{2},\tau)R^4\,, \qquad E_0(\tfrac{5}{2},\tau)d^4R^4$$

- Perturbative expansions in large τ_2 agree with explicit computations

$$E_0(\tfrac{3}{2},\tau) = 2\zeta(3)\tau_2^{\frac{3}{2}} + 4\zeta(2)\tau_2^{-\frac{1}{2}} + \text{instantons}$$

- The coefficient of d^6R^4, $F_0^{(3)}(\tau)$, satisfies inhomogeneous Laplace equation [Green, Vanhove][Yin, Wang]

$$\left(\Delta_{-}^{(0)} - 12\right) F_0^{(3)}(\tau) = -E_0(\tfrac{3}{2},\tau)^2$$

- Will be derived from superamplitudes.

Superamplitudes: 10D helicity spinors

- 10D superamplitudes using helicity spinors:
  - 10D spinor helicity and type IIB SUSY: [Caron-Huot, O'Connell], [Boels, O'Connell] massless momentum

$$p^{BA} := (\gamma^{\mu})^{BA}\, p_{\mu} = \lambda^{Ba}\lambda^{A}_{a}\,.$$

    $A = 1, \ldots, 16$ is the spinor of $SO(9,1)$ and $a = 1, \ldots, 8$ the $SO(8)$ little group index.

  - Supercharges

$$Q^{A}_{n} = \sum_{i=1}^{n} \lambda^{A}_{i,a}\, \eta^{a}_{i}\,, \qquad \bar{Q}^{A}_{n} = \sum_{i=1}^{n} \lambda^{A,a}_{i}\, \frac{\partial}{\partial \eta^{a}_{i}}\,.$$

Superamplitudes: 10D helicity spinors

- The on-shell massless states:

$$\Phi(\eta) = Z + \eta^a \Lambda'_a + \frac{1}{2!}\eta^a \eta^b \phi'_{ab} + \cdots + \frac{1}{8!}(\eta)^8 \bar{Z}\,.$$

with a field redefinition $Z = \frac{\tau - \tau^0}{\tau - \bar{\tau}^0} = \frac{i\delta\tau/\tau_2^0}{1 - i\delta\tau/\tau_2^0}$.

- Assign $U(1)$ charges:

$$q_Z = -2,\ q_{\Lambda'_a} = -\frac{3}{2}, \cdots, q_h = 0, \cdots, q_{\bar{Z}} = 2\,.$$

- The super amplitudes

$$A_n = \delta^{10}\left(\sum_{r=1}^{n} p_r\right)\delta^{16}(Q_n)\,\hat{A}_n(\eta,\lambda)\,, \quad \text{with} \quad \bar{Q}_n^A \hat{A}_n(\eta,\lambda) = 0\,,$$

For supergravity, $U(1)$ is conserved and $\hat{A}_n \sim \eta^{4(n-4)}$.

Superamplitudes: maximal $U(1)$-violating

- Continuous $U(1)$ is broken by α' corrections.
- Maximal $U(1)$-violating amplitudes [Boels]

$$A^{(p)}_{n,i} = F^{(p)}_{n-4,i}(\tau)\delta^{16}(Q_n)\,\hat{A}^{(p)}_{n,i}(s_{ij})\,,$$

  where i denotes a possible degeneracy.
- Maximal $U(1)$-violating amplitudes ($n > 4$) have no poles.
- Therefore $\hat{A}^{(p)}_{n,i}(s_{ij})$ is a degree-p symmetric polynomial of s_{ij}.
  They are super vertices.
- In 4D, they are KLT of $\mathrm{MHV} \otimes \overline{\mathrm{MHV}}$.

Soft axio-dilaton theorems

- The axio-dilaton (Z field) parametrizes the coset space: Higher-point amplitudes are related to lower-point ones by soft limits.
  - The coefficients are related by covariant derivatives,

$$F^{(p)}_{n-4,i}(\tau) \sim \mathcal{D}_{n-5} F^{(p)}_{n-5,i}(\tau)$$

  - The kinematics are related by soft limits (soft Z field)

$$\hat{A}^{(p)}_{n,i}(s_{ij})\big|_{p_n \to 0} \to \hat{A}^{(p)}_{n-1,i}(s_{ij})$$

- Covariant derivative is a result of combination of soft dilaton ($\tau_2 \partial_{\tau_2} A_n$) [Di Vecchia, Marotta, Mojaza] and soft axion ($w \sum_i R_i A_n$).

Soft axio-dilaton theorems

- $\hat{A}^{(0)}_{n,i}(s_{ij}) = 1$ is for P^8 interactions, related to $R^4 Z^{n-4}$, (no degeneracy)

$$F^{(0)}_{n-4}(\tau) \sim \mathcal{D}_{n-5} \cdots \mathcal{D}_0 E_0(\tfrac{3}{2}, \tau) \sim E_{n-4}(\tfrac{3}{2}, \tau)$$

- $\hat{A}^{(2)}_{n,i}(s_{ij}) = \sum_{i<j} s^2_{ij}$ is for P^{12} interactions, related to $d^4 R^4 Z^{n-4}$, (no degeneracy)

$$F^{(2)}_{n-4}(\tau) \sim \mathcal{D}_{n-5} \cdots \mathcal{D}_0 E_0(\tfrac{5}{2}, \tau) \sim E_{n-4}(\tfrac{5}{2}, \tau)$$

- $E_{n-4}(\frac{3}{2}, \tau)$ and $E_{n-4}(\frac{5}{2}, \tau)$ are the non-holomorphic Eisenstein series of weight $n-4$.

Degeneracy at order P^{14} and $n \geq 6$

- $\hat{A}^{(3)}_{n,i}(s_{ij}) \sim s_{ij}^3$, P^{14} interactions ($d^6 R^4 Z^{n-4}$): two independent kinematics (or interaction terms) for $n \geq 6$:

$$\hat{A}^{(3)}_{6,1} = 10 \sum_{i<j} s_{ij}^3 + 3 \sum_{i<j<k} s_{ijk}^3 \,,$$

$$\hat{A}^{(3)}_{6,2} = 2 \sum_{i<j} s_{ij}^3 - \sum_{i<j<k} s_{ijk}^3 \sim \sum_{P} s_{12} s_{34} s_{56} \,.$$

- $\hat{A}^{(3)}_{6,1}$ appears at tree-level [Schlotterer], and goes to $d^6 R^4 Z$ and further to $d^6 R^4$ in soft limits:

$$\mathcal{E}^{(3)}_{2,1} = 2\, \mathcal{D}_1 \mathcal{E}^{(3)}_1 = 4\, \mathcal{D}_1 \mathcal{D}_0 \mathcal{E}^{(3)}_0$$

- $\hat{A}^{(3)}_{6,2}$ is constructed to vanish in soft limit, it starts at one loop. It is genuinely new.

BPS terms and differential eqs.

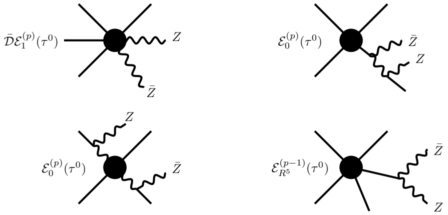
- The BPS terms are further constrained by SUSY, efficiently implemented using superamplitudes. [Yin, Wang] [Chen, Huang, C.W.]
- A simple example: $\mathcal{L}_{4\mathrm{D}} \sim g_3 \phi \partial_\mu \bar{\phi} A^\mu + g_4 \phi^2 \bar{\phi}^2$
  - SUSY implies 4-pt superamplitude is fixed. For $\mathcal{N} = 4$,

$$\mathcal{A}_4 = c \frac{\delta^8(Q)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} .$$

    No supersymmetric contact term can be added.
  - Compute component amplitude $A(\phi, \phi, \bar{\phi}, \bar{\phi})$ from $\mathcal{L}_{4\mathrm{D}}$ and compare it with $\mathcal{A}_4$, we find $g_4 = g_3^2$ (exact result).

BPS terms and differential eqs.

- Contributions to $A_6(h,h,h,h,Z,\bar{Z})$ (order P^8, P^{12}):



6-pt vertex is from expansion around background τ^0: $\mathcal{E}_1^{(p)}(\tau)d^{2p}R^4 Z \to \bar{\mathcal{D}}\mathcal{E}_1^{(p)}(\tau^0)d^{2p}R^4 Z\bar{Z}$ ($p = 0, 2$).

- The absence of supersymmetric contact terms leads to:

$$\bar{\mathcal{D}}\mathcal{E}_1^{(p)} + c_1\mathcal{E}_0^{(p)} + c_2\mathcal{E}_{R^5}^{(p-1)} = 0\,.$$

BPS terms and differential eqs.

$$\bar{\mathcal{D}}\mathcal{E}_1^{(p)} + c_1\mathcal{E}_0^{(p)} + c_2\mathcal{E}_{R^5}^{(p-1)} = 0\,. \tag{1}$$

- 5-graviton amplitude receives contributions from $\mathcal{E}_{R^5}^{(p-1)}\, d^{2p-2}R^5$ and $\mathcal{E}_0^{(p)}\, d^{2p}R^4$. The absence of supersymmetric contact terms requires

$$\mathcal{E}_{R^5}^{(p-1)} + d_1\,\mathcal{E}_0^{(p)} = 0\,. \tag{2}$$

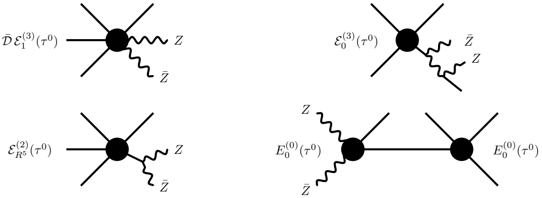
- From Eqs. (1) and (2), we have

$$\bar{\mathcal{D}}\mathcal{E}_1^{(p)} + c_1'\mathcal{E}_0^{(p)} = 0\,.$$

- Soft limit relates 5-points with 4-points: $\mathcal{E}_1^{(p)} = 2\mathcal{D}\mathcal{E}_0^{(p)}$.
- Together, they imply the well-known Laplacian equations.

BPS terms and differential eqs.

- Contributions to $A_6(h,h,h,h,Z,\bar{Z})$ at order P^{14}



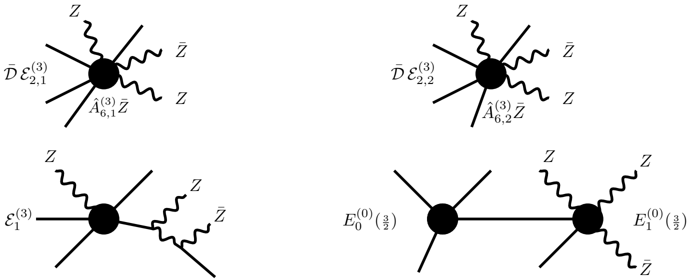
- The absence of supersymmetric contact terms requires

$$\bar{\mathcal{D}}\mathcal{E}_1^{(3)} + c_1\mathcal{E}_0^{(3)} + c_2 E_0(\tfrac{3}{2})E_0(\tfrac{3}{2}) = 0\,.$$

- Leads to inhomogeneous Laplace equation.
- A supersymmetric contact term is allowed at P^{16} (d^8R^4).

BPS terms and differential eqs.

- To study $\mathcal{E}^{(3)}_{2,1}(\tau)$, $\mathcal{E}^{(3)}_{2,2}(\tau)$ of $\hat{A}^{(3)}_{6,1}$, $\hat{A}^{(3)}_{6,2}$, we consider the 7-point amplitude: $A_7(h,h,h,h,Z,Z,\bar{Z})$ at order P^{14}



BPS terms and differential eqs.

- The super-amplitude constraints are

$$\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,1} + a_1\mathcal{E}^{(3)}_{1} + a_2 E_0(\tfrac{3}{2})E_1(\tfrac{3}{2}) = 0\,,$$
$$\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,2} + b_1\mathcal{E}^{(3)}_{1} + b_2 E_0(\tfrac{3}{2})E_1(\tfrac{3}{2}) = 0\,.$$

- $\mathcal{E}^{(3)}_{2,1} = 2\mathcal{D}_1\mathcal{E}^{(3)}_{1}$, knowing $\mathcal{E}^{(3)}_{1}$ from previous study fixes a_1, a_2

$$\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,1} - \frac{1}{2}\mathcal{E}^{(3)}_{1} + \frac{1}{40}E_0(\tfrac{3}{2})E_1(\tfrac{3}{2}) = 0\,.$$

- The diff equation for $\mathcal{E}^{(3)}_{2,2}$ is more interesting.

BPS terms and differential eqs.

- No tree-level term in $\mathcal{E}^{(3)}_{2,2}$ (recall $\hat{A}^{(3)}_{6,2}$ starts at one loop) fixes one constant:

$$\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,2} + c'_1 \left(\mathcal{E}^{(3)}_1 - \frac{1}{12} E_0(\tfrac{3}{2}) E_1(\tfrac{3}{2}) \right) = 0 \,,$$

  and an inhomogeneous Laplace equation

$$\left(\Delta^{(2)}_{-} - 10\right) \mathcal{E}^{(3)}_{2,2} = -c_1 \left(E_0(\tfrac{3}{2}) E_2(\tfrac{3}{2}) - E_1(\tfrac{3}{2}) E_1(\tfrac{3}{2}) \right) \,.$$

- c_1 is determined by the 7-pt superamplitude, or 6-pt string amplitude at 1-loop.
- Solve differential equations (up to overall c_1): all-order predictions for 6-pt string amplitude of order α'^6:

$$\mathcal{E}_{2,2}(\tau) \sim \zeta(2)\zeta(3)\tau_2 - \frac{4}{15}\zeta(2)^2 \tau_2^{-1} + \frac{1}{15}\zeta(6)\tau_2^{-3} + O(e^{-2\pi\tau_2}) \,.$$

Higher-point BPS terms

- Two independent P^{14} n-pt interactions ($d^6 R^4 Z^{n-4}$):

$$\hat{A}^{(3)}_{n,1} = \frac{1}{32}\left((28-3n)\sum_{i<j} s_{ij}^3 + 3\sum_{i<j<k} s_{ijk}^3\right),$$

$$\hat{A}^{(3)}_{n,2} = (n-4)\sum_{i<j} s_{ij}^3 - \sum_{i<j<k} s_{ijk}^3 .$$

- They are constructed such that

$$\hat{A}^{(3)}_{n,1}\Big|_{p_n\to 0} \to \hat{A}^{(3)}_{n-1,1}, \qquad \hat{A}^{(3)}_{n,2}\Big|_{p_n\to 0} \to \hat{A}^{(3)}_{n-1,2}$$

Coefficients $F^{(3)}_{n-4,1}(\tau)$, $F^{(3)}_{n-4,2}(\tau)$ are related to lower-point ones by $\mathcal{D}_{n-5}$.

Conclusion and remarks

- Type IIB interactions are related by soft limits and covariant derivative $\mathcal{D}_w$. Valid for non-BPS terms.
- Consistency of superamplitudes imposes first-order $\bar{\mathcal{D}}_w$ eqs. for BPS terms, (together with $\mathcal{D}_w$ eqs.) $\Rightarrow$ Laplace eqs.
- Predictions for type IIB superstring amplitudes in the α' expansion.
- Rich structures at lower dimensions.
- Implications for correlators in AdS/CFT beyond supergravity: derive the same differential eqs. from CFT?

Thank you!