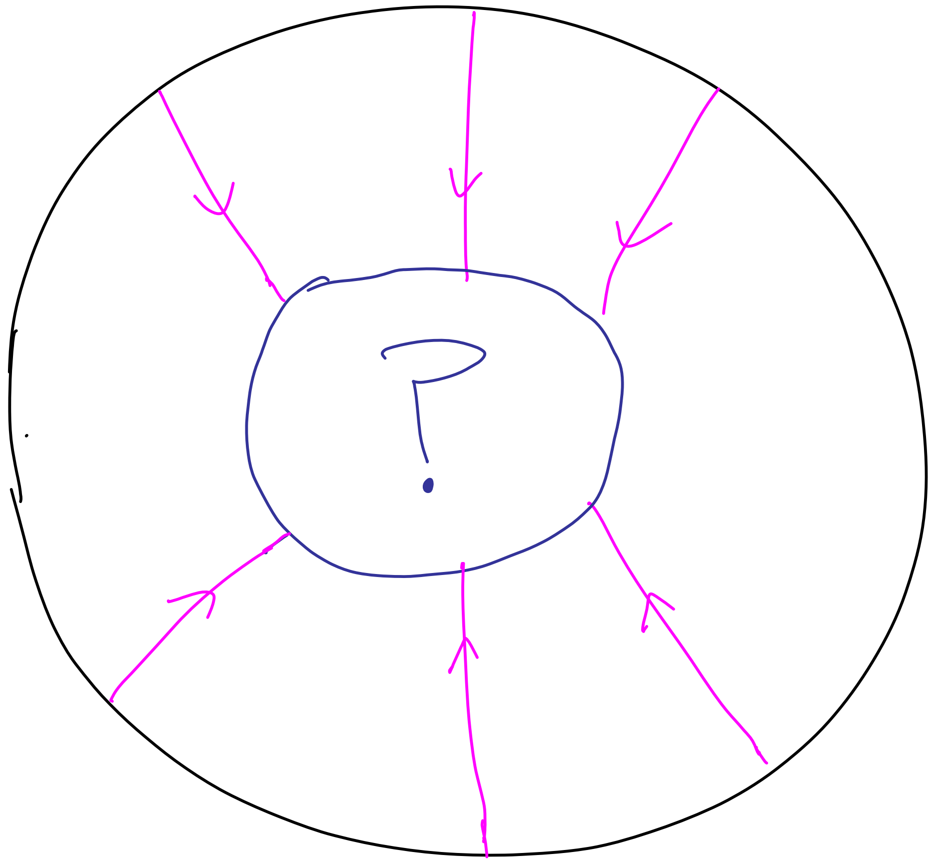
(Binary) Positive Geometry of Causal Diamonds, (Generalized) Particles and Strings

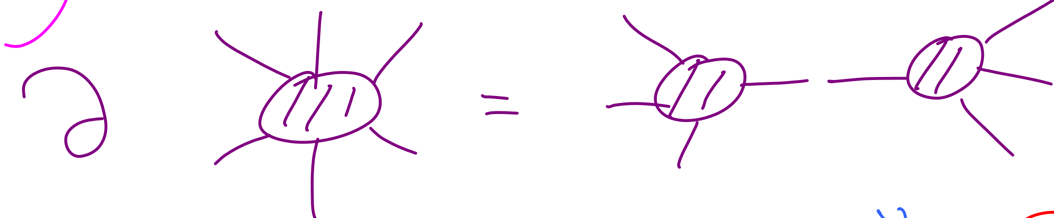


For ~ 10 years
(+ more intensely in last 2 years)
we are seeing

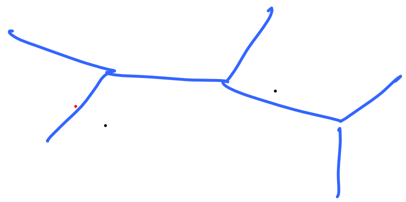
Combinatorial Positive Geometries in Kinematic Space

Big Clue Seen By Closet-Theorist Experimentalist

Amp has poles when $(\sum p)^2 \to 0$, + Factorizes!



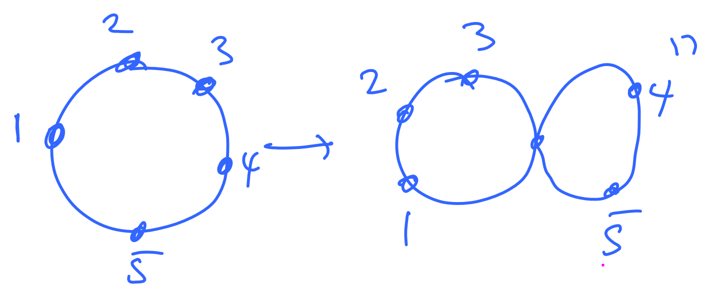
F: "It's



+ … "

Particles in Spacetime

D: "It's



"

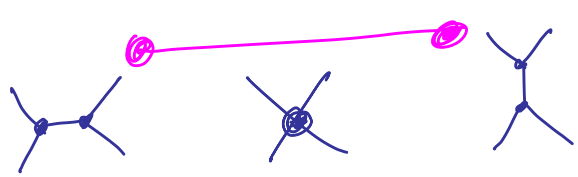
String Worldsheet

Bigger Clue Seen By Lazier but Smarter Experim.

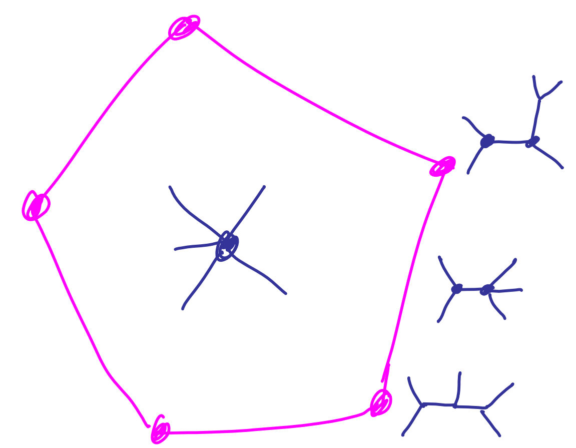
"Only certain patterns of poles come together!

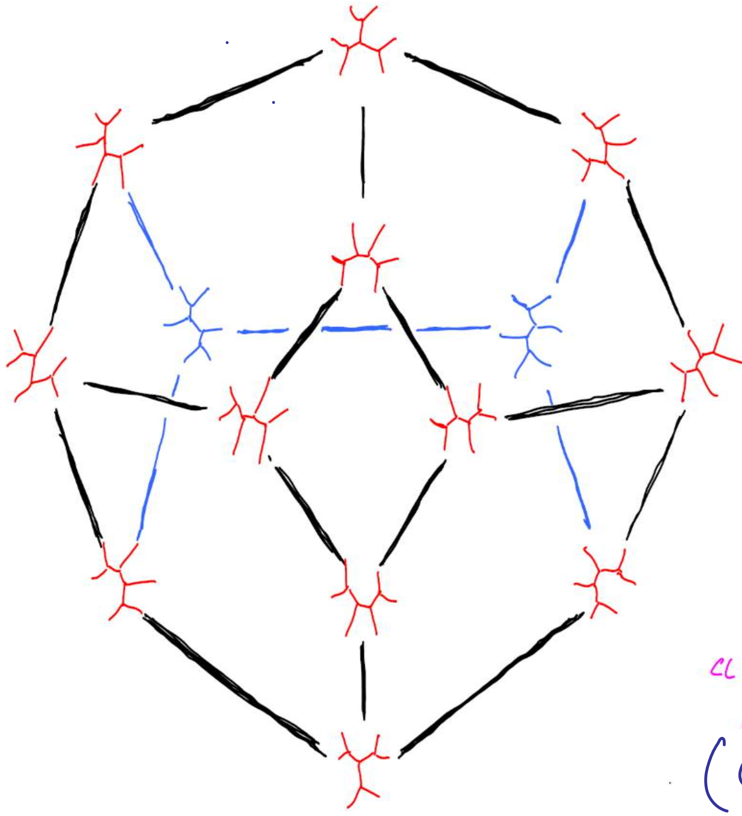
$n = 4$ $\quad s\,,\ t$ but not s, t

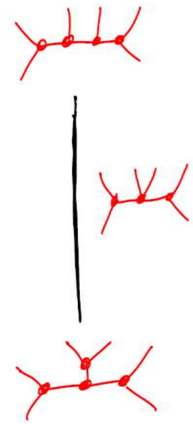
$n = 5$ $\quad s_{12}, s_{123}$ + cyclic but not other comb.



$n = 4$







"Associahedron"
(Combinatorial)

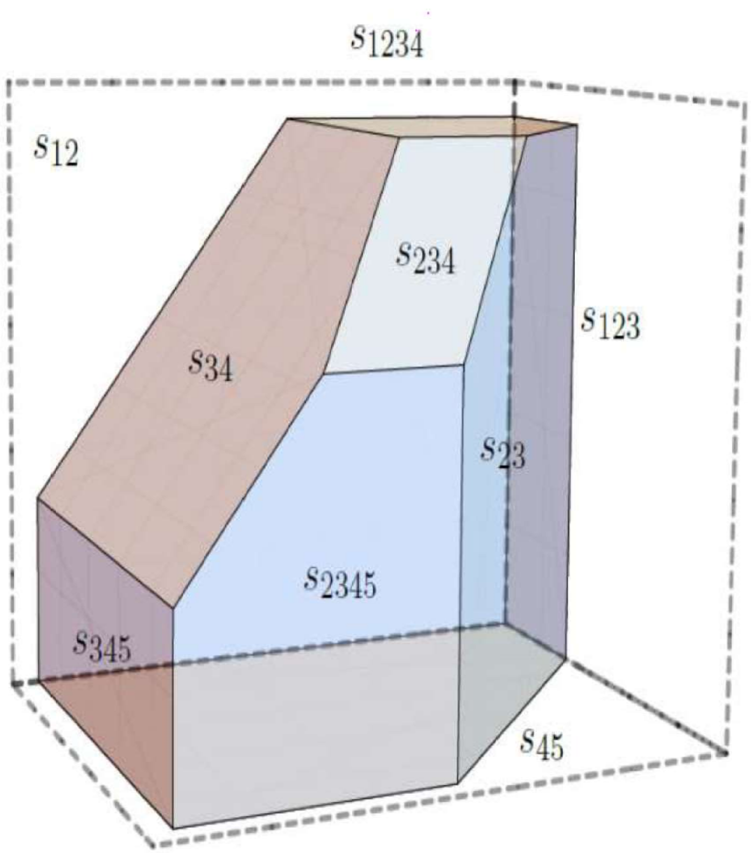
Particle + String picture don't make this obvious!

But Assoc. does see everything since

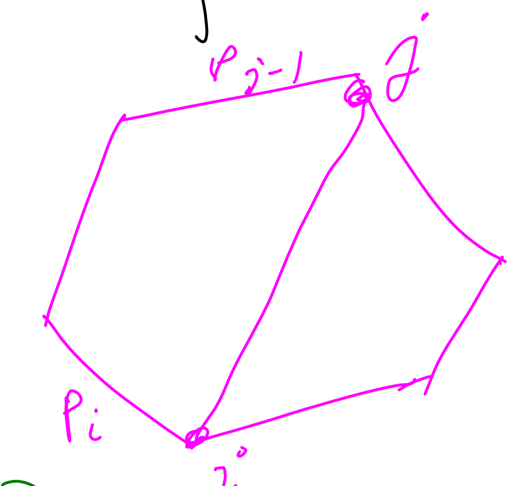
$\partial A_n = A_{n_1} \times A_{n_2}$

HOW CAN WE MAKE THIS OBVIOUS?

A Qualitative Clue to the "theory" in Kinematic Space



ABHY Associahedron in Kinematic Space



$X_{i,j} = (p_i + \dots + p_{j-1})^2$

HOW CAN WE MAKE THIS OBVIOUS?

A Qualitative Clue to the "theory" in Kinematic Space

... Pursuing this question over the last year has led to a striking convergence of several lines of research:

* Positive Geometries Grassmannians/Amplituhedra
* Cluster Algebras + Polytopes
* String Amplitudes
* CHY + Scatt. eqns
* $\mathcal{N}=4$ symbol cluster geometry

together with generalizations of open + closed string/particle amplitudes, as well as practically useful new links to arithmetic geometry

Outline

1. Causal Diamonds + Cluster Polytopes: ABCD's of ϕ^3_{BA} thru 1-loop
2. All-loop "projective invariance" of ϕ^3_{BA} scatt. form
3. Stringy Canonical Forms + SE
4. Binary Positive Cluster Geometry: $\mathcal{U}$ space + "orderings"
5. Cluster generalized string + particle Amplitudes
6. Arithmetic geometry of amp. varieties and Amps over $\mathbb{F}_{p^m}$
7. $G(k,n)/T$; $G(4,n)/T$ + $\mathcal{N}=4$

(2) All-loop "projective invariance" of Φ^3_{BA} scatt. form

↑

Taming "Infinite" Clusters

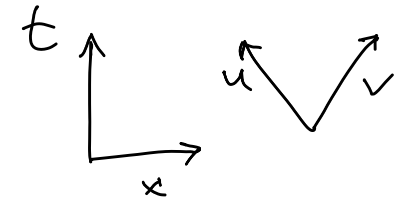
↓

(7) $G(k,n)/T$; $G(4,n)/T$ + $\mathcal{N}=4$

Causal Diamonds, Cluster Polytopes + ϕ^3_{BA} Amplitudes

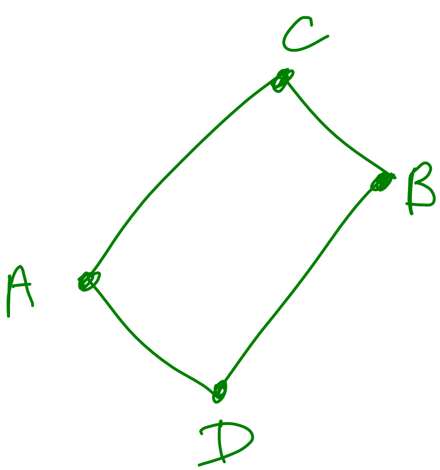
w/ Song He
Hugh Thomas
Giulio Salvatori

Wave Equation in (1+1)-d



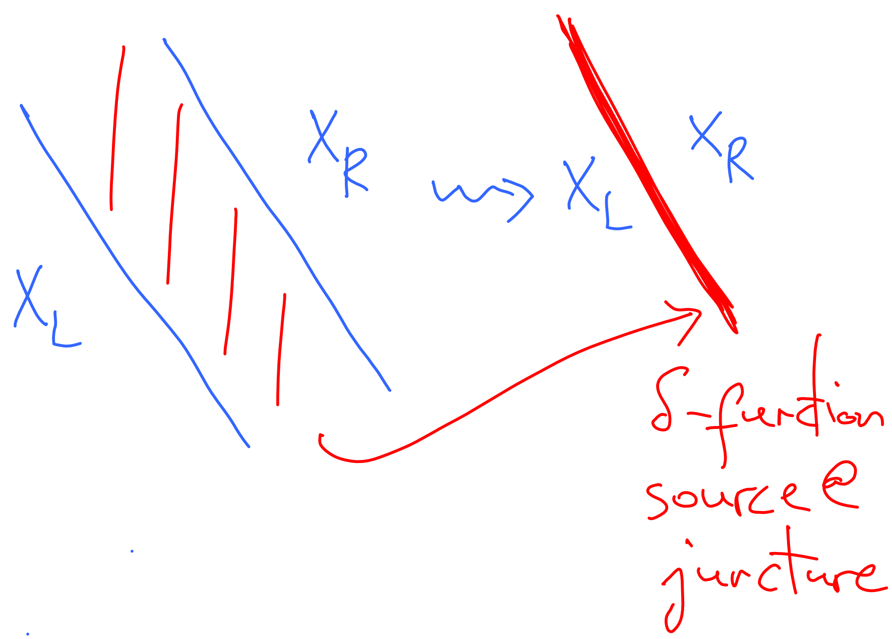
$(\partial_t^2 - \partial_x^2) X = j \; ; \; \partial_u \partial_v X = j$

Gauss Law for Causal Diamonds

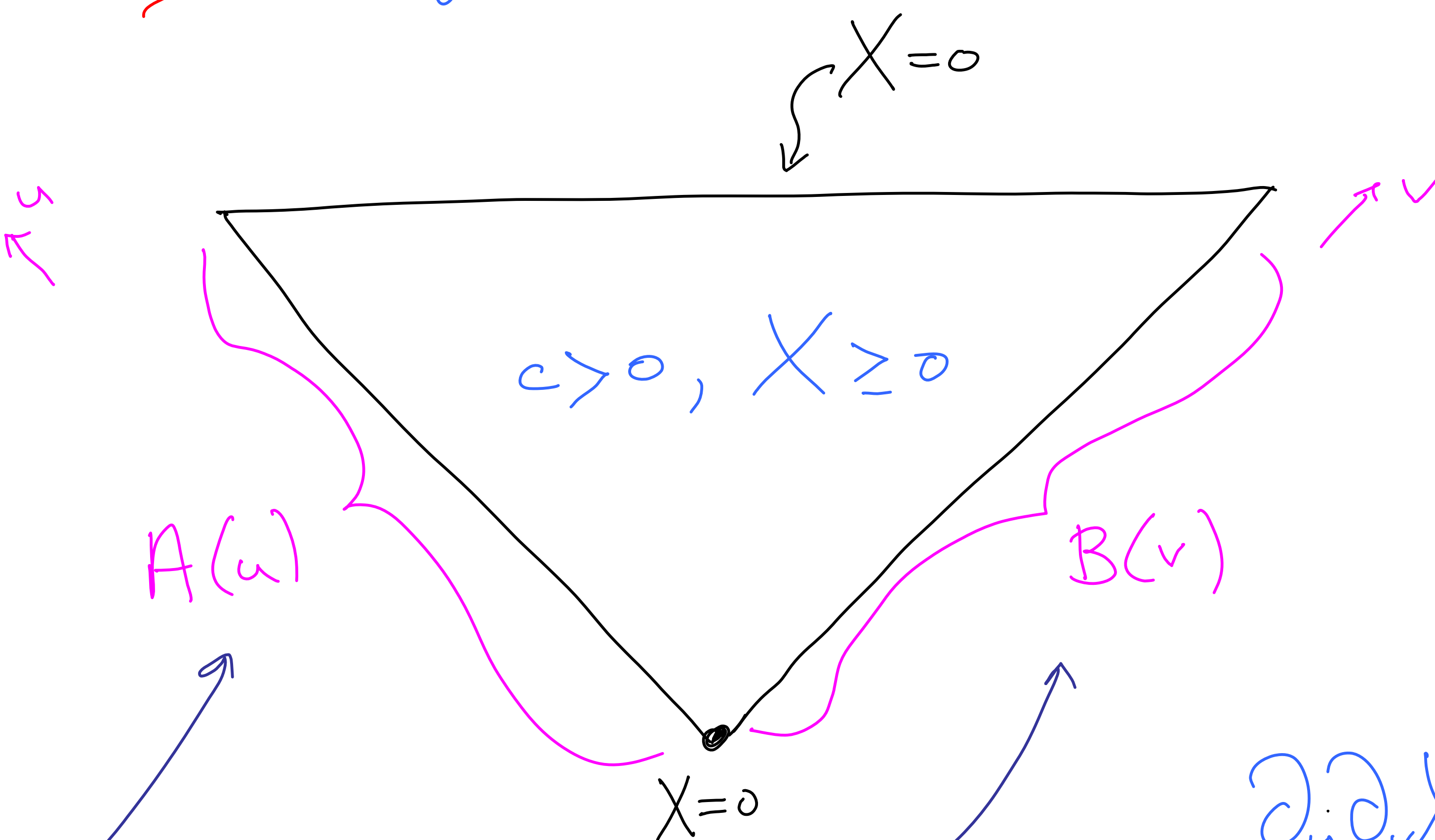


$A + B - C - D = J$

"Scrunching"

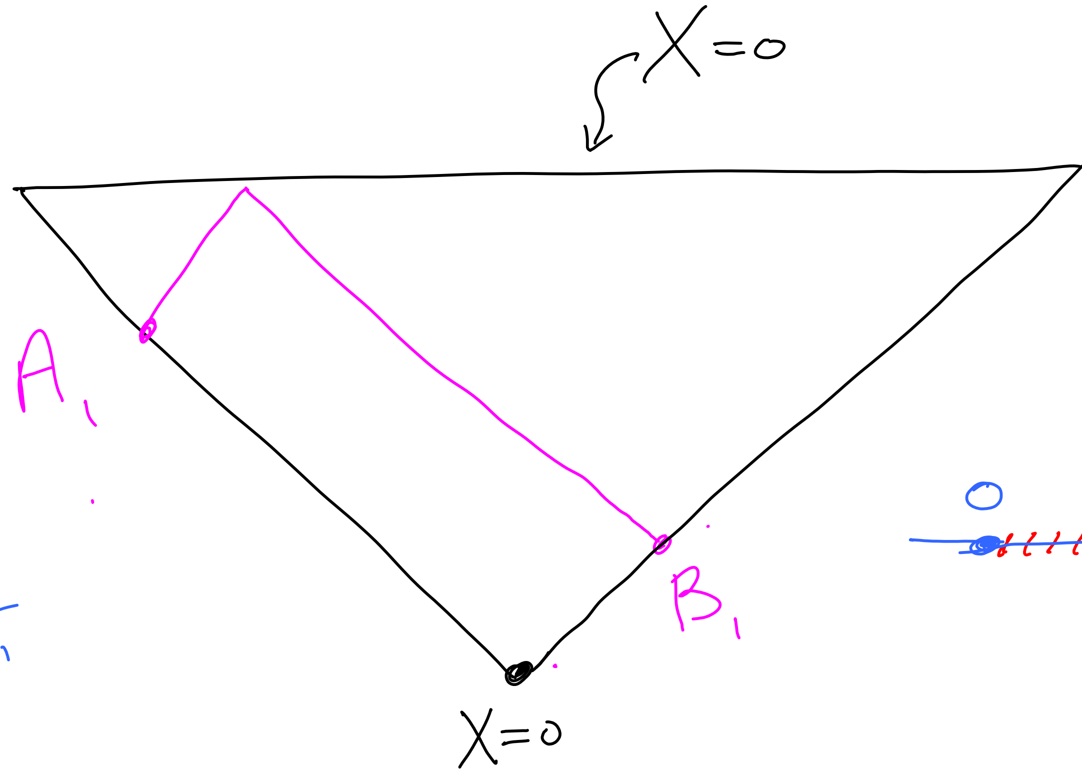


Positivity + Wave Equation

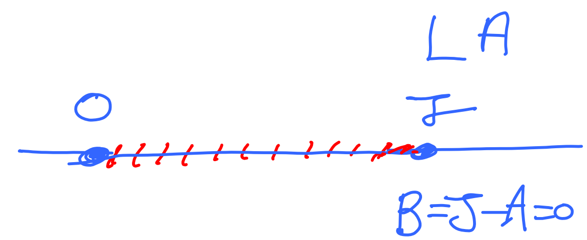


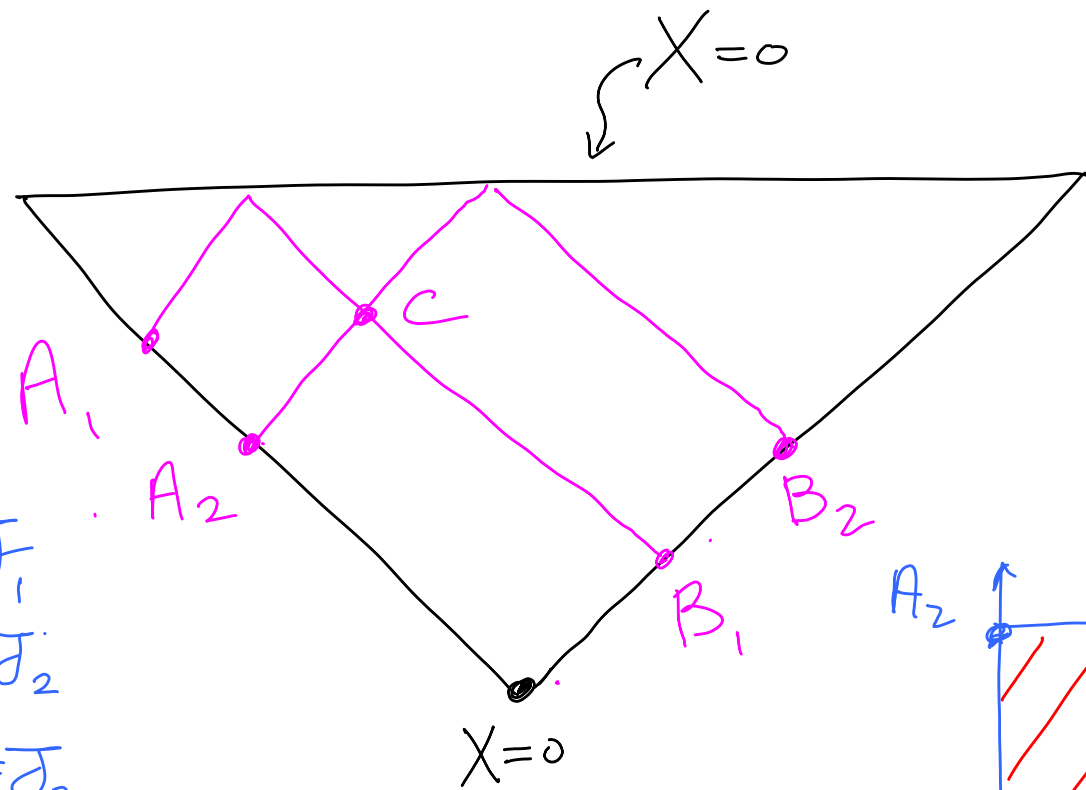
$\partial_u \partial_v X = c$

What ∂ $A(u), B(v)$ guarantee $X \geq 0$ for $c>0$?



$A_1 + B_1 = J_1$

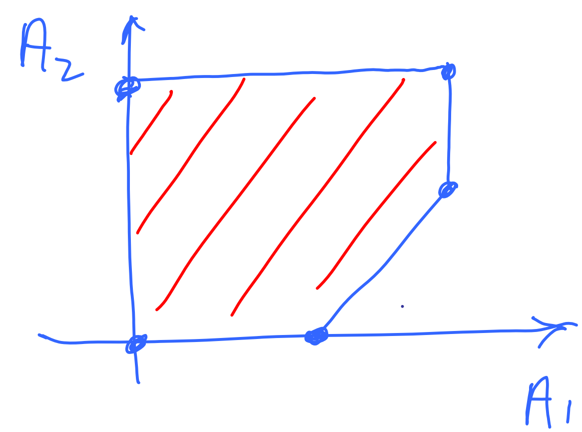




$A_1 + C - A_2 = J_1$

$C + B_2 - B_1 = J_2$

$A_2 + B_1 - C = J_3$



$A_1 + C - A_2 = J_1$

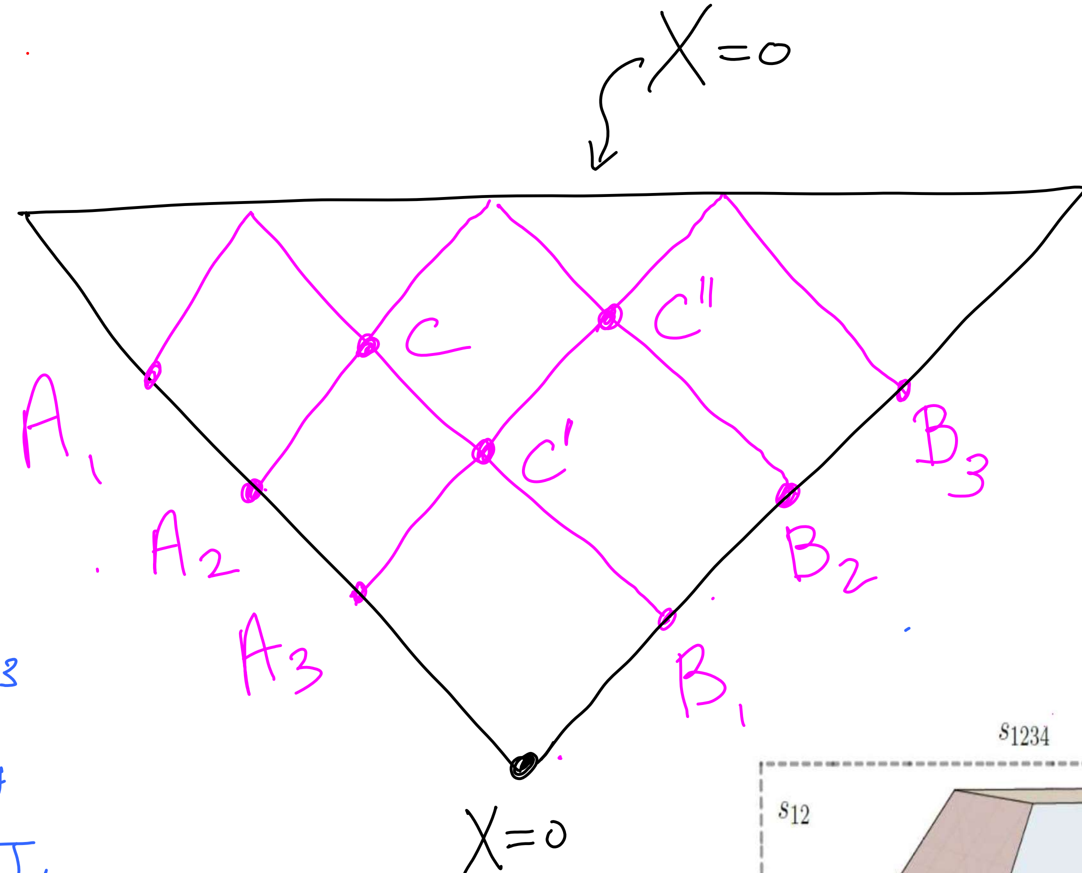
$C + C'' - C' = J_2$

$C'' + B_3 - B_2 = J_3$

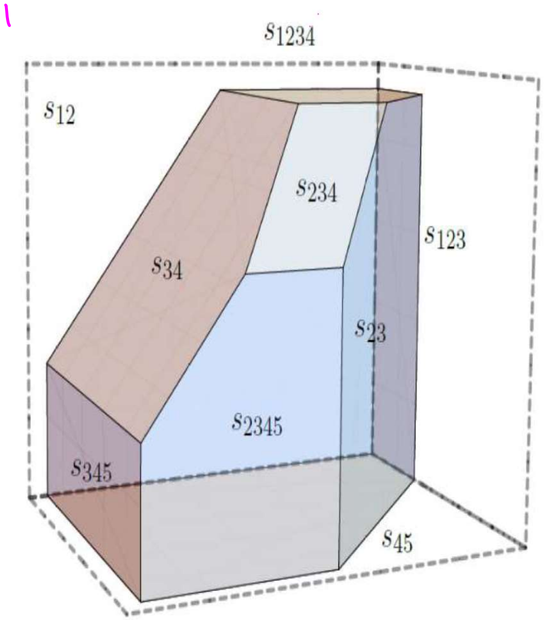
$A_2 + C' - C = J_4$

$C' + B_2 - C'' = J_5$

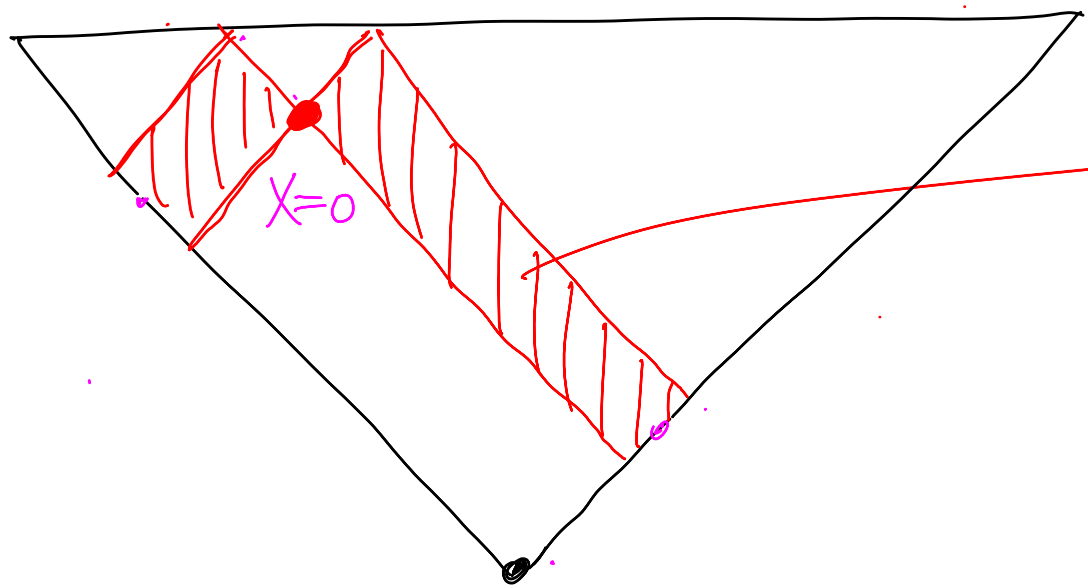
$A_3 + B_1 - C' = J_6$



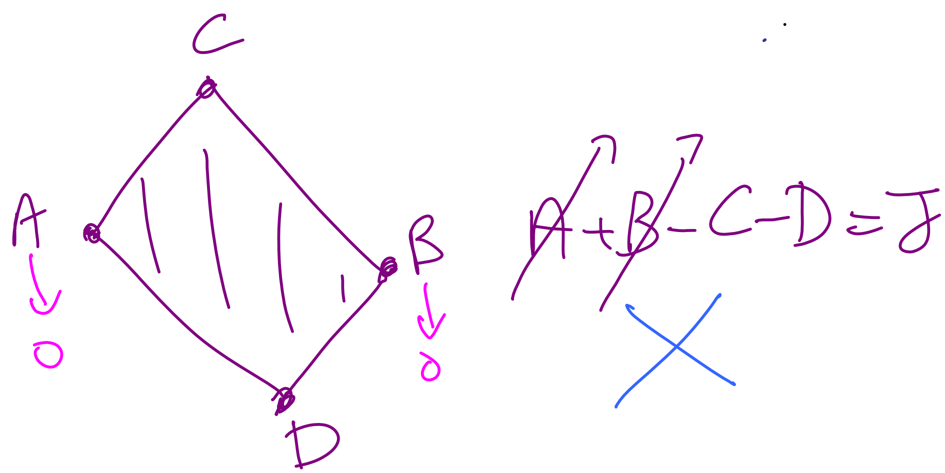
ABHY Assoc!



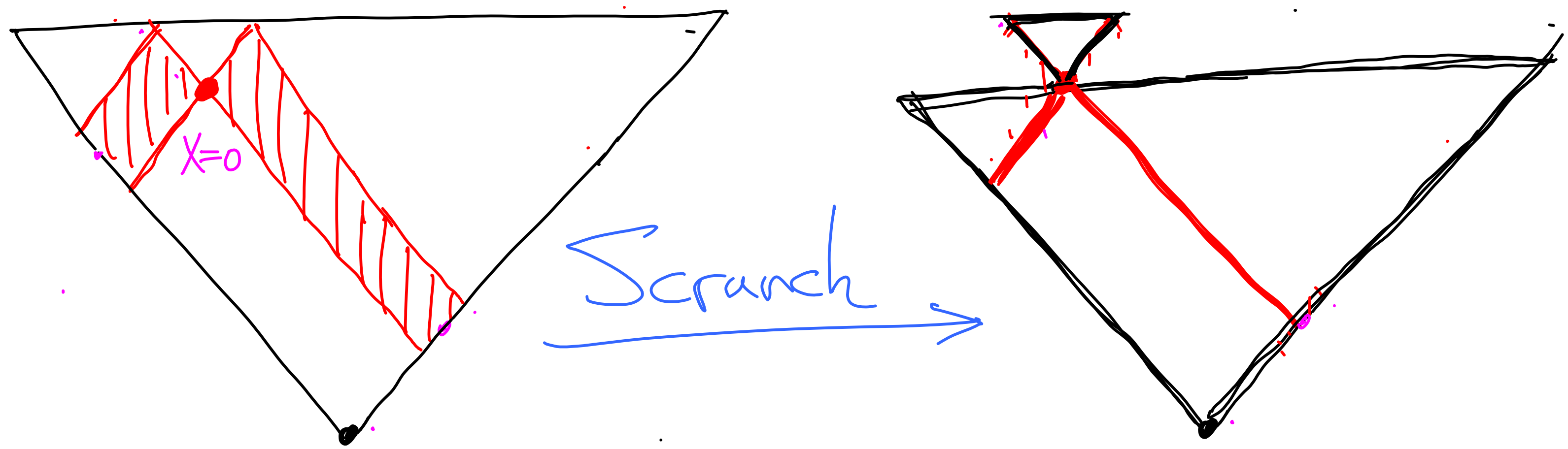
Boundary Structure



Can't put $X \to 0$ in spacelike separated but causally connected regions!

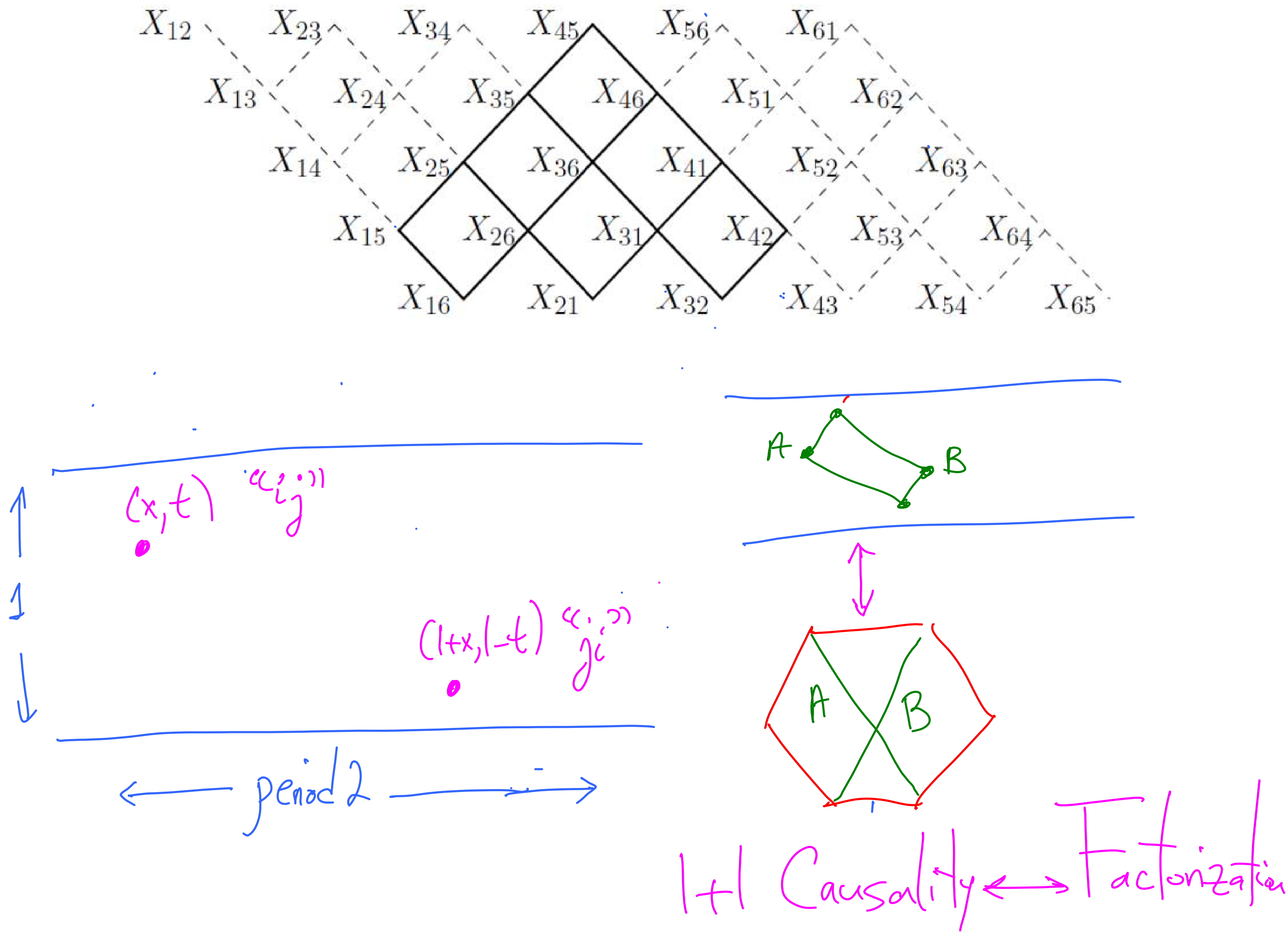


Factorization is Obvious!



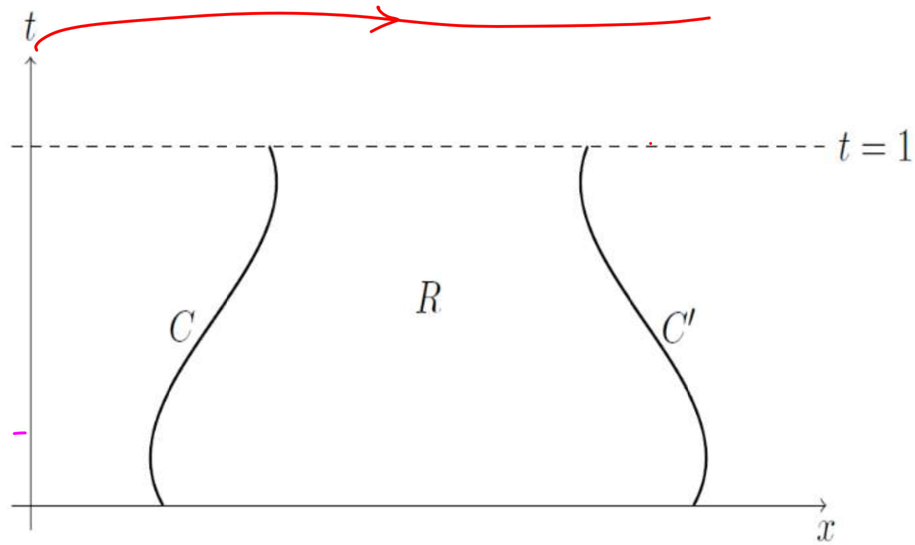
Polytopal Structure AND Factorization is Manifest!

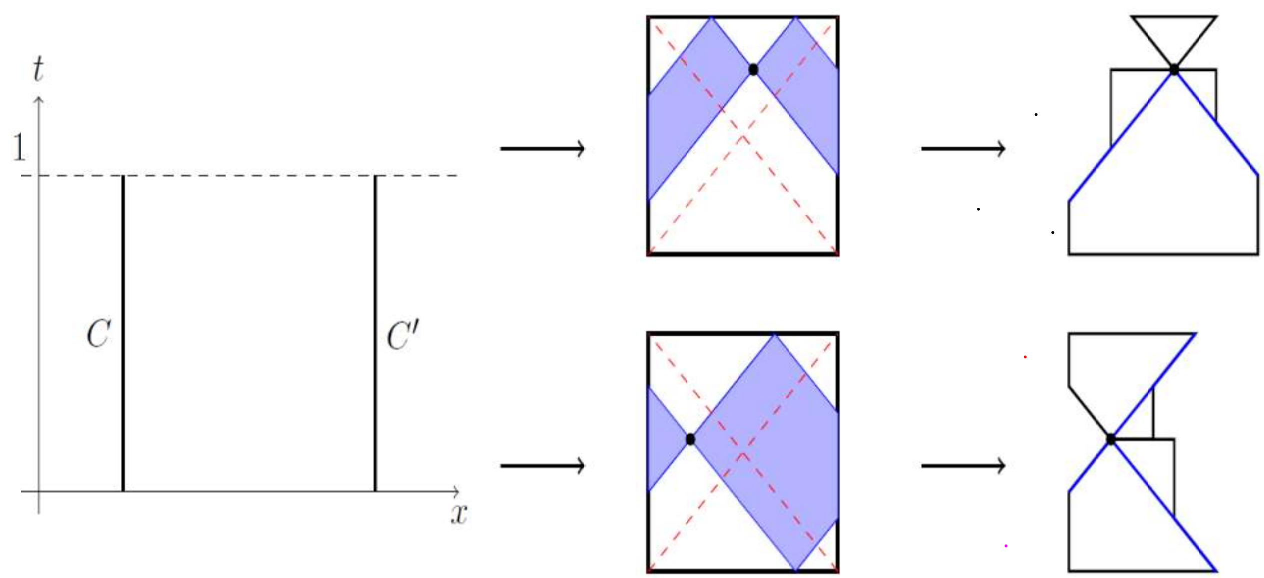
(Contrast w/ Particle or String Picture)



X_{12} X_{23} X_{34} X_{45} X_{56} X_{61}
X_{13} X_{24} X_{35} X_{46} X_{51} X_{62}
X_{14} X_{25} X_{36} X_{41} X_{52} X_{63}
X_{15} X_{26} X_{31} X_{42} X_{53} X_{64}
X_{16} X_{21} X_{32} X_{43} X_{54} X_{65}
(x,t) "i"
(1+x,1-t) "i"
1
period 2
A
B
A
B
1+1 Causality ⟷ Factorization

General Meshes





Note: Polytope makes Projective Invariance of canonical f-m manifest:

$\Omega[X]$ no pole @ ∞, invariance under $X_{ij} \to f(X)\, X_{ij}$!

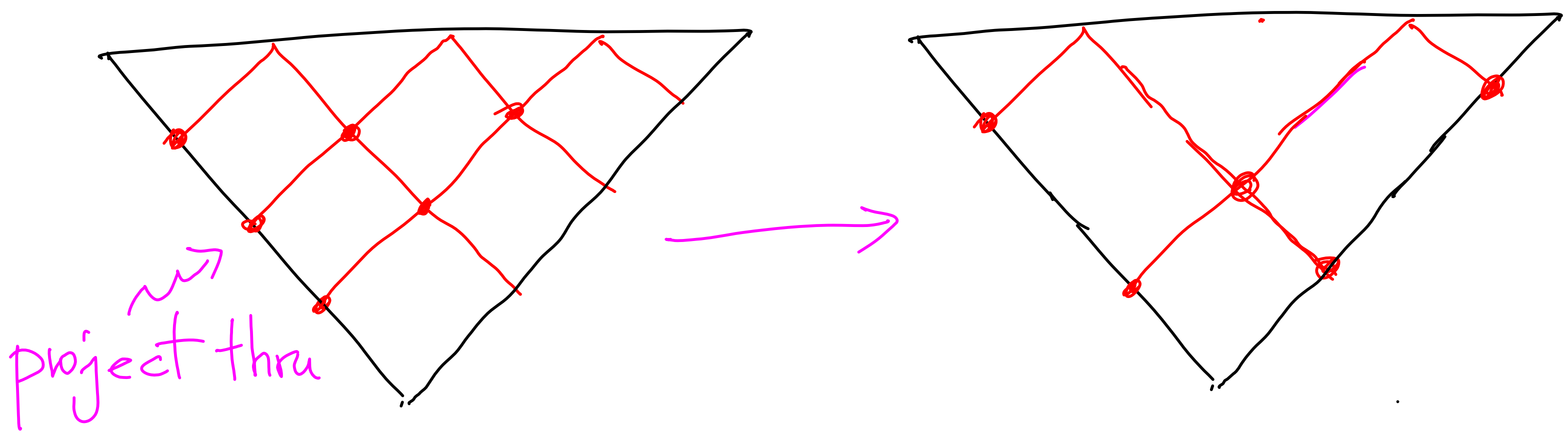
Feynman Diagram Triang: $\Omega = \sum_{\text{diag.}} \pm \prod_{ij} d\log X_{ij}$

doesn't manifest this.

Proj inv ∴ ϕ^3_{BA} as Dual Conf Symm : $\mathcal{N}=4$ SYM

Hidden Symmetry of ϕ^3_{BA}

Adding/Forgetting + New Recursion Relation



$$A_{n+1} \xrightarrow{\text{1-d Fiber}} A_n$$

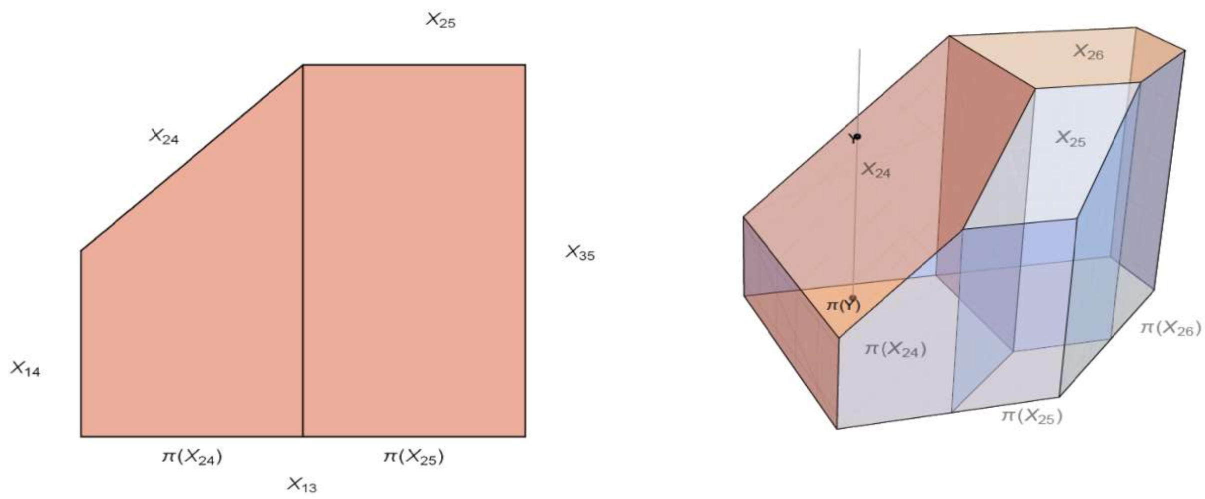
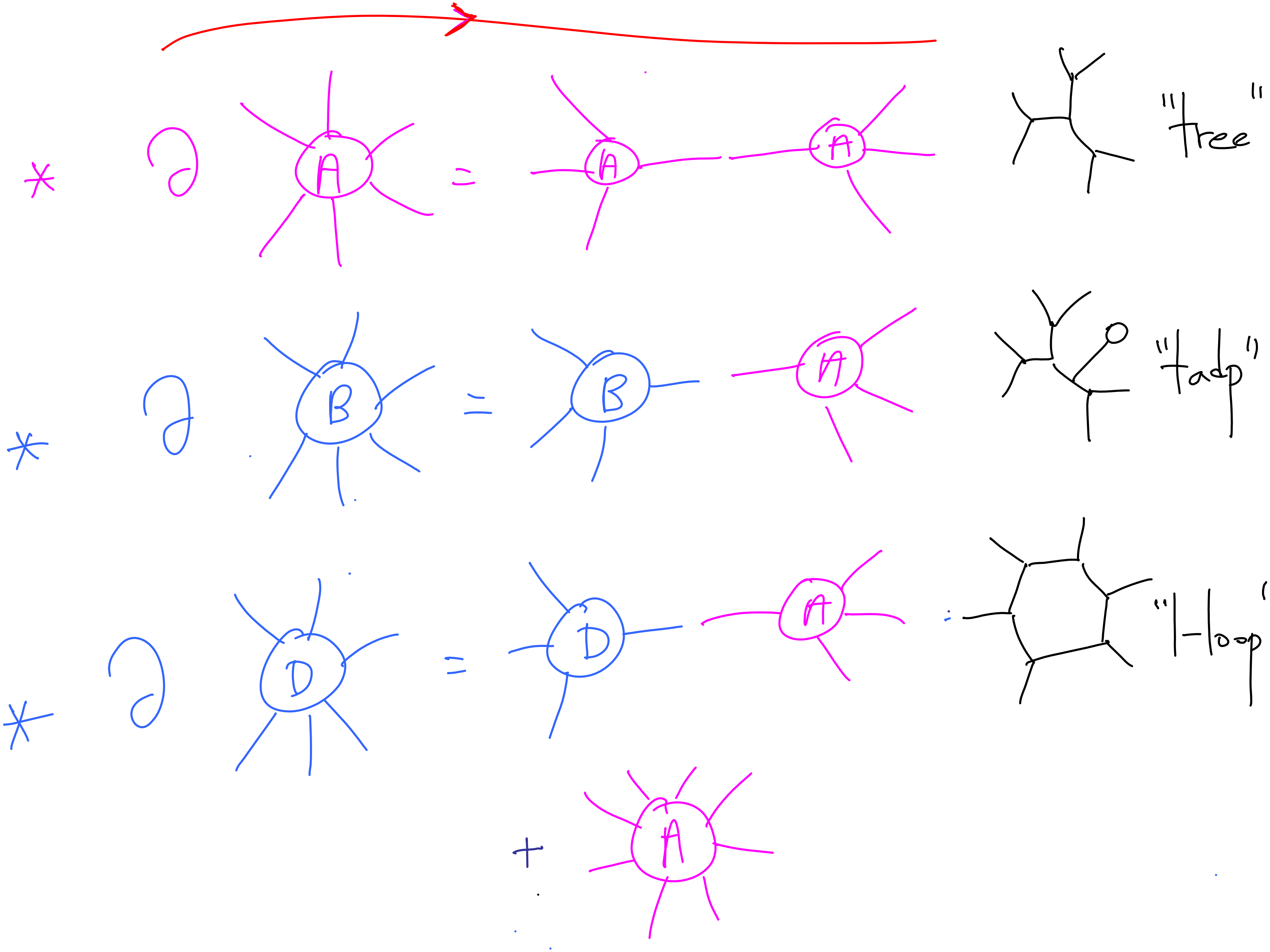


Figure 11: The *soft limit* triangulation for $n = 5$ (left) and $n = 6$ (right). The projection is done through a point at infinity in the y direction and z direction, respectively.

$$A_5 = \left(\frac{1}{X_{1,4}} + \frac{1}{c_{1,4} - X_{1,4}}\right)\left(\frac{1}{X_{1,3}} + \frac{1}{c_{1,3} - X_{1,3} + X_{1,4}}\right) + \left(\frac{1}{c_{1,4} + c_{2,4} - X_{1,4}} + \frac{1}{X_{1,4} - c_{1,4}}\right)\left(\frac{1}{X_{1,3}} + \frac{1}{c_{1,3} + c_{1,4} - X_{1,3}}\right).$$

General n : $(n-3)$ term recursion!!

Other Factorizations



Other (1+1) Meshes

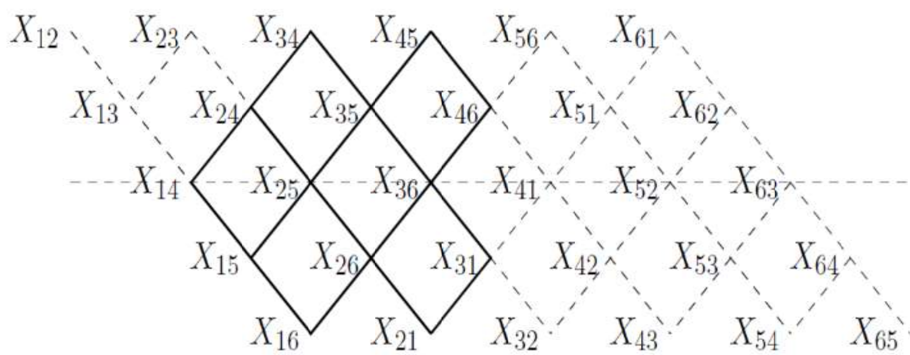
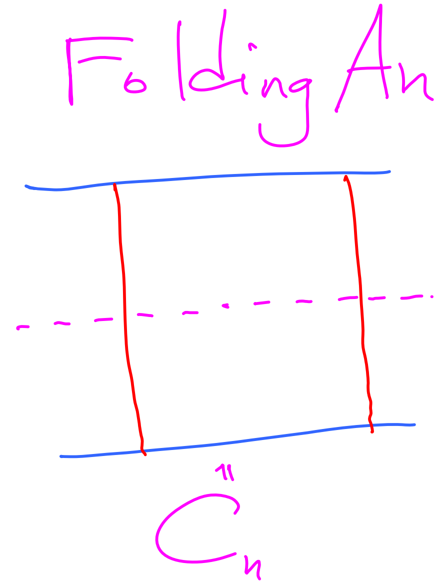
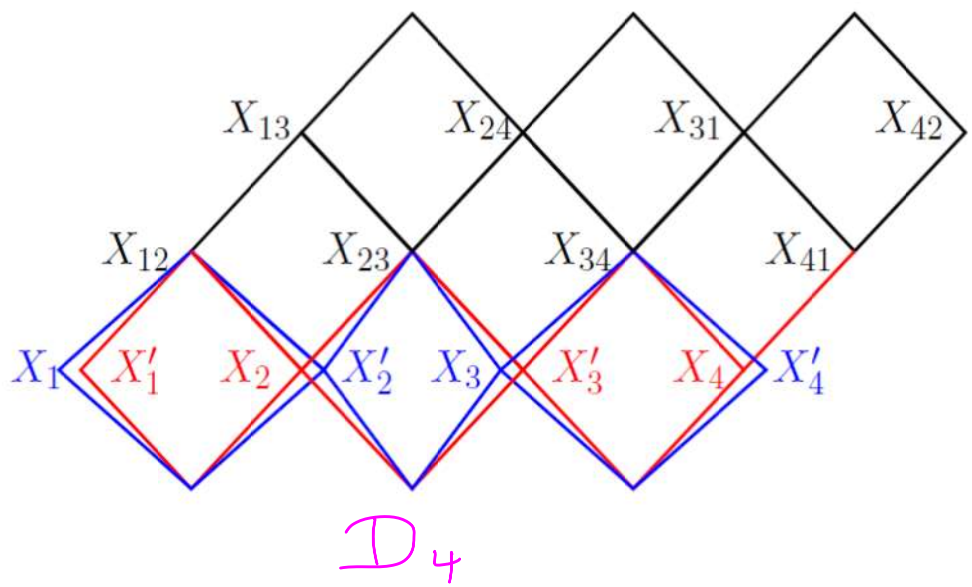


Figure 14: The mesh diagram of $\mathcal{C}_2$ from folding a centrally-symmetric one of $\mathcal{A}_3$.





= Dn

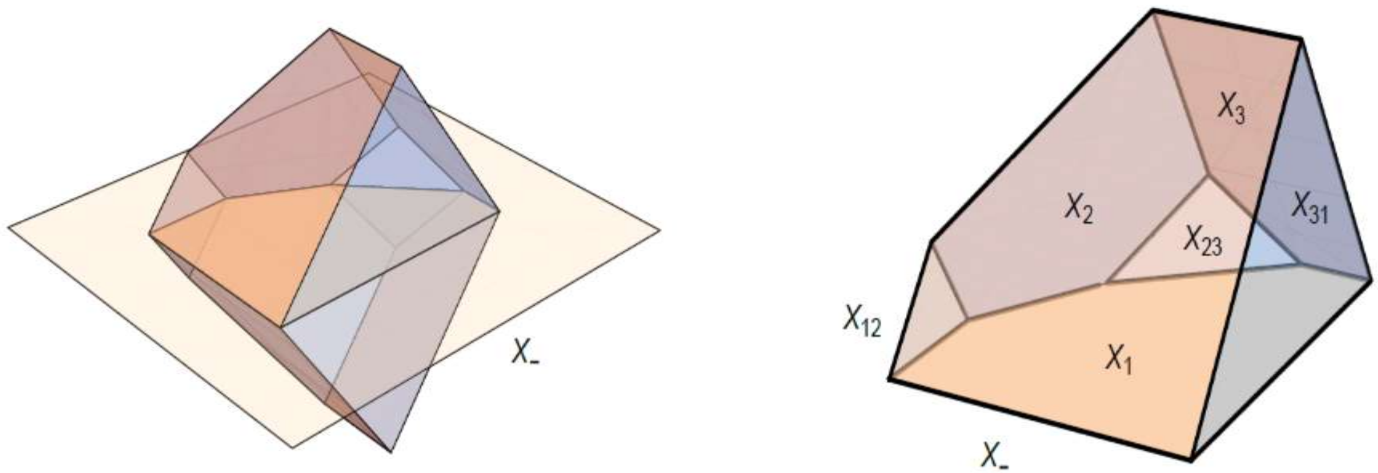
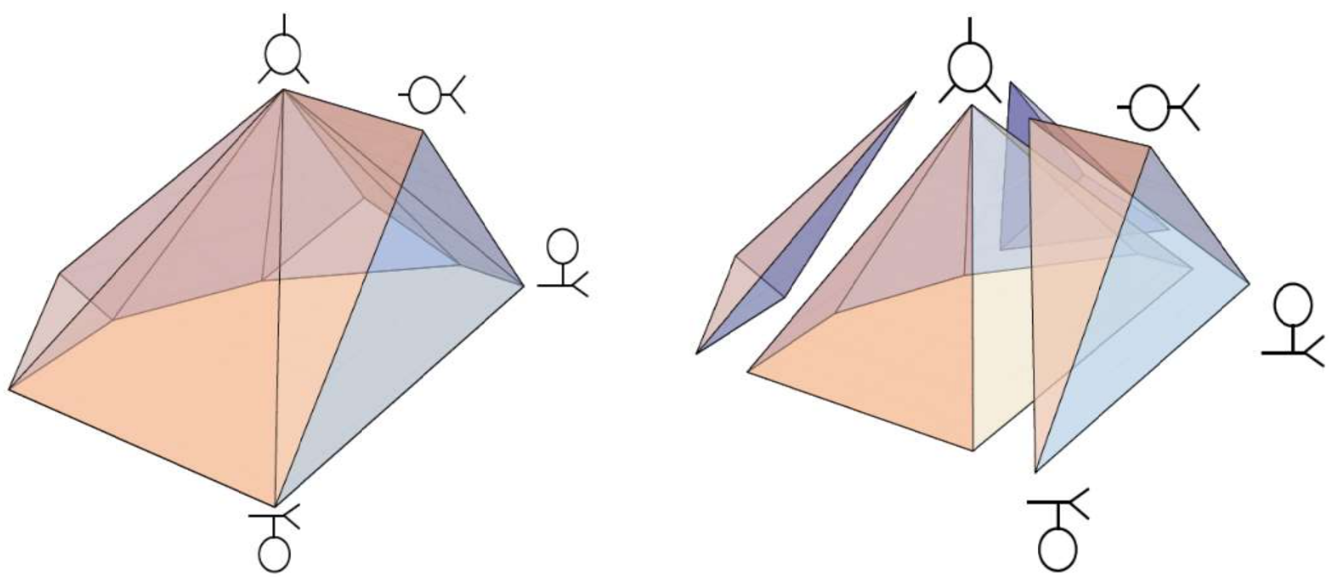
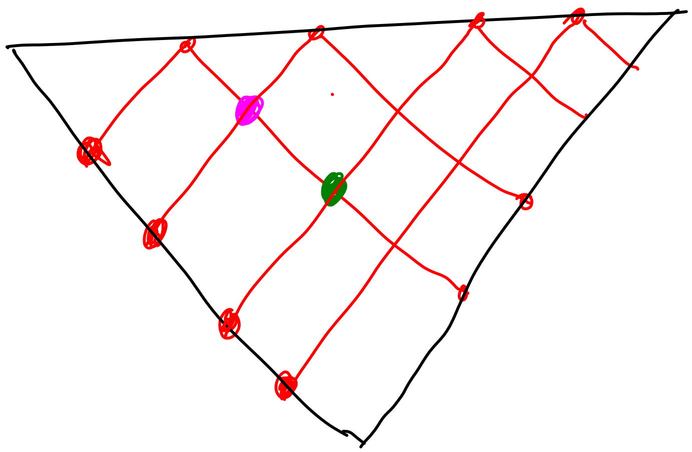


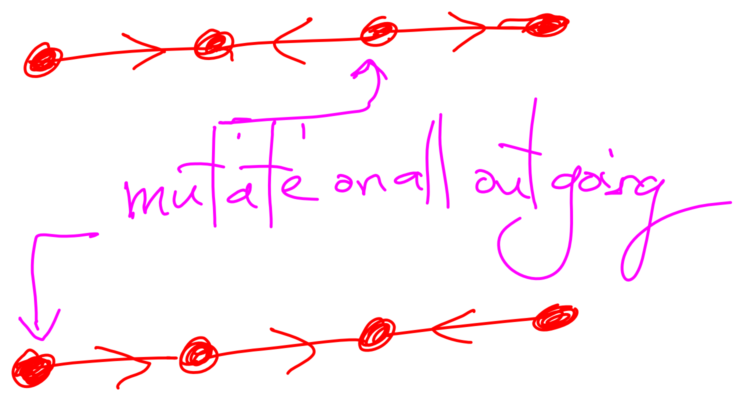
Figure 20: Slicing $\mathcal{D}_3$ with the tadpole plane produces two copies of $\overline{\mathcal{D}}_3$



Abstract Away Rules

Walk thru Spacetime $\longrightarrow$ Walk through Quivers





$$X_v' + X_v = \sum_{v' \to v} X_{v'} + c$$

Can Define For Any Tree Quiver

$Q \xrightarrow[@ i]{\text{mutate}} Q' \Longrightarrow X' = A_i X$

A_i generators of Coexeter Group of Q!

ABHY Polytope Finite ⟶ Coexeter Grp Finite

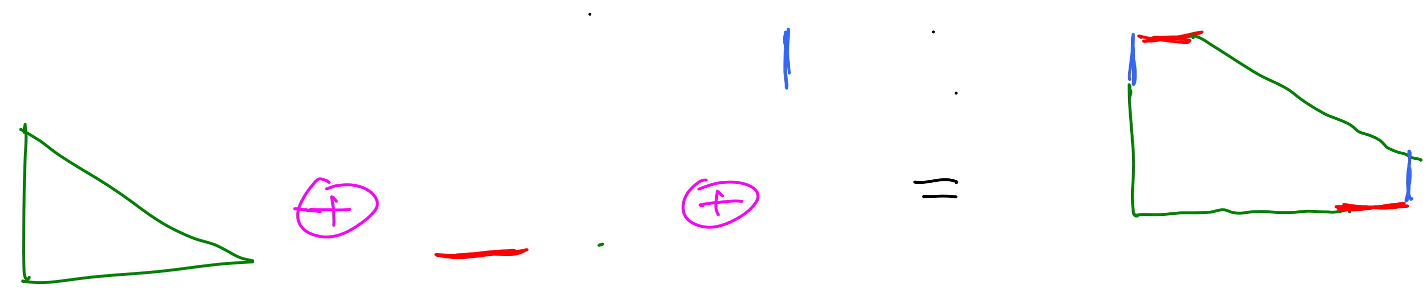
⟶ Dynkin Classification ⟶ ABHY Real.

of all Finite Cluster Algebra Polytopes,

à la Hugh Thomas + students!

Note for Later

All the Polytopes are naturally given as a "Minkowski sum" $P = c_1 P_1 \oplus \cdots \oplus c_m P_m$ where each P_i is the polytope given by setting a mesh constants other than $c_i \to 0$, and $c_i \to 1$.

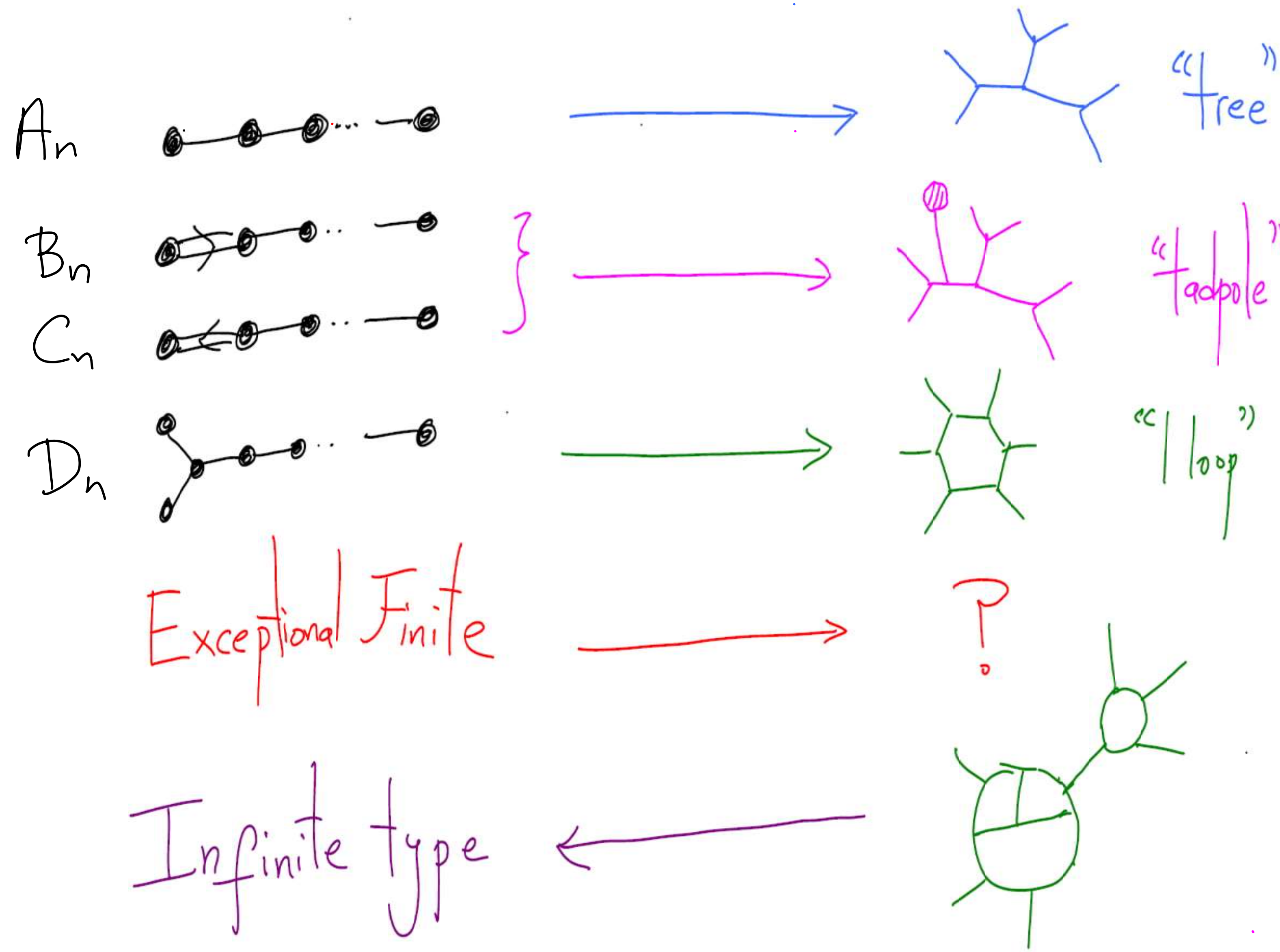


Summary

* (1+1)-d causal meshes give us a simple origin of "amplituhedra" for ϕ^3_{BA} thru 1-loop, as ABHT realizations of ABCD Cluster Polytopes à la Thomas et. al.

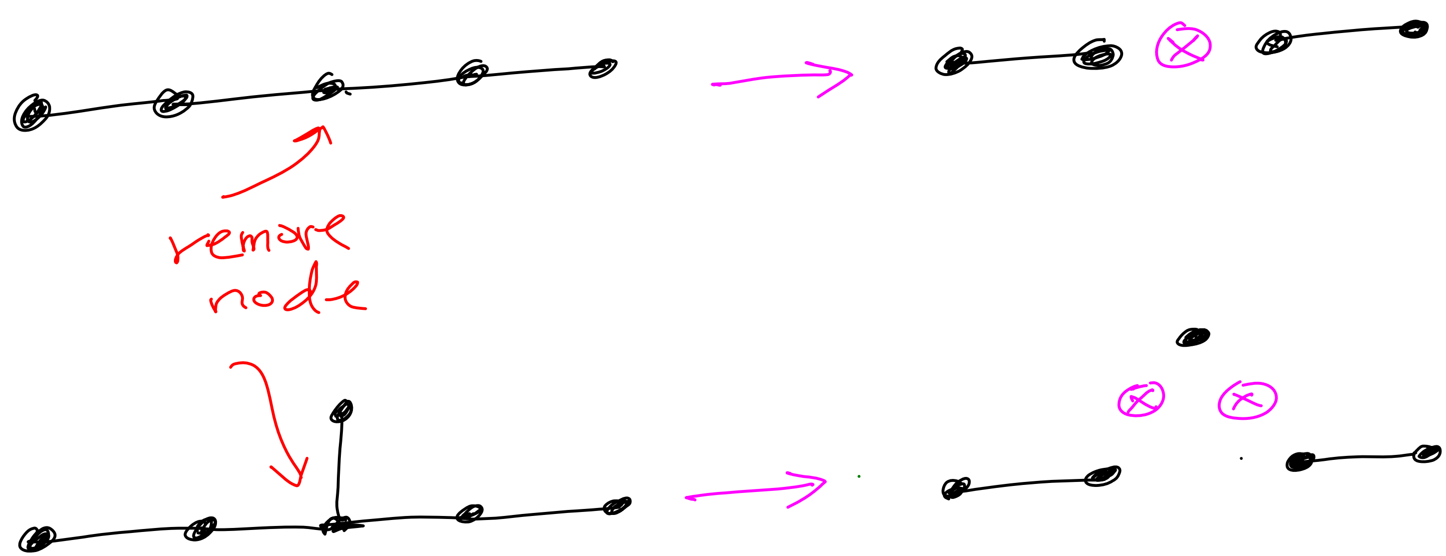
* Hidden symmetries exposed by geometry led to new recursion relations thru 1-loop, even for dumb-old ϕ^3_{BA}!

* What is this (1+1) structure? Some Lorentzian W.S. in disguise?



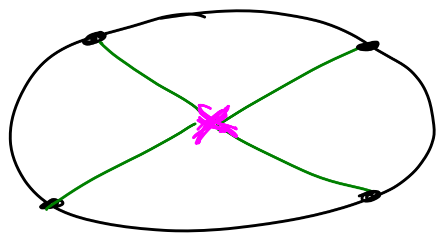
A_n
B_n
C_n
D_n
"tree"
"tadpole"
"loop"
Exceptional Finite
?
Infinite type

Note: all these cluster polytopes + "amplitudes" Factorize on facets/poles.

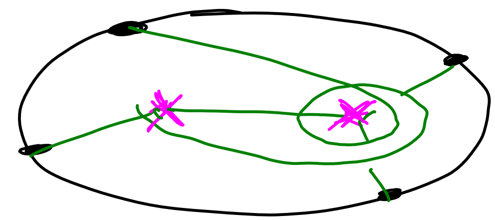


"Generalized Particle Amplitudes"

All-Loop Scatt. Form



All finite quiver-type, but →



"is "dumbly" infinite due to "windings""

* Highly non-trivial claim (w/ H. Frost, G. Salvatori)

Take $\Omega = \prod d\log X$'s, mutate to fix signs, then identify X's that differ "dumbly" ⟹

Form is projectively invariant + finite!

* Hidden symmetry of ϕ^3_{BA} @ all loops!

* What is the Polytope?

Stringy Canonical Forms

w/ Song He

Thomas Lam

Newton Polytopes + Pushforward Map

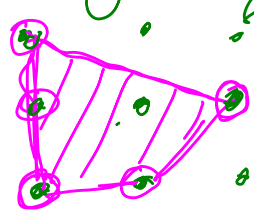
$$P = \sum a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

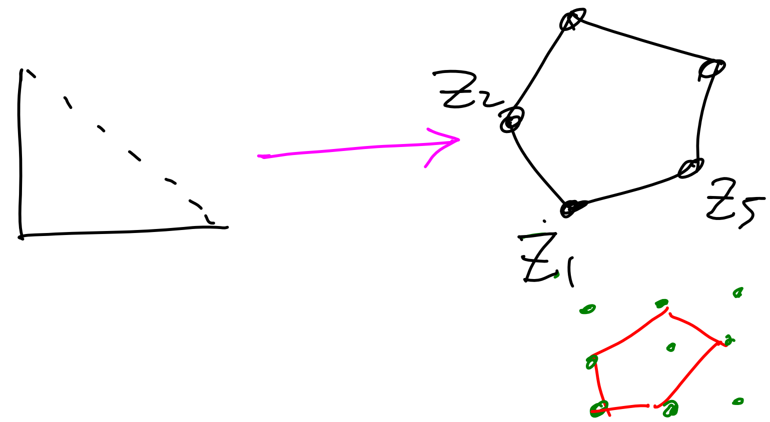
$$\equiv \sum_v a_v x^v$$

$\mathbb{P}$: convex hull of all v's

$P : 3 + x + 2y + x^2 y + y^3$

$\mathbb{P}$:





$$Y^I = \sum_v (x^{v_i}) Z_i^I$$

$$\frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \xrightarrow[\text{forward}]{\text{Push}} \Omega^{can}(P)$$

(w/ Bai, Lam)

Stringy Canonical Form

$$I^{\alpha'}[P;X,c] \equiv \alpha'^{n} \int_0^\infty \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \left(\prod_i x_i\right)^{\alpha' X_i} P(x)^{-\alpha' c}$$

$$\Omega^{\alpha'}[P;X,c] \equiv I^{\alpha'}\, dX_1 \cdots dX_n$$

I converges iff $\mathcal{P}$ is top-dimensional and $X = (X_1, \ldots, X_n)$ lies inside $c\mathcal{P}$.

- As $\alpha' \to 0$, $\Omega^{\alpha' \to 0} = \Omega^{can}[X, c\mathcal{P}]$,
- For general α', meromorphic, self-factorizing!

Pushforward + "Scattering Equations"/"CHY"

"Gross-Mende": critical points as $\alpha' \to \infty$ of

$$\int_0^\infty \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \left(\prod_i x_i\right)^{\alpha' X_i} P(x)^{-\alpha' C}$$

Precisely gives Newton Polytope map from $\{x\}$ into $\mathcal{P}$ in X space!

Pushforward gives $\Omega^{can}[\mathcal{P}]$, which is $\alpha' \to 0$ limit of I!

Trivial Generalizations

$$I = \alpha'^n \int \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \left(\prod_i x_i^{\alpha' X_i} \right) P_1^{-\alpha' c_1} \cdots P_d^{-\alpha' c_d}$$

Stringy canonical form / "Scatt eqn" / CHT for $c_1 \mathbb{P}_1 \oplus c_2 \mathbb{P}_2 \oplus \cdots \oplus c_d \mathbb{P}_d$ (Minkowski Sum)

{ "Closed String" gen. also easy + natural }

$\alpha' \to 0$: Can. Form, Pushforward via S.E.

$\alpha' \neq 0$ self-factorizes; TOTALLY GENERAL PHENOMENA

"Cluster" String Amplitudes: "Binary" + V. Special

* A_n realized as ABHY, Minkowski sum:
  stringy canonical form = K-N string integral!
* Strings from comb. geometry in kin-sp; no standard W.S.!
* For general ABHY realized cluster polytopes: stringy can. form = "string amps" for all cluster algebras

Stringy Magic Generalizes

* All meromorphic
* Remarkably all factorize on "massless poles" even @ finite α'!
* "Channel duality"
* Exp. soft in "UV"
* BCD give 1-loop ϕ^3_{BA} as $\alpha' \to 0$

What is physical int. at finite α'?

* In general, stringy canonical forms define various compactifications of interesting spaces

General Cluster "$G(k,n)/T$"

"mom cons." for T inv.

$$\int \underbrace{\left(\frac{dx_i}{x_i}\right)}_{\text{one cluster}} \prod_i A_i^{\alpha' a_i}$$

$$\int \left(\frac{dx_i}{x_i}\right) \prod (i_1 \cdots i_k)^{\alpha' a_{i_1 \cdots i_k}}$$

(see also Freddy et al.)

Full Cluster

D_4 $[G(3,6)]$ $(1, 50, 100, 66, 16, 1)$

E_6 $[G(3,7)]$ $(1, 833, 2499, 2856, 1547, 399, 42, 1)$

E_8 $[G(3,8)]$ $(1, 25080, 100320, 163856, 140448, 67488, 17936, 2408, 128, 1)$

"$G(k,n)/T$"

$G(3,6)/T$ $(1, 48, 98, 66, 16, 1)$

$G(3,7)/T$ $(1, 693, 2163, 2583, 1463, 392, 42, 1)$

$G(3,8)/T$ $(1, 13612, 57768, 100852, 93104, 48544, 14088, 2072, 120, 1)$

Different Compactifications of "$G_+(k,n)/T$"

Summary

* There is a "stringy canonical form" for any (rationally realizable) polytope P; $\Omega^{\alpha'}[P]$

* Simple conceptual understanding of "CHY/Scatt eqn" pushforward map for any P — this has nothing to do with strings per se.

* Especially nice "string amplitudes" for all finite-type cluster algebras.

* (Something special about usual SE: $(n-3)!$ is tiniest # of solutions needed!)

Binary Positive Cluster Geometry

and

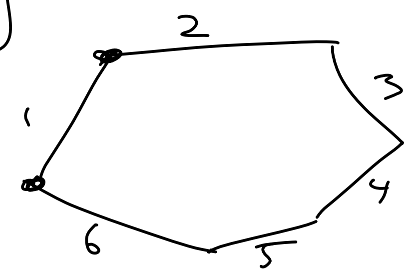
Generalized Cluster String Integrals

w/ Song He
Thomas Lam
Hugh Thomas

Binary Positive Geometry

* Polytopes still "floppy" realization of combinatorics

$W_1 \to 0$,

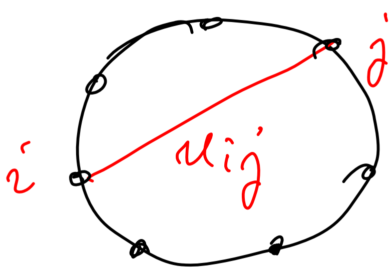


$W_{3,4,5}$ can't $\to 0$, but vary along 1

* Binary, curvy geometry possible?

$$u_\alpha + \prod_{\beta \text{ not on } \alpha} u_\beta = 1 \, ?$$

Amazingly, (conj) only Associahedra!



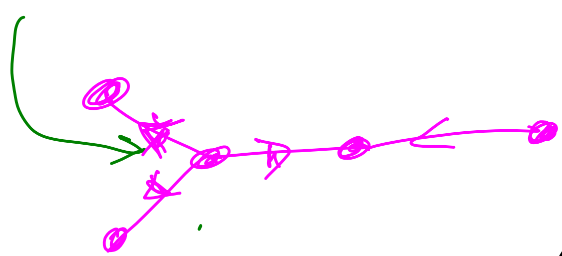
$$u_{ij} + \prod_{\substack{kl \\ \text{crossing } ij}} u_{kl} = 1$$

Binary Cluster Geometry

$$u_\alpha + \prod_\beta u_\beta^{n_{\alpha|\beta}} = 1$$

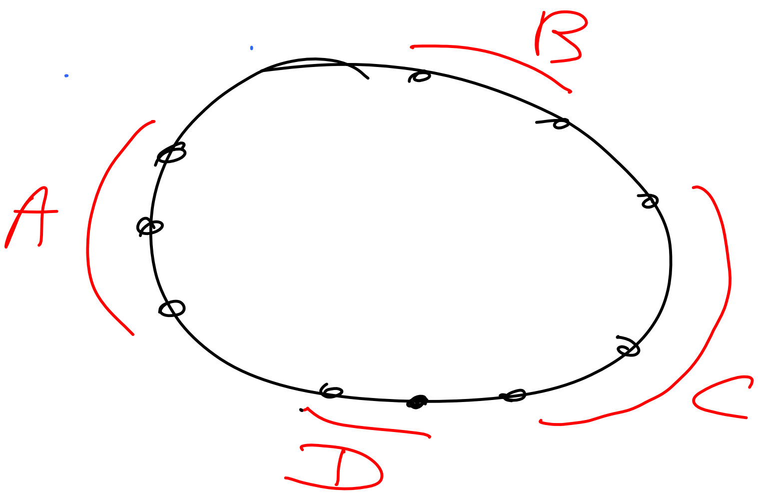
$n_{\alpha|\beta} = 0, 1, 2, 3$

* $u = \dfrac{x}{1+x}$ for special x cluster var encountered in "walk through clusters"



* As many u's as A var, + multiplicatively independent!

A_n: from $u + uu \cdots u = 1$, we derive more generally



$U_{AC} + U_{BD} = 1$

$\Downarrow$

u's are cross-ratios!

$\Downarrow$

G.I. des. of W.S.

Generalizes to all Dynkins

"Orderings"

Apparently, u_{ij}'s need an ordering...

But look at all possible sign patterns of $\mho + \mho$ equations $\Longrightarrow$ $(n-1)!/2$ of them $\longrightarrow$ + equations have an action of S_n on them under sign change!

Real $\mho$ space sees all orderings!

* All this carries over to general finite type: $U + U$ equation sign patterns define other "orderings", but new shapes appear! E.g.

B_2/C_2: 16 orderings; 4 Hexagons, 12 Pent.

G_2: 25 orderings; 1 Octagon, 4 Hex. 20 Pent.

Generalized "open" + "closed" string in U language

e.g. $$I^{open} = \int_{u_i > 0} \omega_{can} \cdot \prod_i u_i^{\alpha' \sigma_i}$$

Also natural origin (+ gen.) of u's from general stringy can form....

But "binary" is special, generically get

$$u + u \cdots u = \text{"}P/Q\text{"}$$ where $P = Q$ but not $= 1$

$$G(k,n)/T + \mathcal{N} = 4$$

w/ Thomas Lam

Mark Spradlin

* Space determining $\mathcal{N}=4$ integrand:

Amplituhedron in mom-twistor space

$G_k(m=4,n)$

"winding/flip".

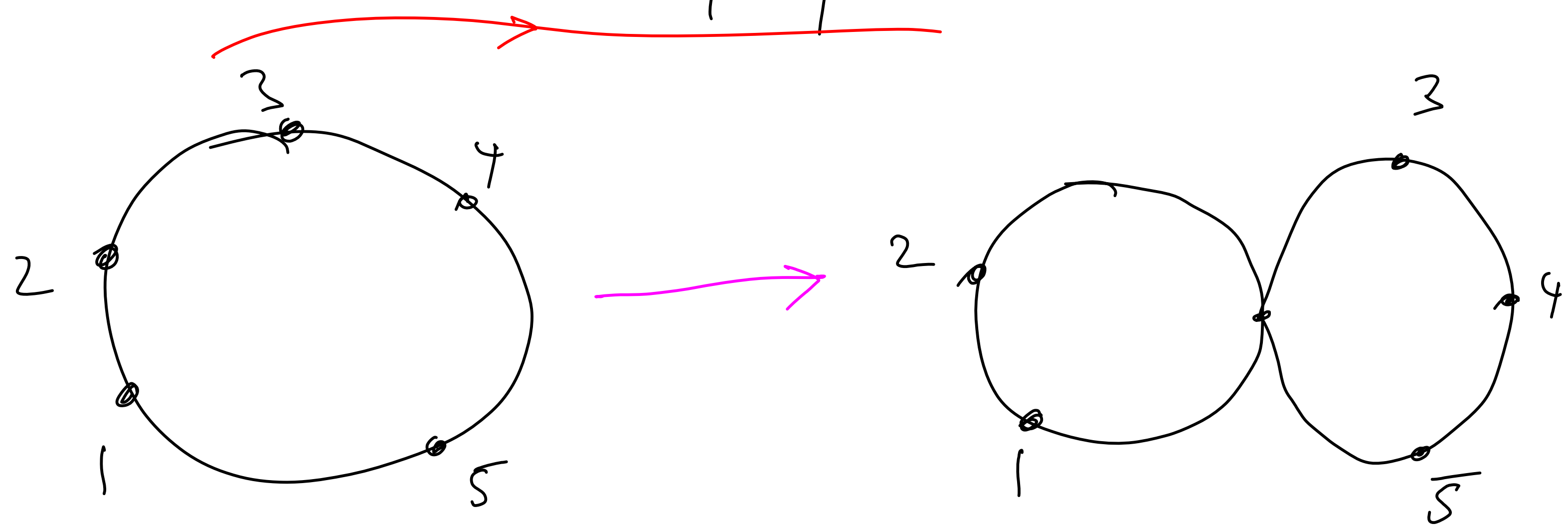
* Natural guess for Amplitude itself

$G_k(m=4,n)/T$ = Little Group

* Already mysterious for $k=0$ (MHV)!

* Supposedly $G_+(4,n)/T$ is just the "cluster $\mathcal{X}$ variety of $G_+(4,n)$", but this is infinite already for $n = 8$! But of course $G_+(4,n)$ itself is not infinite! What's going on?

$G_+(2,n)/T$



"Bubbling Picture"

What is the natural gen. of this?

* Consider a point in $G(k,n)$, and look @ its minors in $\mathbb{P}^{\binom{n}{k}-1}$ plucker space.

* Now little group scale all columns $c_i \to t_i c_i$
+ look at the entire orbit for $t_i \in (0,\infty)$.

* By Newton-Polytopology, the image projects to a Hypersimplex

Hypersimplex

Consider an n-dimensional vector of 0's + 1's with exactly k 1's. Conv. hull of all these guys is **Hypersimplex**

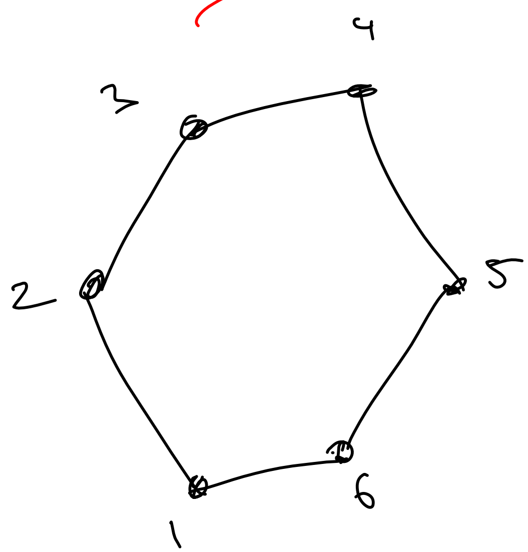
Positroid Polytope

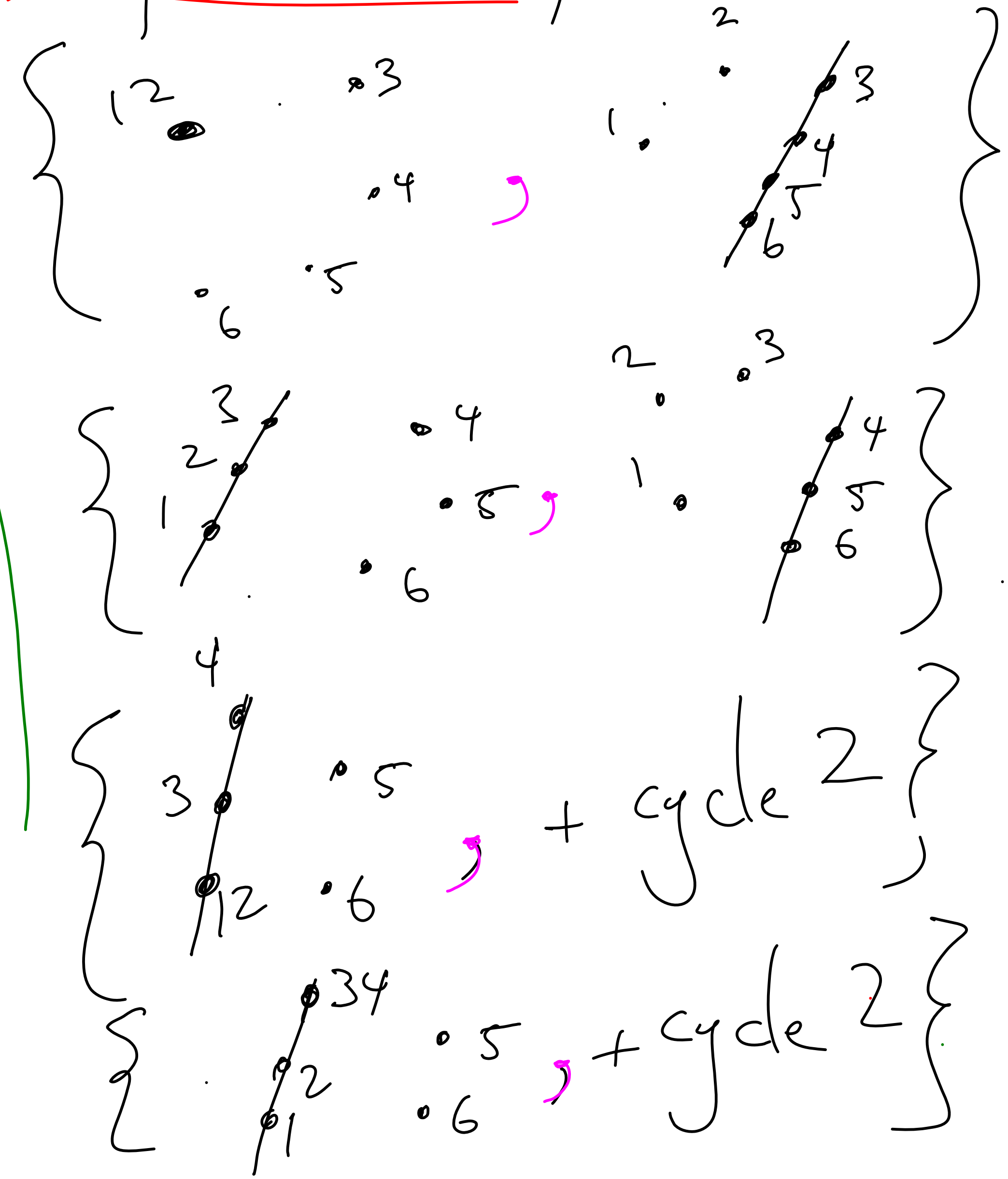
Take any cell Γ of $G_+(k,n)$; only some minors are non-vanishing. Take (0,1) vectors with 1's in columns of non-vanishing minors. Conv. hull of these is **Positroid Polytope of Γ**.

Boundaries of $G_+(k,n)/T$

* Just before you hit a bndy, torus orbit = Hypersimplex
* When you hit a bndy — "triangulate" Hypersimplex into non-overlapping positroid polytopes! General "bubbling" picture

Boundaries of $G_+(3,6)/T$





* This definition of $G(k,n)/T$ is also what we get from the stringy canonical form already mentioned, with

$$\mathcal{I} = \int \frac{d^{k\times n} C / GL(k)\times T}{(12\ldots k)\cdots(n1\ldots k-1)} \prod (i_1\ldots i_k)^{\alpha' a_{i_1\ldots i_k}}$$

Choose your favorite pos. co-ordinates + do Minkowski sum of Newton Polytopes of $(i_1\ldots i_k)$ to get the polytope

* This is manifestly finite, though big!
* $G_+(4,8)/T$ has 360 faces, 90608 vertices!
* Is this the right space for $\mathcal{N}=4$ amps? Still investigating...
* Note: even though $G_+(4,8)$ is parity invariant, $G_+(4,8)/T$ subtly breaks parity

* There are even smaller, manifestly parity invariant compactifications of $G_+(4,n)/T$, e.g.

$$I = \int \omega \prod_{i,j} (i\, i{+}1\, j\, j{+}1)^{\alpha' a_{ij}} \, (\bar{i}\, j)^{\alpha' b_{ij}}$$

For $n=7$, $(595, 1918, 2373, 1393, 385, 42)$

vs. $G_+(4,7)/T$, $(693, 2163, 2583, 1463, 392, 42)$

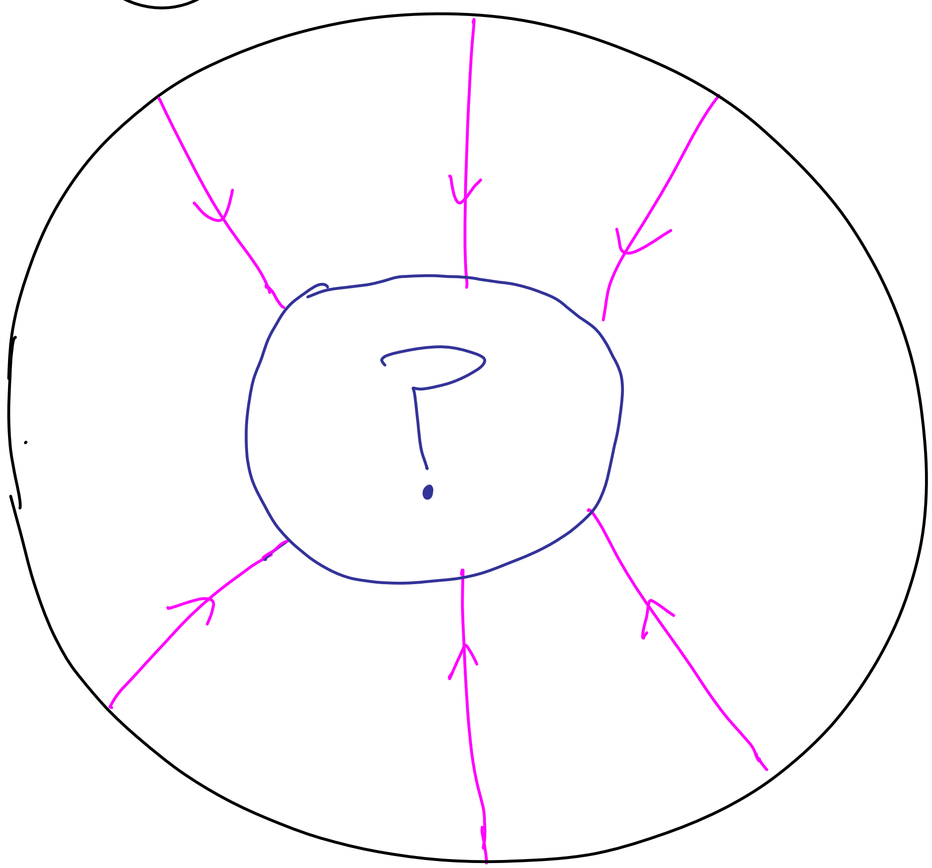
* Clearly there are a number of ways of "taming the ∞" here –
but which if any are right?

* Beyond guess-work + pattern recognition

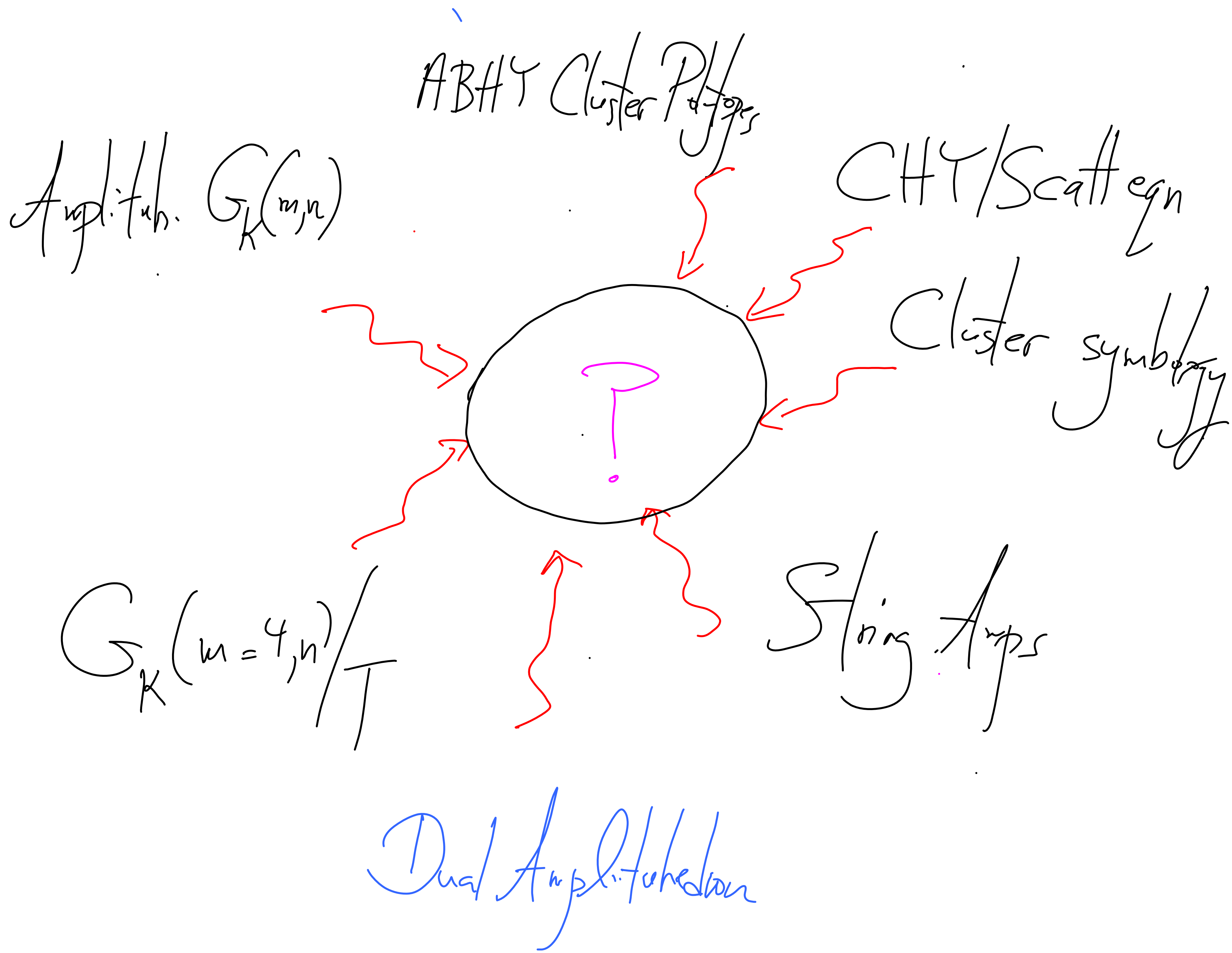
– understand the <u>miraculous</u> link between $G_+(4,n)$ positivity of data + <u>absence</u> of solutions to Landau eqns!

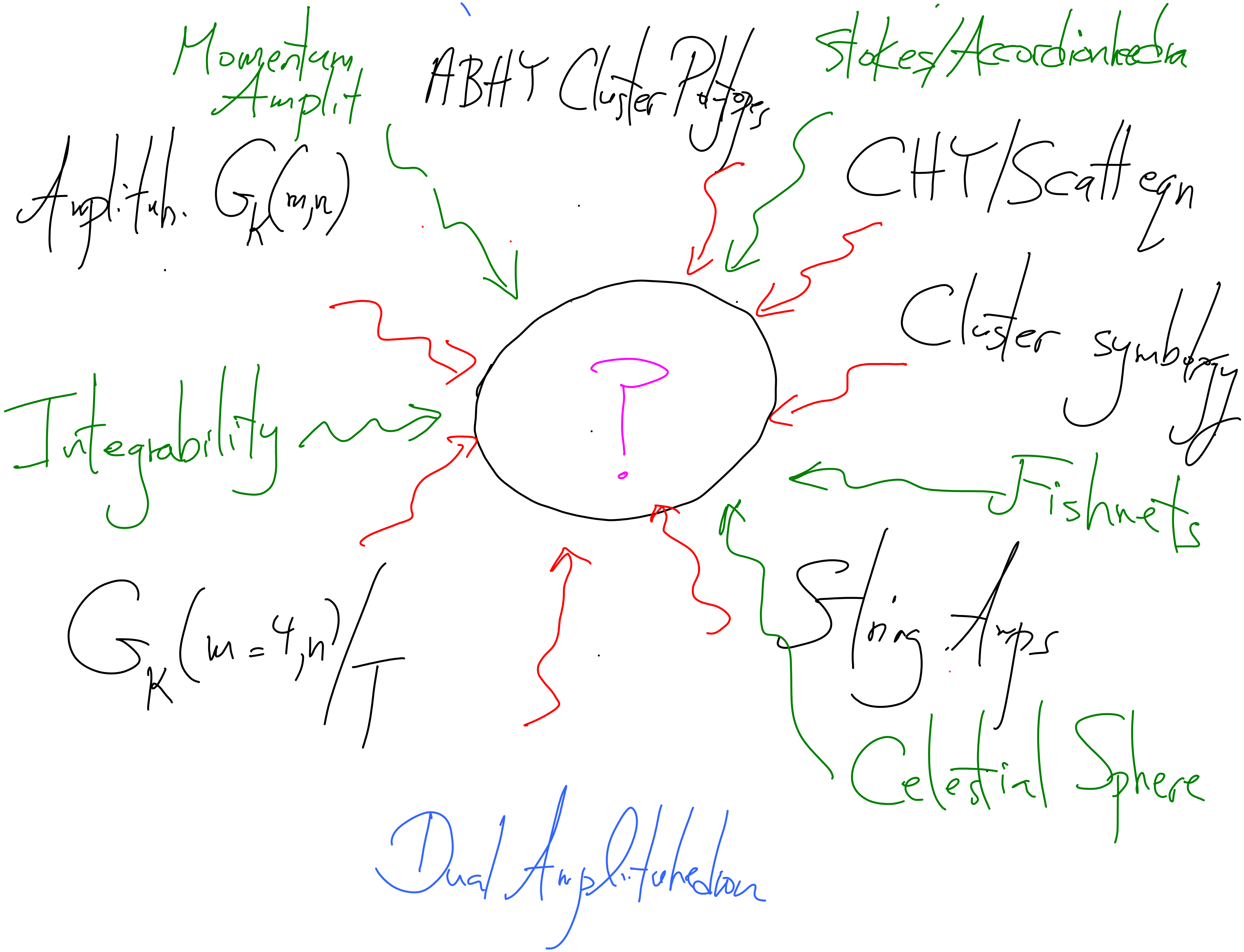
Outlook

Big Picture Issue Remains:



What is the intrinsic, non-pert Q in Kin Space to which A is the answer?


Amplituh. $G_k(m,n)$
ABHY Cluster Polytopes
CHY/Scatt eqn
Cluster symbology
?
$G_k(m=4,n)/T$
String Amps
Dual Amplituhedron


Momentum Amplit
ABHY Cluster Polytopes
Stokes/Accordionhedra
Amplituh. $G_K(m,n)$
CHY/Scatt eqn
Cluster symbology
Integrability
?
Fishnets
$G_K(m=4,n)/T$
String Amps
Celestial Sphere
Dual Amplituhedron

Every Amplitudes meeting I've
been @ since 2009 has
been buzzing with amazing
results — thanks to all
of you for making this
an exhilirating journey so far,

and onwards to Amps 2020!