Loop recursion relation and dual conformal symmetry for form factors

based on 1812.09001 and 1812.10468 with A. Brandhuber, R. Panerai and G. Travaglini

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Amplitudes, Trinity College Dublin

Motivation and definitions

- Progress for the computation of scattering amplitudes using on-shell techniques.
- Many of these developments were triggered by the study of planar $\mathcal{N} = 4$ SYM.
- Can we extend these techniques to off-shell quantities?

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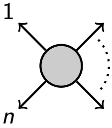
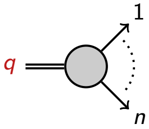
Scattering amplitudes

$$A_n = \langle 1, \cdots, n | 0 \rangle$$

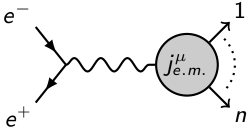


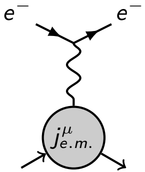
Form factors

$$F_n(q) = \int d^4x \; e^{iqx} \langle 1, \cdots, n | \mathcal{O}(x) | 0 \rangle$$



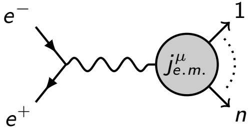
- e^+e^- annihilation and deep inelastic scattering

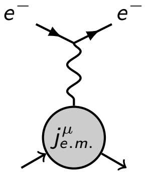




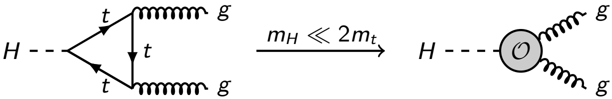
Examples of form factors

- e^+e^- annihilation and deep inelastic scattering





- Higgs effective theory



$$\mathcal{O} = \mathrm{Tr}\{F^{\mu\nu}F_{\mu\nu}\}$$

Form factors in $\mathcal{N} = 4$ SYM

- ✓ Strong coupling [Alday, Maldacena, 2007; Maldacena, Zhiboedov, 2010]
- ✓ Tree-level BCFW recursion relation [Brandhuber, Gurdogan, Mooney, Travaglini, Yang, 2011]
- ✓ Generalized unitarity [Brandhuber, Spence, Travaglini, Yang, 2010]
- ✓ Color-kinematic duality [Boels, Kniehl, Tarasov, Yang, 2012]
- ✓ On-shell diagrams and Grassmanian [Frassek, Meidinger, Nandan, Wilhelm, 2015]
- ✓ Twistor-space formulation [Koster, Mitev, Staudacher, Wilhelm, 2016]
- ✓ Scattering equations [He, Zhang, 2016; Brandhuber, Hughes, Panerai, Spence, Travaglini, 2016]
- ✓ Non-protected operators [Loebbert, Nandan, Sieg, Wilhelm, Yang, 2015-16; Brandhuber, Kostacinska, Penante, Travaglini, 2017-18]

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Results for chiral primary Tr(ϕ^2) in $\mathcal{N} = 4$ SYM

loops \ legs	2	3	4	5	6
0	✓	✓	✓	✓	✓
1	✓	✓	✓	MHV	MHV
2	✓	✓			
3	✓				
4	✓				

[van Neerven, 1986; Brandhuber, Gurdogan, Mooney, Travaglini, Yang, 2011; Brandhuber, Spence, Travaglini, Yang, 2010; Bork, Kazakov, Vartanov, 2010; Bork, 2012; Brandhuber, Travaglini, Yang, 2012; Gehrmann, Henn, Huber, 2012; Brandhuber, Penante, Travaglini, Wen, 2014; Boels, Huber, Yang, 2018]

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- ✓ Loop-level recursion relation [LB, Brandhuber, Panerai, Travaglini, 2018]
- ✓ Dual conformal invariance [LB, Brandhuber, Panerai, Travaglini, 2018]

Loop recursion and dual conformal invariance

- The discovery of dual conformal invariance for scattering amplitudes goes back 13 years [Drummond, Henn, Smirnov, Sokatchev, 2006; Alday, Maldacena, 2007; Drummond, Henn, Korchemsky, Sokatchev, 2007; Brandhuber, Heslop, Travaglini, 2007, 2008]
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Why now?

- Form factors are inherently non-planar.
- Definition of the loop integrand $\leftrightarrow$ Region variables
- Presence of triangle integrals

$$I_4 = \int \mathrm{d}^4 x_0 \, \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \qquad I_3 = \int \mathrm{d}^4 x_0 \, \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2}$$

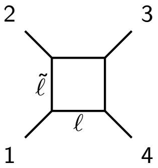
Inversion: $x_i \to x_i / x_i^2 \Rightarrow x_{0i}^2 \to \dfrac{x_{0i}^2}{x_0^2 x_i^2}$

Change of variable: $x_0 \to x_0 / x_0^2 \Rightarrow \mathrm{d}^4 x_0 \to \mathrm{d}^4 x_0 / x_0^8$

Part I

Loop Recursion Relation

- The main issue with the extension of BCFW recursion relation at loop level is the definition of the integrand.



$$= \int d^4\ell \frac{1}{\ell^2(\ell+p_1)^2(\ell+p_{12})^2(\ell-p_4)^2} = \int d^4\tilde{\ell} \frac{1}{\tilde{\ell}^2(\tilde{\ell}+p_2)^2(\tilde{\ell}+p_{23})^2(\tilde{\ell}-p_1)^2}$$

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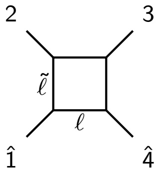

$$= \int d^4\ell \frac{1}{\ell^2(\ell+\hat{p}_1)^2(\ell+\hat{p}_{12})^2(\ell-\hat{p}_4)^2} = \int d^4\tilde{\ell} \frac{1}{\tilde{\ell}^2(\tilde{\ell}+p_2)^2(\tilde{\ell}+p_{23})^2(\tilde{\ell}-\hat{p}_1)^2}$$

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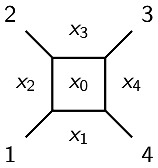
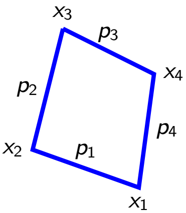
$$p_i = x_i - x_{i+1}$$



$$= \int d^4 x_0 \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \longleftrightarrow$$



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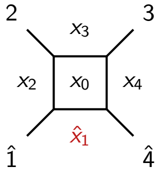
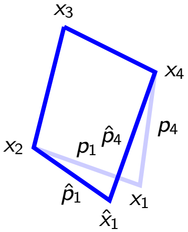
- The BCFW shift is a shift of x_1

$$\hat{x}_1 = x_1 - z\lambda_n\tilde{\lambda}_1$$



$$= \int \mathrm{d}^4 x_0 \, \frac{1}{\hat{x}_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \quad \longleftrightarrow$$



Non-planarity

- Region variable assignment is natural for planar diagrams.
- For amplitudes they correspond to the vertices of the dual polygon Wilson loop.
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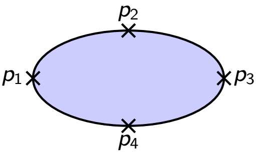
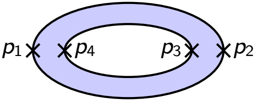
- For amplitudes they correspond to the vertices of the dual polygon Wilson loop.
- The strong coupling picture suggests that the Wilson loop dual should be periodic.
- Similarities with double trace part of amplitudes. [Ben-Israel, Tumanov, Sever, 2018]

Colour ordering

$$A_4 = N\ \mathrm{Tr}\{T^{a_1}T^{a_2}T^{a_3}T^{a_4}\}\mathcal{A}_4 + \mathrm{Tr}\{T^{a_1}T^{a_2}\}\mathrm{Tr}\{T^{a_3}T^{a_4}\}\mathcal{A}_4^{\mathrm{dt}} + \text{non-cyclic perms}$$



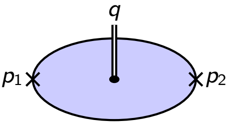
Single trace



Double trace

Form factor and cutting [LB, Brandhuber, Panerai, Travaglini]

- The form factor is similar since the gauge invariant operator is a trace on its own.



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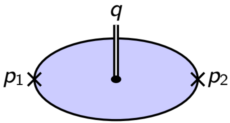
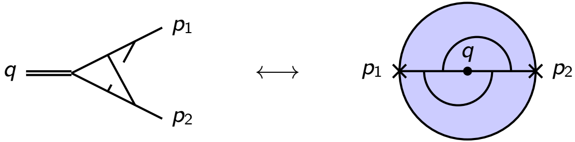


- On the punctured disk diagrams are planar



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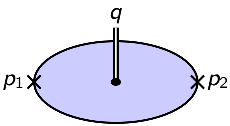

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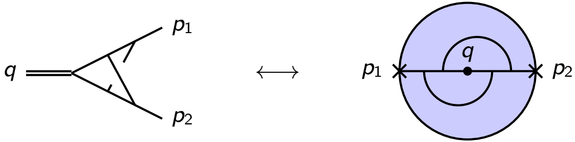
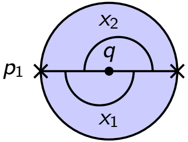


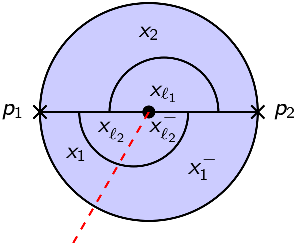
- Can I assign region variables on the punctured disk? No, unless I cut it.



$\Rightarrow \quad p_1 = x_{12}, \quad p_2 = x_{21} \Rightarrow$ wrong!

Region variable assignment [LB, Brandhuber, Panerai, Travaglini]

- In this picture the operator insertion looks like a branch point.



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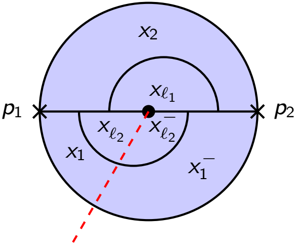

- There is a discontinuity across the cut.

$$x_1^- = x_1 - q$$

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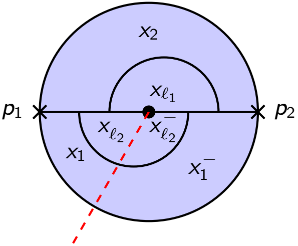

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- The integrand is (notice $x_1 - x_{\ell_2} = x_1^- - x_{\ell_2}^-$)

$$\int d^4x_{\ell_1} d^4x_{\ell_2} \frac{1}{x_{2\ell_2}^2 x_{2\ell_1}^2 (x_{1\ell_1}^-)^2 x_{1\ell_2}^2 x_{\ell_2\ell_1}^2 (x_{\ell_2\ell_1}^-)^2}$$

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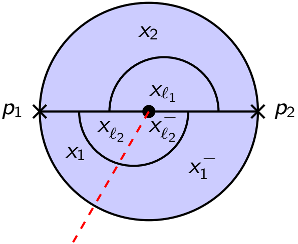

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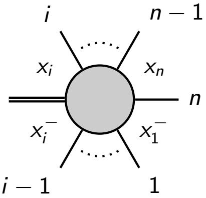
- Nothing changes if we shift all the variables by q: periodicity.

Region variable assignment [LB, Brandhuber, Panerai, Travaglini]

- We can give a prescription to assign region variables diagram by diagram.



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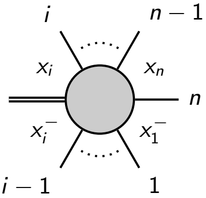
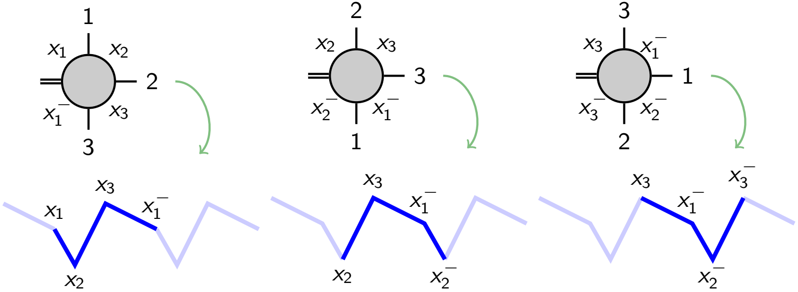


- This is like choosing where to start in the periodic configuration.



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- We define the one-loop integrand

$$F_{n,k}^{(1)} = \int \mathrm{d}^d x_0 \, \mathcal{F}_{n,k}^{(1)}(\{x_i\}; x_0)$$

- $\mathcal{F}_{n,k}^{(1)}(\{x_i\}; x_0)$ is actually a function only of x_{ij} and x_{0i}.
- If all x_i are shifted by q, one can compensate by shifting x_0.

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- We can define equivalence classes

$$\mathcal{F}^{(1)}_{n,k}(\{x_i\}; x_0) \sim \mathcal{F}^{(1)}_{n,k}(\{x_i\}; x_0 + m\, q) \qquad m \in \mathbb{Z}$$

- Of course they yield the same result after integration.

BCFW for the integrand [LB, Brandhuber, Panerai, Travaglini]

- Define the shifted integrand $\mathcal{F}^{(1)}_{n,k}(\{\hat{x}_i\};x_0) \equiv \widehat{\mathcal{F}}^{(1)}_{n,k}(z)$
- Use residue theorem

$$0 = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}z}{z}\, \widehat{\mathcal{F}}^{(1)}_{n,k}(z) = \mathcal{F}^{(1)}_{n,k}(\{x_i\};x_0) + \sum_{z_i\neq 0} \operatorname*{Res}_{z=z_i} \frac{\widehat{\mathcal{F}}^{(1)}_{n,k}(z)}{z}$$

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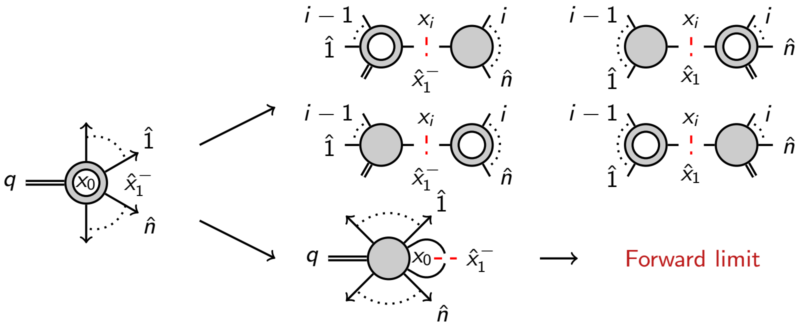
Two types of poles [Caron-Huot, 2010; Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka, 2010]



Loop recursion relation

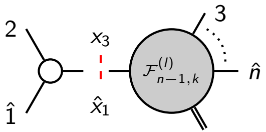
Result

$$\mathcal{F}_{n,k}^{(l)} = F_{n,k}^{(0)}\,\tilde{\mathcal{F}}_{n-1,k}^{(l)}(\hat{x}_1, x_3, \ldots, x_n, x_0)$$
$$+ \frac{1}{x_{01}^2} \int \mathrm{d}^4\eta_\ell\, \mathcal{F}_{n+2,k+1}^{(l-1)}(\hat{x}_1, \ldots, x_n, \hat{x}_1^-, x_0^-)$$
$$+ \sum_{l_\mathrm{L}, i, k_\mathrm{L}} \int \mathrm{d}^4\eta_\ell \Bigg[\mathcal{F}_{i,k_\mathrm{L}}^{(l_\mathrm{L})}(\hat{x}_1, \ldots, x_i)\,\frac{1}{(x_{i1}^+)^2}\,\mathcal{A}_{n-i+2,k_\mathrm{R}}^{(l_\mathrm{R})}(\hat{x}_1, x_i, \ldots, x_n)$$
$$+ \mathcal{A}_{i,k_\mathrm{L}}^{(l_\mathrm{L})}(\hat{x}_1, \ldots, x_i)\,\frac{1}{(x_{i1})^2}\,\mathcal{F}_{n-i+2,k_\mathrm{R}}^{(l_\mathrm{R})}(\hat{x}_1, x_i, \ldots, x_n)\Bigg]$$

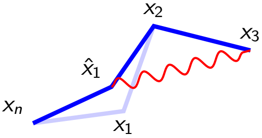
Loop recursion relation

Result

$$
\begin{aligned}
\mathcal{F}^{(l)}_{n,k} = {} & F^{(0)}_{n,k}\, \tilde{\mathcal{F}}^{(l)}_{n-1,k}(\hat{x}_1, x_3, \dots, x_n, x_0) \\
& + \frac{1}{x_{01}^2} \int \mathrm{d}^4\eta_\ell\, \mathcal{F}^{(l-1)}_{n+2,k+1}(\hat{x}_1, \dots, x_n, \hat{x}_1^-, x_0^-) \\
& + \sum_{l_\mathrm{L}, i, k_\mathrm{L}} \int \mathrm{d}^4\eta_\ell \Bigg[\mathcal{F}^{(l_\mathrm{L})}_{i,k_\mathrm{L}}(\hat{x}_1, \dots, x_i)\, \frac{1}{(x^+_{i1})^2}\, \mathcal{A}^{(l_\mathrm{R})}_{n-i+2,k_\mathrm{R}}(\hat{x}_1, x_i, \dots, x_n) \\
& \qquad + \mathcal{A}^{(l_\mathrm{L})}_{i,k_\mathrm{L}}(\hat{x}_1, \dots, x_i)\, \frac{1}{(x_{i1})^2}\, \mathcal{F}^{(l_\mathrm{R})}_{n-i+2,k_\mathrm{R}}(\hat{x}_1, x_i, \dots, x_n) \Bigg]
\end{aligned}
$$



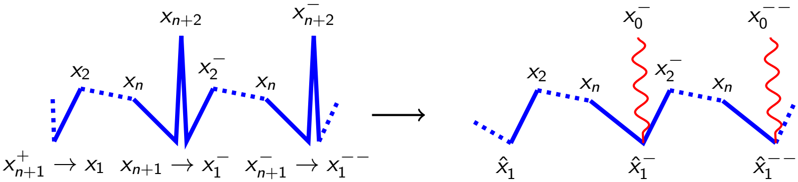
$\longleftrightarrow$



Loop recursion relation

Result

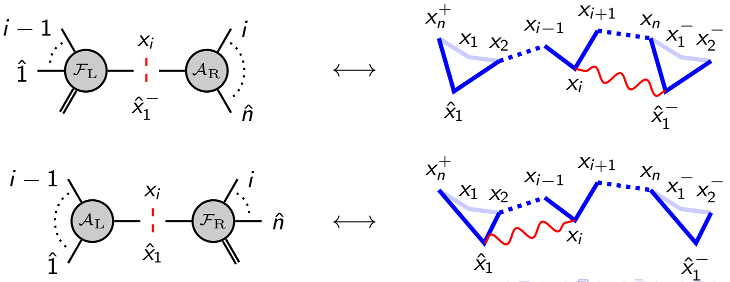
$$\mathcal{F}^{(l)}_{n,k} = F^{(0)}_{n,k}\, \tilde{\mathcal{F}}^{(l)}_{n-1,k}(\hat{x}_1, x_3, \ldots, x_n, x_0)$$
$$+ \frac{1}{x_{01}^2} \int \mathrm{d}^4\eta_\ell\, \mathcal{F}^{(l-1)}_{n+2,k+1}(\hat{x}_1, \ldots, x_n, \hat{x}_1^-, x_0^-)$$
$$+ \sum_{l_\mathrm{L}, i, k_\mathrm{L}} \int \mathrm{d}^4\eta_\ell \Big[\mathcal{F}^{(l_\mathrm{L})}_{i,k_\mathrm{L}}(\hat{x}_1, \ldots, x_i)\, \frac{1}{(x_{i1}^+)^2}\, \mathcal{A}^{(l_\mathrm{R})}_{n-i+2,k_\mathrm{R}}(\hat{x}_1, x_i, \ldots, x_n)$$
$$+ \mathcal{A}^{(l_\mathrm{L})}_{i,k_\mathrm{L}}(\hat{x}_1, \ldots, x_i)\, \frac{1}{(x_{i1})^2}\, \mathcal{F}^{(l_\mathrm{R})}_{n-i+2,k_\mathrm{R}}(\hat{x}_1, x_i, \ldots, x_n) \Big]$$



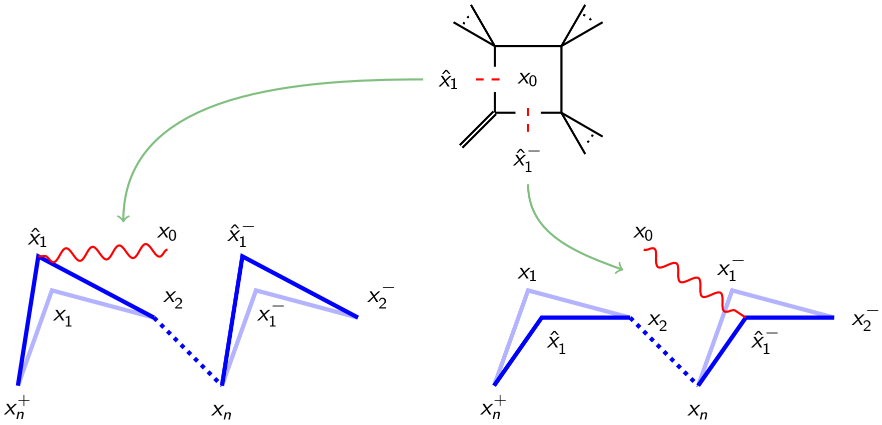
Loop recursion relation

Result

$$
\begin{aligned}
\mathcal{F}^{(l)}_{n,k} &= F^{(0)}_{n,k}\, \tilde{\mathcal{F}}^{(l)}_{n-1,k}(\hat{x}_1, x_3, \ldots, x_n, x_0) \\
&\quad + \frac{1}{x_{01}^2} \int \mathrm{d}^4\eta_\ell\, \mathcal{F}^{(l-1)}_{n+2,k+1}(\hat{x}_1, \ldots, x_n, \hat{x}_1^-, x_0^-) \\
&\quad + \sum_{l_\mathrm{L}, i, k_\mathrm{L}} \int \mathrm{d}^4\eta_\ell \Big[\mathcal{F}^{(l_\mathrm{L})}_{i,k_\mathrm{L}}(\hat{x}_1, \ldots, x_i)\, \frac{1}{(x^+_{i1})^2}\, \mathcal{A}^{(l_\mathrm{R})}_{n-i+2,k_\mathrm{R}}(\hat{x}_1, x_i, \ldots, x_n) \\
&\qquad\qquad + \mathcal{A}^{(l_\mathrm{L})}_{i,k_\mathrm{L}}(\hat{x}_1, \ldots, x_i)\, \frac{1}{(x_{i1})^2}\, \mathcal{F}^{(l_\mathrm{R})}_{n-i+2,k_\mathrm{R}}(\hat{x}_1, x_i, \ldots, x_n) \Big]
\end{aligned}
$$



A peculiarity



Subtleties and features

- The forward limit in general is singular, but for supersymmetric theories the singularity cancels in the sum over states.

- As for amplitudes, the result contains spurious poles.

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- It is very important, as a matter of principles, to know that the integrand can be determined recursively.
- We checked for $n = 2, 3$ at one loop that the formula works.
- We also derived an all-line recursion formula.
- It gives the same result of generalized unitarity after integration for two reasons:
  1. As for amplitudes, it agrees with unitarity up to parity odd terms that integrate to zero. [Cachazo, 2008; Bourjaily, Caron-Huot, Trnka, 2013]
  2. Furthermore, for form factors, agreement is obtained up to shifts $x_0 \to x_0 + nq$.

Part II

Dual conformal symmetry

Periodic momentum twistors

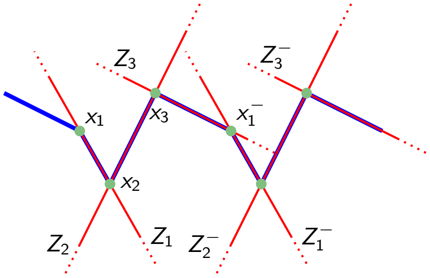
- Supermomentum variables are periodic

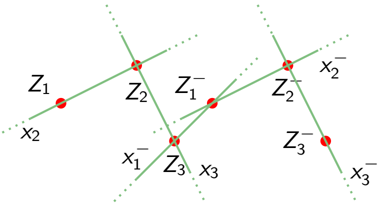
$$x_i^{[m]} = x_i + m\, q \qquad (\theta_i^{[m]})^{A\alpha} = (\theta_i)^{A\alpha} + m\, \mathrm{q}^{A\alpha}$$

- One can introduce a periodic configuration for supertwistors

$$\mathcal{Z}_i^{[m]M} = \begin{pmatrix} Z_i^{[m]\hat{A}} \\ \chi_i^{[m]A} \end{pmatrix}$$

$$Z_i^{[m]\hat{A}} = \begin{pmatrix} \lambda_i^{\alpha} \\ \mu_i^{[m]\dot{\alpha}} \end{pmatrix} \qquad \mu_i^{[m]\dot{\alpha}} = (x_i^{[m]})^{\dot{\alpha}\alpha} \lambda_{i\alpha} \qquad \chi_i^{[m]A} = (\theta_i^{[m]})^{A\alpha} \lambda_{i\alpha}$$





Dual conformal invariance at tree level

- Usual superconformal invariant

$$[a,b,c,d,e] = \frac{\delta^{(4)}(\langle a,b,c,d\rangle\,\chi_e + \text{cyclic})}{\langle a,b,c,d\rangle\,\langle b,c,d,e\rangle\,\langle c,d,e,a\rangle\,\langle d,e,a,b\rangle\,\langle e,a,b,c\rangle}$$

$$\langle i,j,k,l\rangle = \epsilon_{\hat{A}\hat{B}\hat{C}\hat{D}} Z_i^{\hat{A}} Z_j^{\hat{B}} Z_k^{\hat{C}} Z_l^{\hat{D}}$$

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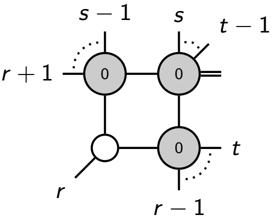
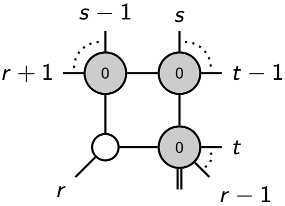
- Using BCFW one can express all tree level ratios $\tilde{F}^{(0)}_{n,k}$ as combinations of

$R'_{rst} =$



$R''_{rst} =$



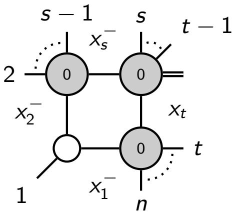
- For example, for the NMHV four-point four factor

$$\tilde{F}^{(0)}_{4,1} = R'_{133} + R'_{134} + R'_{144} + R''_{131}$$

Dual conformal invariance at tree level [Bork, 2014]

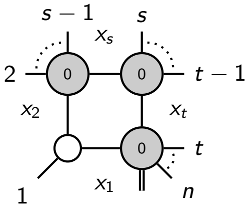
- General configurations are dual superconformal invariant

$R'_{1st} =$



$= [(s-1)^-, s^-, t-1, t, 1^-]$

$R''_{1st} =$



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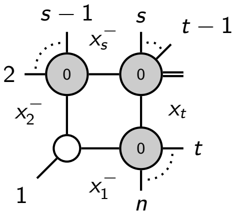

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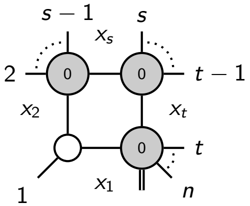
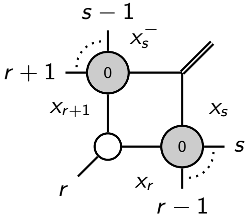


$= [s-1, s, t-1, t, 1]$

- There is a special case

$R'_{rss} =$



$= \frac{\langle r,(s-1)^-,s^-,s-1\rangle\langle r,(s-1)^-,s^-,s\rangle}{\langle r^+,s-1,s,r\rangle\langle s,s^-,s-1,(s-1)^-\rangle}[(s-1)^-,s^-,s-1,s,r]$

Dual conformal invariance at one loop

- At one-loop the dual conformal anomaly reads [Drummond, Henn, Korchemsky, Sokatchev, 2007]

$$\mathrm{K}^{\mu} A_{n,k}^{(1)} \;=\; -4\, A_{n,k}^{(0)} \sum_{i=1}^{n} \frac{x_{i+1}^{\mu} \left(-x_{ii+2}^{2}\right)^{-\epsilon}}{\epsilon}$$

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$$\mathrm{K}^{\mu} F^{(1)}_{n,k} \;=\; -4\, F^{(0)}_{n,k} \sum_{i=1}^{n} \frac{x^{\mu}_{i+1}\left(-x^{2}_{ii+2}\right)^{-\epsilon}}{\epsilon}$$

- Relation with the IR divergence

$$F^{(1)}_{n,k}\Big|_{\mathrm{IR}} = -F^{(0)}_{n,k} \sum_{i=1}^{n} \frac{\left(-x^{2}_{ii+2}\right)^{-\epsilon}}{\epsilon^{2}} \qquad \Rightarrow \qquad \mathrm{K}^{\mu} F^{(1)}_{n,k} = \; 4\,\epsilon\, x^{\mu}_{i+1}\; F^{(1)}_{n,k}\Big|_{\mathrm{IR}}$$

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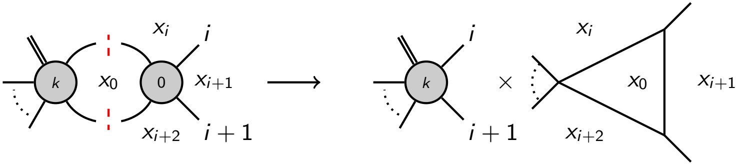
- This can be seen by looking at the only IR-divergent cut [Brandhuber, Heslop, Travaglini, 2009]



- The IR-singular part of this cut is given entirely by the forward configuration

$$\ell_1 = -p_i \qquad \ell_2 = -p_{i+1} \qquad x_0 = x_{i+1}$$

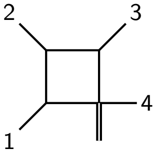
NHMV example

- Let us focus on the finite part

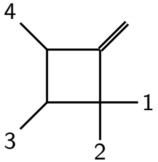
$$K^\mu \; F^{(1)}_{n,k}\Big|_{\text{fin}} = -2\, F^{(0)}_{n,k} \sum_{i=1}^{n} p_i^\mu \log\left(\frac{x^2_{i i+2}}{x^2_{i-1\, i+1}}\right)$$

Result for NMHV at 4 points

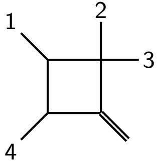
$\tilde{F}^{(1)}_{4,1} = b^{1\text{m}}$



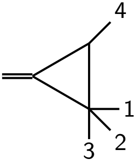
$+ \, b_1^{2\text{mh}}$



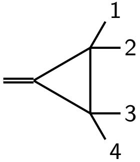
$+ \, b_2^{2\text{mh}}$



$+ \, c^{2\text{m}}$



$+ \, c^{3\text{m}}$



$+$ cyclic ,

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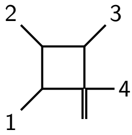
Result for NMHV at 4 points

$\tilde{F}^{(1)}_{4,1} = \frac{c^{2\mathrm{m}}}{2} \Bigg($



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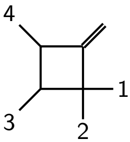

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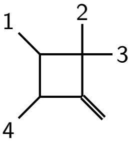
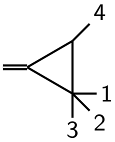
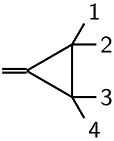


$+\ 2$



$\Bigg)$

$+\ c^{3\mathrm{m}}$



$+\ \text{cyclic}\,.$

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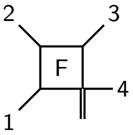
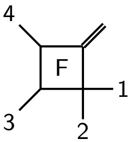
Finite part

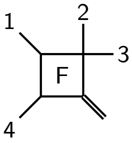
$$\tilde{F}^{(1)}_{4,1}\Big|_{\rm fin} = \frac{c^{2\rm m}}{2}\Bigg($$



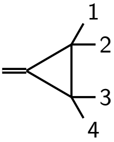
$$+$$



$$+$$



$$\Bigg) + c^{3\rm m}$$



$$+ \text{cyclic}\,.$$

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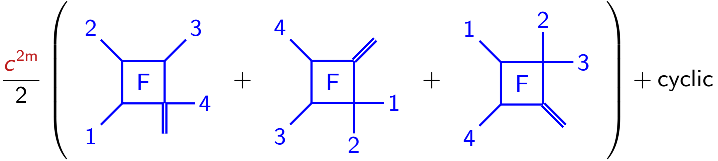
Dual conformal variations



$$K^\mu c^{2m} = 0 \qquad \sum_{\text{cyclic}} \frac{c^{2m}}{2} K^\mu\,(\text{Boxes}) = -2\, F^{(0)}_{4,1} \sum_{i=1}^{4} p_i^\mu \log\left(\frac{x^2_{i\,i+2}}{x^2_{i-1\,i+1}}\right)$$

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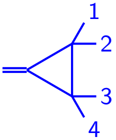
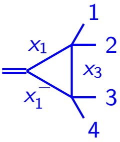
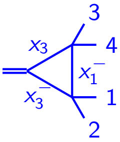
Dual conformal variations

c^{3m} 

$+$ cyclic

$K^\mu c^{3m} = 0$ $\qquad$ $K^\mu \Bigg($ 

$+$ 

$\Bigg) = 0$

Conclusion and outlook

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- We provided a natural prescription to assign region variables in the perturbative computation of form factors.
- This allowed to define the loop integrand and to derive a loop recursion relation.
- We also found that the dual variables representation provided by our assignment exhibits dual conformal invariance.
- We explicitly checked dual conformal symmetry for the MHV and NMHV one-loop amplitude, finding that triangle integrals do not contribute to the anomaly.

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- We explicitly checked dual conformal symmetry for the MHV and NMHV one-loop amplitude, finding that triangle integrals do not contribute to the anomaly.

Outlook

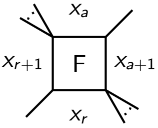
- It would be desirable to have a more direct recipe to go from momenta to dual variables
- The integrands provided by generalized unitarity or loop recursion are not ideal. One would like a local integrand representation like that provided by prescriptive unitarity. [Bourjaily, Herrmann, Trnka, 2017]
- Wilson loop dual and finite coupling.

Examples

$$K^\mu \; F^{(1)}_{n,k}\Big|_{\rm fin} = -2\, F^{(0)}_{n,k} \sum_{i=1}^{n} p_i^\mu \log\left(\frac{x^2_{i\,i+2}}{x^2_{i-1\,i+1}}\right)$$

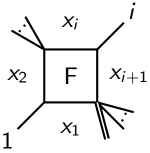
MHV

$$F^{(1)}_{n,0} = F^{(0)}_{n,0}\left(-\sum_{i=1}^{n} \frac{(-x^2_{i\,i+2})^{-\epsilon}}{\epsilon^2} + \sum_{r,a} \right.$$



$$\left.\right)$$

$$\sum_i K^\mu \Bigg($$



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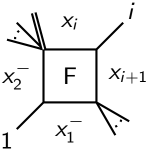
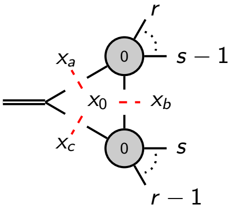


$$\Bigg) \sim -2\, p_1^\mu \; \log\left(\frac{x^2_{1\,3}}{(x^-_{2\,n})^2}\right)$$

Triangles



$$c^{3\mathrm{m}} = \mathcal{R}_{r,s}(\ell_2)\frac{\sqrt{uv}}{\Delta}\ ,$$

$$\mathcal{R}_{r,s}(\ell_2) = [\ell_2, r, r-1, r^-, (r-1)^-]\,\frac{\langle \ell_2, r, r-1, r^-\rangle\,\langle \ell_2, r^-, (r-1)^-, r-1\rangle}{\langle \ell_2, r, r-1, s-1\rangle\,\langle \ell_2, r^-, (r-1)^-, s\rangle}$$
$$\times\ \frac{\langle s-1, s, r-1, r\rangle^{\frac{1}{2}}\,\langle s-1, s, (r-1)^-, r^-\rangle^{\frac{1}{2}}}{\langle r-1, r, (r-1)^-, r^-\rangle}\ .$$

$$\mathcal{Z}^M_{\ell_2} = \begin{pmatrix} Z^{\hat{A}}_{\ell_2} \\ \theta_b^{A\alpha}\lambda_{\ell_2\alpha}\end{pmatrix}\ , \qquad Z^{\hat{A}}_{\ell_2} = \begin{pmatrix} \lambda^{\alpha}_{\ell_2} \\ x_b^{\dot{\alpha}\alpha}\lambda_{\ell_2\alpha}\end{pmatrix}$$