Monodromy relations from twisted homology

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July 5, 2019

Amplitudes '19, Trinity College Dublin

with S. Mizera and P. Tourkine (to appear)

Motivation

Tree-level open string amplitude

$$\mathcal{A}(1\cdots n)=\int_{\Delta(1\cdots n)}\prod_{j=1}^{n-3}dz_j\prod_{1\leq j<l\leq n}(z_l-z_j)^{\alpha' k_j\cdot k_l}\,\varphi(z_j)$$

- $\varphi(z_j)$ theory-dependent, rational function of $\{z_i\}$. Can also depend on $\{g_i, k_j^{\mu}, \varepsilon_j^{\mu}\}$.
- $\Delta(1\cdots n) := \{z_1 < z_2 < \cdots < z_n\}$ is a top dimensional cycle on $\mathcal{M}_{0,n}(\mathbb{R})$.

Twisted homology[1] is a homology theory for multivalued functions. Cycles are paired up with choice of branches for

$$\prod_{1\leq j<l\leq n}(z_l-z_j)^{\alpha' k_j\cdot k_l}$$

[1] Steenrod '43; Aomoto '77; Aomoto, Kita '11

- Twisted (co)homology is the natural setting for tree-level (genus 0) string amplitudes and ambitwistor string[2], conveniently packaging information about multivaluedness of the functions involved
- Provides regularization for tree-level amplitudes
- KLT relations reinterpreted as topological features of twisted homology, i.e. KLT as intersection numbers[2]
- Also useful for describing tree-level field theory amplitudes, CHY[3] and loop Feynman integrals[4]

[2] Mizera '17, '18

[3] Mizera '18, '19

[4] Mastrolia, Mizera '18; Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi, Mizera '19

Motivation

Objective is to generilize this framework for higher genus string amplitudes, aiming towards the analogue of the quadratic KLT relations at higher genus as intersection on twisted homology

On the way, several subtleties must be addressed and we must make sure that we're on the right track.

Check by computing the monodromy relations for the string which have a simple interpretation as boundary relations on the twisted homology, and compare them with results in the literature[5].

Field theory limit of loop-level monodromy relations should shed light on color-kinematics and double copy at loops.

[5] Plathe '70; Tourkine, Vanhove '16; Stieberger, Hohenegger '17

Results

As a consequence of defining higher genus twisted homology we:

- Simplify the computations monodromy relations
- Argue we have found *all* the (generic) string theory monodromy relations for all genus
- Give new, explicit expressions for all genus and solve discrepancy in the literature
- Obtain field theory relations which work without the need to shift the loop momentum (BCJ 'triples' without loop momentum shift)

Outline

1. Twisted homology and tree-level string amplitudes
2. Monodromy relations as boundary relations
3. Genus 1
4. Higher genus
5. Field theory limit and BCJ triples

Tree-level String

$$\mathcal{A}(\alpha) := \int_{\Delta(\alpha)} \prod_{j=1}^{n-3} dz_j \prod_{1\leq j<l\leq n} |z_l - z_j|^{\alpha' k_j \cdot k_l} \varphi(z_j)$$

- $\varphi(z_j)$ theory-dependent, rational function of $\{z_i\}$. Can also depend on $\{g_i, k_j^\mu, \varepsilon_j^\mu\}$.
- $\Delta(\alpha) := \{z_{\alpha(n)} < z_{\alpha(1)} < z_{\alpha(2)} < \cdots < z_{\alpha(n)}\}$ is a top dimensional cycle on $\mathcal{M}_{0,n}(\mathbb{R})$.

Consider the integrand as a function of z_1 keeping all other punctures fixed

Tree-level String

Analytic continue in z_1 by considering it a coordinate on $\Sigma_{0,n-1}$. The relevant part of the integrand is:

$$T_0(z_1) := \prod_{j=2}^{n} (z_j - z_1)^{k_1 \cdot k_j}$$

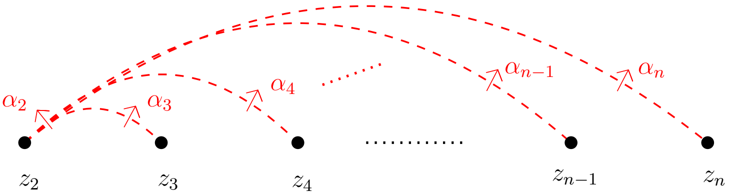
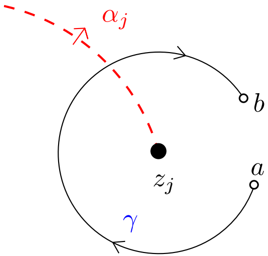


Figure: Choice of branch cuts for the loading $T_0(z_1)$ with monodromies $\alpha_i = e^{2\pi i \alpha' k_1 \cdot k_i}$. Note that $\alpha_2 = \prod_{j=3}^{n} \alpha_j^{-1}$ is not independent.

Twisted homology

Twisted chain

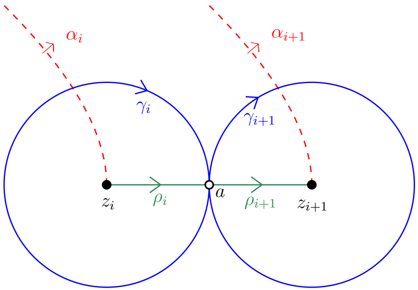


Here γ is twisted by a section of the line bundle $\mathcal{L}_0$.
Twisted boundary operator

$$\delta\gamma = \alpha_j b - a$$

Twisted homology

Twisted cycles



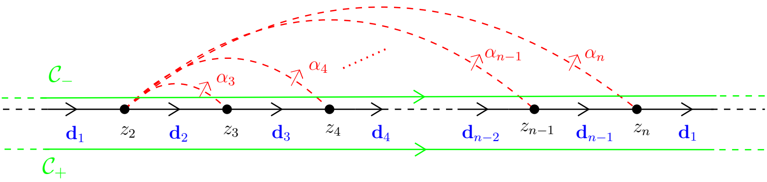
Twisted boundary operator

$$\delta\left(\frac{\gamma_i}{(\alpha_i - 1)} - \frac{\gamma_{i+1}}{(\alpha_{i+1} - 1)}\right) = a - a = 0$$

$$\delta(\rho_i + \rho_{i+1}) = a - a = 0$$

both represent the same element of $H_1(\Sigma_{0,n-1}, \mathcal{L}_0)$

Tree-level monodromy relations



The contours $\mathcal{C}_\pm$ are zero in the twisted homology giving the relations

$$\mathbf{d}_1 + \sum_{i=2}^{n-1} \prod_{j=i+1}^{n-1} \alpha_j^{-1} \, \mathbf{d}_i = 0 = \sum_{i=1}^{n-1} \mathbf{d}_i$$

$\mathrm{Dim} H_1(\Sigma_{0,n-1}, \mathcal{L}_0) = (n-3) =$ number of independent $\mathbf{d}$ contours

Tree-level monodromy relations

Relation to physical branch:

$$\int \prod_{i=2}^{n-3} dz_i \int_{\mathbf{d}_i} dz_1 \; T_0(z_1)\,\varphi = \int \prod_{i=2}^{n-3} dz_i \int_{\mathbf{d}_i} dz_1 \; \prod_{j=2}^{n} (z_j - z_1)^{k_1\cdot k_j}\,\varphi$$

$$= e^{\pi i k_1\cdot\sum_{j=2}^{i} k_j} \int \prod_{i=2}^{n-3} dz_i \int_{\mathbf{d}_i} dz_1 \prod_{j=2}^{n} |z_j - z_i|^{k_i\cdot k_j}\varphi$$

$$=: e^{\pi i k_1\cdot\sum_{j=2}^{i} k_j} \mathcal{A}(2\cdots i, 1, i{+}1\cdots n)$$

Tree-level monodromy relations

Therefore

$$\mathbf{d}_1 + \sum_{i=2}^{n-1} \prod_{j=i+1}^{n-1} \alpha_j^{-1} \mathbf{d}_i = 0 = \sum_{i=1}^{n-1} \mathbf{d}_i$$

$$\downarrow$$

$$\mathcal{A}(12\cdots n) + e^{\pm\pi i k_1\cdot k_2}\mathcal{A}(213\dots n) + \cdots$$
$$+ e^{\pm\pi i k_1\cdot\sum_{j=2}^{i}}\mathcal{A}(2,\dots,i,1,i{+}1,\dots,n)$$
$$+\cdots + e^{\pm\pi i k_1\cdot\sum_{j=2}^{n-1}}\mathcal{A}(2,\dots,n{-}1,1,n) = 0$$

$\mathrm{Dim}\, H_{n-3}(\mathcal{M}_{0,n},\mathcal{L}_0) = (n-3)! =$ number of independent tree-level amplitudes

Genus 1 monodromy relations

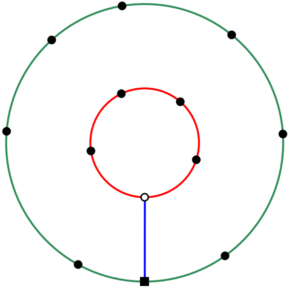
Genus 1 string integrand

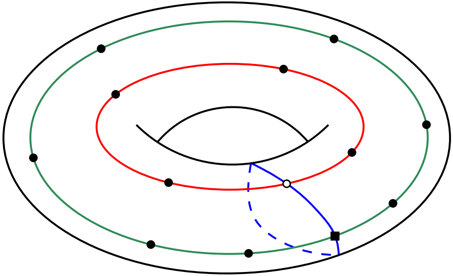
$$\mathcal{I}(\alpha|\beta) := \int_0^\infty dt \int_{\Delta(\alpha|\beta)} \prod_{\substack{j=1 \\ j\neq m}}^{n} dz_j \, e^{-\pi t \alpha' \ell^2} e^{-2\pi\alpha' \ell\cdot\sum_{j=1}^{n} k_j y_j} \times \prod_{1\leq j<l\leq n} |G_1(z_j, z_l)|^{\alpha' k_j\cdot k_l} \, \varphi(\ell^\mu, z_j, \tau)$$

$$G_1(z_j, z_k) = \begin{cases} \log\left|\dfrac{\vartheta_1(iy_j - iy_k)}{\vartheta_1'(0)}\right| & \text{for} \quad |x_j - x_k| = 0, \\ \log\left|\dfrac{\vartheta_2(iy_j - iy_k)}{\vartheta_1'(0)}\right| & \text{for} \quad |x_j - x_k| = \dfrac{1}{2}. \end{cases}$$

Genus 1 monodromy relations

Analityc continuation and dissection





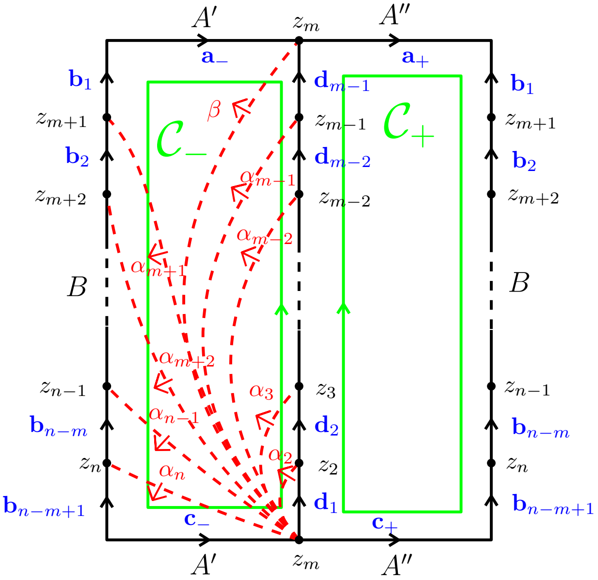
Genus 1 monodromy relations

Fixing all particles except z_1, the relevant part of the integrand is analytic continued as

$$T_1(z_1) := e^{2\pi i k_1\cdot\ell\, z_1} \prod_{j=2}^{m} (-i\vartheta_1(iy_j - z_1))^{k_1\cdot k_j} \prod_{j=m+1}^{n} \vartheta_2(iy_j - z_1)^{k_1\cdot k_j}.$$

where the analytic continued integrand is $\mathcal{I}(1\cdots m|m+1\cdots n)$

Genus 1 monodromy relations



Genus 1 monodromy relations

Contours $\mathcal{C}_\pm$ are zero in the twisted homology and give the relations

$$\sum_{i=1}^{m-1} \prod_{j=2}^{i} \alpha_j \, \mathbf{d}_i - \beta \prod_{j=2}^{m-1} \alpha_j \, (\mathbf{a}_- + \mathbf{b}_1)$$
$$- \beta \prod_{j=2}^{m-1} \alpha_j \sum_{i=2}^{n-m} \prod_{k=m+1}^{m+i-1} \alpha_k \, \mathbf{b}_i - \beta \prod_{\substack{j=2\\ j\neq m}}^{n} \alpha_j \, (\mathbf{b}_{n-m+1} - \mathbf{c}_-) = 0$$

$$\sum_{i=1}^{m-1} \mathbf{d}_i + \mathbf{a}_+ - \sum_{i=1}^{n-m+1} \mathbf{b}_i - \mathbf{c}_+ = 0$$

With

$$\alpha_j = e^{-2\pi i k_1 \cdot k_j}, \qquad \beta = e^{-2\pi i k_1 \cdot \ell} e^{-2\pi i k_1 \cdot k_m}.$$

Genus 1 monodromy relations

Relation to physical branch

$$\int_{\mathbf{d}_i} dz_1\, T_1(z_1)\,\varphi = e^{\pi i k_1\cdot\sum_{j=2}^{i} k_j}\,\widehat{\mathcal{I}}(2,3\ldots,i,1,i{+}1,\ldots,m|m{+}1,\ldots,n)$$

$$\int_{\mathbf{b}_{i-m+1}} dz_1\, T_1(z_1)\,\varphi$$
$$= -e^{\pi i k_1\cdot\ell} e^{\pi i k_1\cdot\sum_{j=2}^{i} k_j}\,\widehat{\mathcal{I}}(2,\ldots,m|m{+}1,\ldots,i,1,i{+}1,\ldots,n)$$

$$\int_{\mathbf{a}_\pm} dz_1\, T_1(z_1)\,\varphi =: e^{\pi i k_1\cdot\ell} e^{\pi i k_1\cdot\sum_{j=2}^{m} k_j}\,\widehat{\mathcal{J}}_{\mathbf{a}_\pm}(2,\ldots,m|1,m{+}1,\ldots,n)$$

$$\int_{\mathbf{c}_\pm} dz_1\, T_1(z_1)\,\varphi =: e^{\pi i k_1\cdot\ell}\,\widehat{\mathcal{J}}_{\mathbf{c}_\pm}(2,\ldots,m|m{+}1,\ldots,n,1)$$

Genus one monodromy relations

The relations in homology gives the n-term identity

$$\sum_{i=1}^{m-1} e^{\pm\pi i k_1\cdot\sum_{j=2}^{i} k_j}\,\mathcal{I}(2,3,\ldots,i,1,i{+}1,\ldots,m|m{+}1,\ldots,n)$$
$$+\sum_{i=m+1}^{n-1} e^{\pm\pi i k_1\cdot(\ell+\sum_{j=2}^{i} k_j)}\,\mathcal{I}(2,\ldots,m|m{+}1,\ldots,i,1,i{+}1,\ldots,n)$$
$$+\,e^{\pm\pi i k_1\cdot(\ell+\sum_{j=2}^{m} k_j)}\mathcal{K}_{\pm}(2,\ldots,m|1,m{+}1,\ldots,n)$$
$$+\,e^{\pm\pi i k_1\cdot\ell}\mathcal{K}_{\pm}(2,\ldots,m|m{+}1,\ldots,n,1)=0$$

$\mathrm{Dim} H_{n-1}(F_{1,n},\mathcal{L}_1)=(n-1)!$; and $\mathrm{Dim} H_{n-1}(\tilde{F}_{1,n},\mathcal{L}_1)=n!$

Higher genus monodromy relations

Relevant part of the analytic continued string integrand

$$T_g(z_1) = \prod_i^n E(z_i, z_1)^{k_i \cdot k_1} e^{-2\pi i \sum_{I=1}^{g} \ell_I \cdot k_1 \int_P^{z_1} \omega_I}.$$

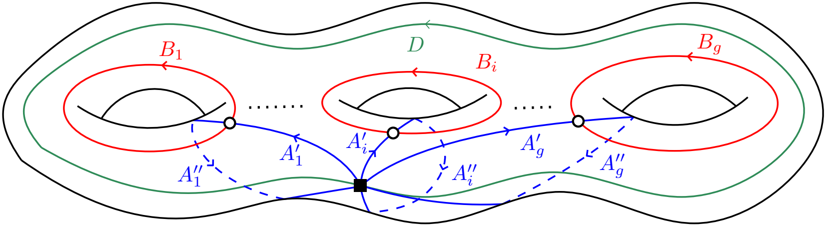
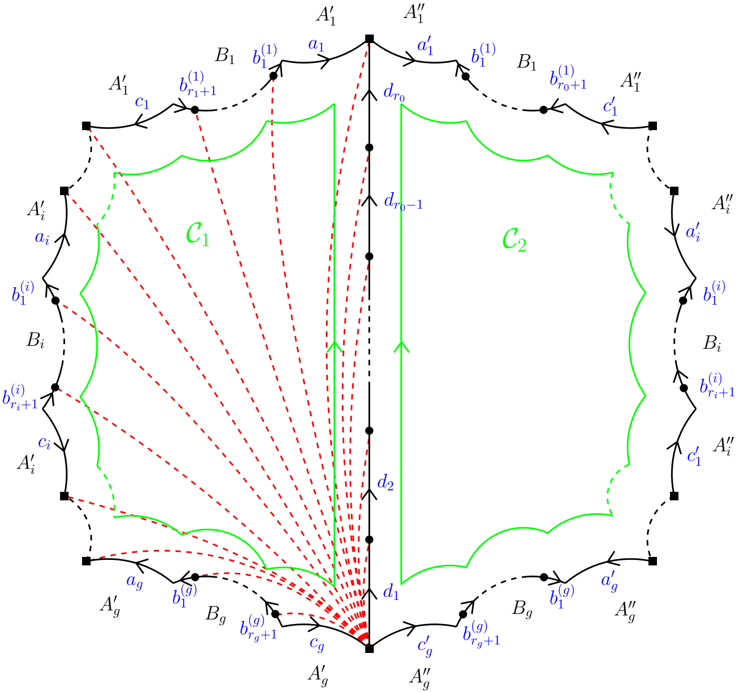


Figure: Genus g surface is cut along red and blue cycles.



A_1'
A_1''
a_1
a_1'
$b_1^{(1)}$
B_1
$b_{r_1+1}^{(1)}$
$b_{r_0+1}^{(1)}$
c_1
c_1'
d_{r_0}
d_{r_0-1}
d_2
d_1
A_i'
A_i''
a_i
a_i'
$b_1^{(i)}$
B_i
$b_{r_i+1}^{(i)}$
c_i
$\mathcal{C}_1$
$\mathcal{C}_2$
A_g'
A_g''
a_g
a_g'
$b_1^{(g)}$
B_g
$b_{r_g+1}^{(g)}$
c_g
c_g'

Higher genus monodromy relations

$$\mathbf{d}_1 + \sum_{i=2}^{r_0} \prod_{j=2}^{i} \alpha_j \mathbf{d}_i - \sum_{j=1}^{g} \prod_{q=1}^{j} \beta_q \tilde{\alpha}_{q-1} (\mathbf{a}_i + \mathbf{b}_1^{(j)})$$

$$- \sum_{i=1}^{g} \prod_{q=1}^{i} \beta_q \tilde{\alpha}_{q-1} \sum_{j=2}^{r_i} \prod_{l=\tilde{r}_i+1}^{j+\tilde{r}_i-1} \alpha_l \mathbf{b}_j^{(i)} - \sum_{i=1}^{g} \prod_{q=1}^{i} \beta_q \tilde{\alpha}_q \tilde{\alpha}_0 (\mathbf{b}_{r_i+1}^{(i)} - \mathbf{c}_i) = 0$$

with $\tilde{r}_i = \sum_{j=0}^{i} r_j$ and

$$\tilde{\alpha}_i = \begin{cases} \prod_{j=2}^{r_0} \alpha_j & \text{if } i = 0 \\ \prod_{j \in \{B_i\}}^{r_i} \alpha_j & \text{if } i > 0 \end{cases}$$

and

$$\sum_{i=1}^{m-1} \mathbf{d}_i + \sum_{i=1}^{g} (\mathbf{a'}_i - \mathbf{c'}_1) + \sum_{j=1}^{g} \sum_{i=1}^{r_j+1} \mathbf{b}_i^{(j)} = 0$$

Higher genus monodromy relations

$$\mathcal{I}(1,2,\cdots,r_0|\cdot)+\sum_{i=2}^{r_0}e^{\mp i\pi k_i\cdot k_{2\dots i}}\mathcal{I}(2,\dots,i,1,i+1\dots,r_0|\cdot)$$

$$+\sum_{j=1}^{g}e^{\pm i\pi k_1\cdot l_j}\sum_{i=\tilde{r}_{j-1}}^{\tilde{r}_j}e^{\pm i\pi k_1\cdot k_{i+1\dots\tilde{r}_j}}\mathcal{I}(\cdot|\tilde{r}_{j-1}+1,\dots,i,1,i+1,\dots,\tilde{r}_j|\cdot)$$

$$+\sum_{j=1}^{g}e^{\pm i\pi k_1\cdot l_j}\mathcal{K}_\pm(\cdot|1,\tilde{r}_{j-1}+1,\dots,\tilde{r}_j|\cdot)$$

$$+\sum_{j=1}^{g}e^{\pm i\pi k_1\cdot l_j}e^{\pm i\pi k_1\cdot k_{\tilde{r}_{j-1}+1\dots\tilde{r}_j}}\mathcal{K}_\pm(\cdot|\tilde{r}_{j-1}+1,\dots,\tilde{r}_j,1|\cdot)=0$$

$$\mathrm{Dim}\,H_{\mathrm{middle}}(F_{g,n},\mathcal{L}_g)=\frac{(2g+n-3)!}{(2g-3)!}$$

Field theory limit and BCJ triples

The monodromy relations have the generic form

$$\sum_j e^{i\alpha'\phi_j}\mathcal{I}_j + \sum_J e^{i\alpha'\theta_J}\mathcal{K}_J = 0$$

Relations for field theory integrands come from terms of order $O(1)$ and $O(\alpha')$ of the above relations. The behaviour of the two contributions is different

$$\mathcal{I} = \mathcal{I}^{FT} + \mathcal{I}^1 + O(\alpha'^2) \qquad \mathcal{K} = \mathcal{K}^{FT}\alpha' + O(\alpha'^2)$$

In the limit $\mathcal{I}$ gives tree-valent graphs but $\mathcal{K}$ gives contact terms

Field theory limit and BCJ triples

In the limit recover well-known field theory relations at one-loop[6]

$$\mathcal{A}(\alpha|\beta) = (-1)^{\beta} \sum_{\gamma \in (\alpha \text{shuffle} \beta)} \mathcal{A}(\gamma)$$

$$\sum_{i=2}^{p-1} k_1 \cdot k_{2\dots i}\, \mathcal{I}(2 \cdots i, 1, i+1 \cdots p | p+1 \cdots n)$$

$$+ \sum_{i=p}^{n} k_1 \cdot (l + k_{i+1\dots n}) \mathcal{I}(2 \cdots p | p+1 \cdots i, 1, i+1 \cdots n) \simeq 0$$

$$\sum_{i=1}^{n-1} k_1 \cdot (l + k_{1\dots i}) \mathcal{I}(2 \cdots i, 1, i+1 \dots n) \simeq 0$$

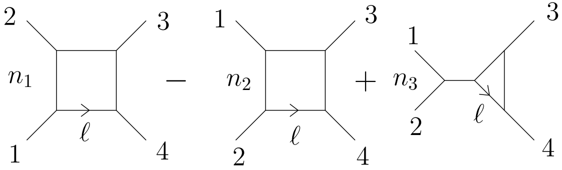
and at higher loops[7]

[6] Bern, Dixon, Dunbar, Kosower '94; Feng, Huang, Jia '10, '11; Boels, Iserman '11; Brown, Naculich '16

[7] Chiodaroli, Gunaydin, Johansson, Roiban '17; Feng, Huang, Jia '11

Field theory limit and BCJ triples

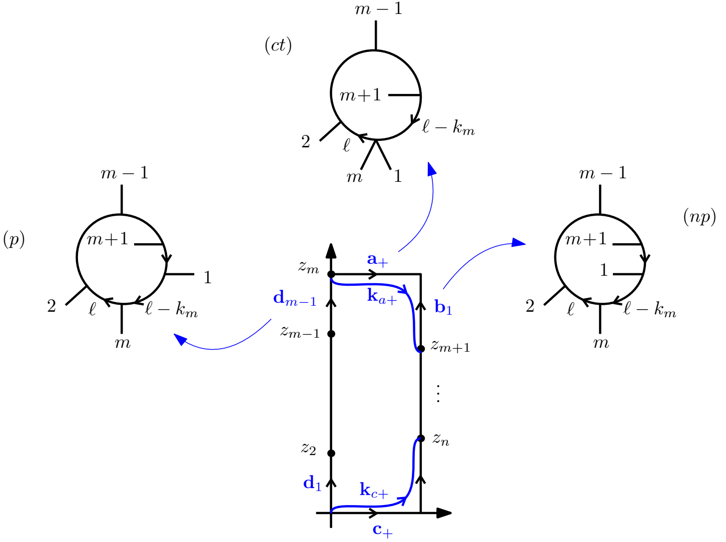
In the field theory limit, diagrams can be organized into BCJ triples[8]



But in some of them there's a mismatch due to shift in loop momenta. Taking into account the diagrams from the horizontal contours corrects the BCJ triples by a contact term, and the field theory relations work without the need to shift the loop momenta

[8] Ochirov, Tourkine, Vanhove '17

Field theory and BCJ triples



The pairing of the horizontal contours with the edge contours is natural from their origin as twisted cycles

Summary and future directions

Done:

- Generalized the twisted homology to higher genus and obtained all the generic monodromy relations
- Found interesting interpretation for non-physical contours in the field theory limit as correction terms for BCJ triples

To do:

- Systematize the field theory limit including the non-physical cycles
- Extra particles along the non-physical cycles to complete the twisted homology basis on $F_{g,n}$ (relations with higher point contact terms)
- Twisted intersection theory on higher genus surfaces $\rightarrow$ KLT (some preliminary results)