Cutting and sewing of amplitudes

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I: **What we got right on the first shot**

II: **What we got wrong on the first shot**
(and got right now)

III: **How it works at l-loops**

Goal

In D spacetime dimensions an l-loop amplitude with n external particles involves

$$D \cdot l$$

integrations.

Have also real emission contributions with fewer loops and more external particles, down to 0 loops and $n + l$ external particles. These involve

$$(D-1) \cdot l$$

integrations beyond the integrations for the Born contribution.

We would like to cancel all divergences at the integrand level, take $D = 4$ and integrate numerically.
We don't want to work with individual graphs, but with amplitude-like objects.

Loop-tree duality

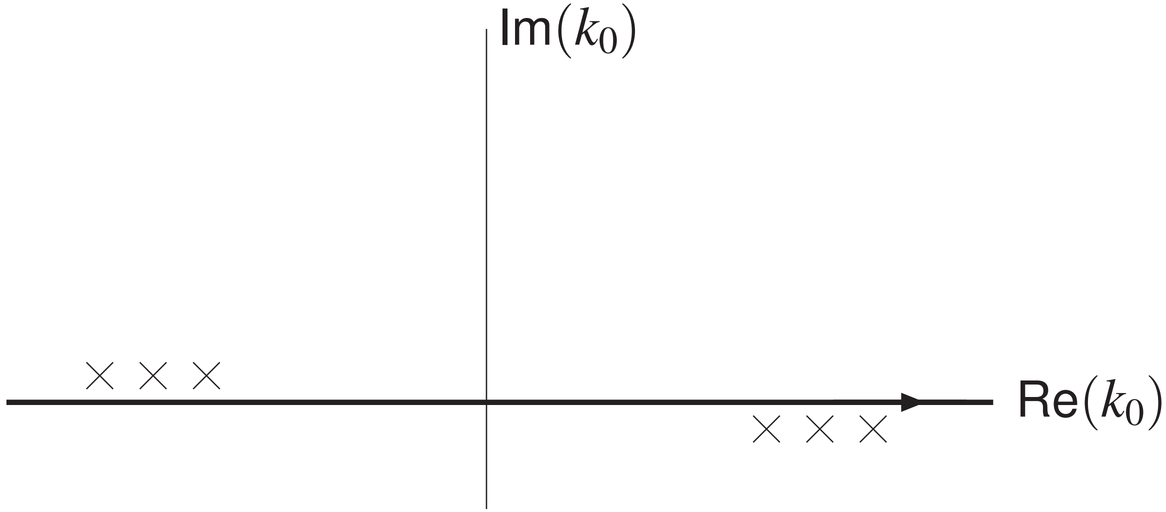
For each loop, do the energy integration with the help of Cauchy's residue theorem.

This leaves

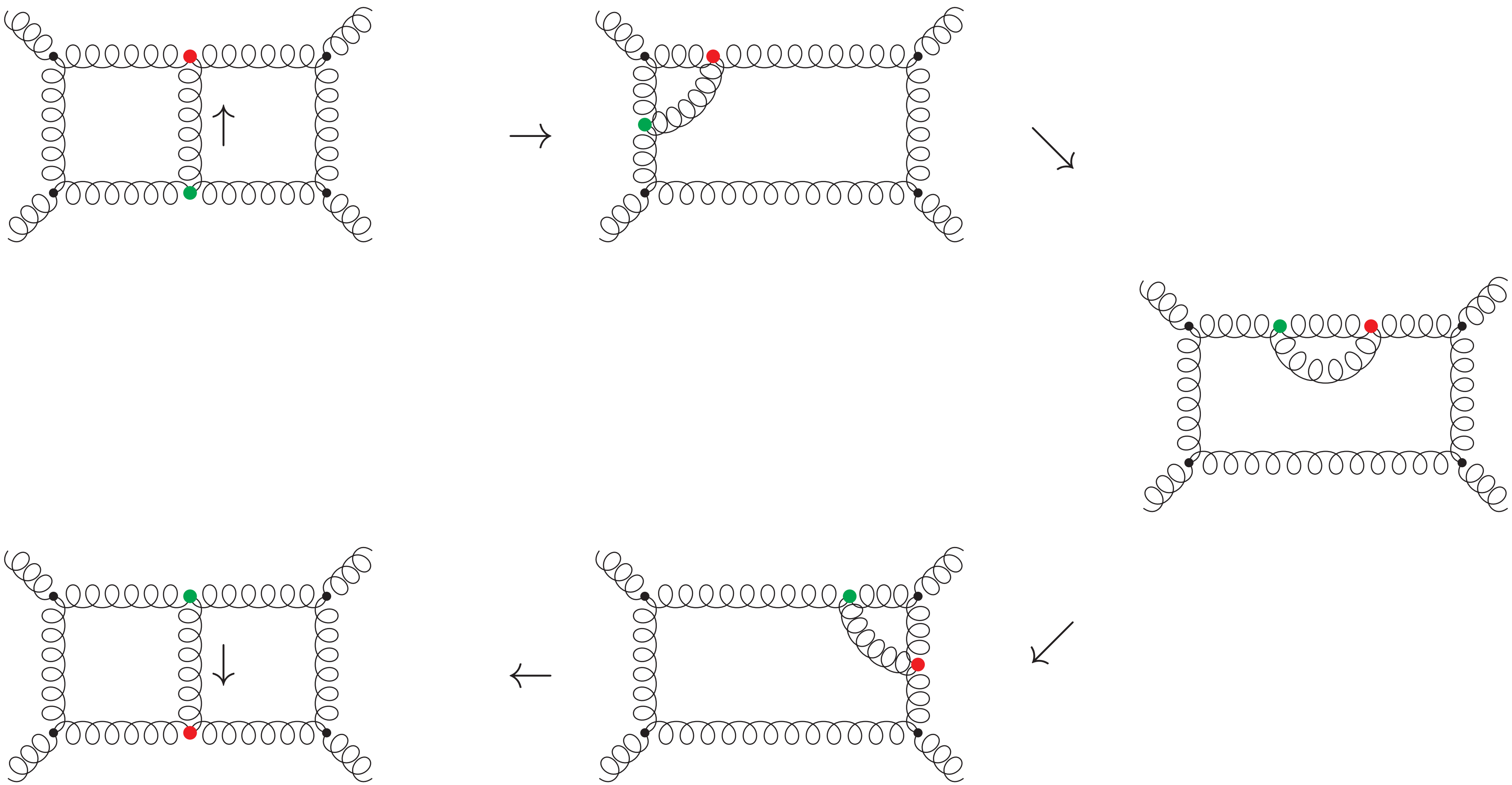
$$(D-1)\cdot l$$

integrations at l-loops.

Can close the contour below or above.



Global definition of the loop momenta ?



Loop-tree duality at one-loop

- Modified causal $i\delta$-prescription

  Catani, Gleisberg, Krauss, Rodrigo, Winter, '08

- UV-counterterms

  Becker, Reuschle, S.W., '10

- Contour deformation

  Gong, Nagy, Soper, '08

- Dual cancellations

  Buchta, Chachamis, Draggiotis, Malamos, Rodrigo, '14

- Local cancellation of infrared divergences

  Sborlini, Driencourt-Mangin, Hernandez-Pinto, Rodrigo, '16, Seth, S.W., '16

Works not only for graphs, but also for amplitudes.

Modified causal $i\delta$-prescription at one-loop

A one-loop integral

$$I_n = \int \frac{d^D k}{(2\pi)^D} \frac{P(k)}{\prod_{j=1}^{n} \left(k_j^2 - m_j^2 + i\delta\right)}.$$

can be written with Cauchy's theorem as

$$I_n = -i \sum_{i=1}^{n} \int \frac{d^{D-1} k}{(2\pi)^{D-1}\, 2E_i} \frac{P(k)}{\prod_{\substack{j=1 \\ j\neq i}}^{n} \left[k_j^2 - m_j^2 - i\delta\,(E_j - E_i)\right]} \Bigg|_{E_i = \sqrt{\vec{k}_i^2 + m_i^2}},$$

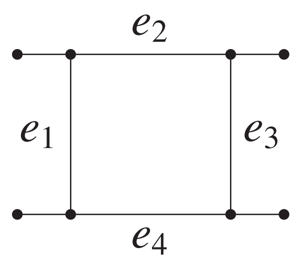
Catani, Gleisberg, Krauss, Rodrigo, Winter, '08

Loop-tree duality beyond one-loop

- Modified causal $i\delta$-prescription
- Absence of higher poles in the on-shell scheme
- Combinatorial factors
- From graphs to amplitude-like objects

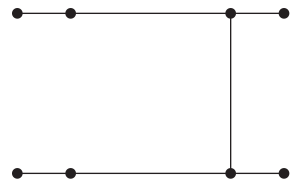
Spanning trees and cut trees

Spanning tree: Sub-graph of Γ, which contains all the vertices and is a connected tree graph.

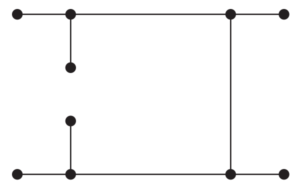


Obtained by deleting l internal edges.

Denote by $\sigma = \{\sigma_1, ..., \sigma_l\}$ the set of indices of the deleted edges and by $\mathcal{C}_\Gamma$ the set of all such sets of indices.



Cut tree: Each σ defines also a cut graph, obtained by cutting each of the l internal edges e_{σ_j} into two half-edges.



The $2l$ half-edges become external lines and the cut graph is a tree graph with $n+2l$ external lines.

l-fold residue

Consider an l-loop graph Γ. Choose an orientation for each internal edge. This defines positive energy / negative energy:

$$k_j^2 - m_j^2 + i\delta \quad = \quad \left(E_j - \sqrt{\vec{k}_j^2 + m_j^2 - i\delta}\right)\left(E_j + \sqrt{\vec{k}_j^2 + m_j^2 - i\delta}\right)$$

$\mathcal{C}_\Gamma$ set of all spanning trees / cut trees.

$\sigma = (\sigma_1, ..., \sigma_l) \in \mathcal{C}_\Gamma$: indices of the cut edges

$\alpha = (\alpha_1, ..., \alpha_l) \in \{1, -1\}^l$: energy signs

$$\mathrm{Cut}\left(\sigma_1^{\alpha_1}, ..., \sigma_l^{\alpha_l}\right) \quad = \quad (-i)^l \left(\prod_{j=1}^{l} \alpha_j\right) \mathrm{res}\,(...)$$

Modified causal $i\delta$-prescription

All uncut propagators have a modified $i\delta$-prescription:

$$\frac{1}{\prod\limits_{j \notin \sigma} \left(k_j^2 - m_j^2 + i s_j(\sigma)\delta\right)}, \qquad s_j(\sigma) = \sum_{a \in \{j\} \cup \pi} \frac{E_j}{E_a}.$$

The set σ defines a cut tree. Cutting in addition edge e_j will give a two-forest (T_1, T_2).

We orient the external momenta of T_1 such that all momenta are outgoing.

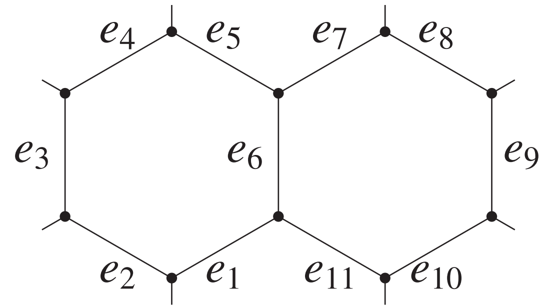
Let π be the set of indices corresponding to the external edges of T_1 which come from cutting the edges e_{σ_i}.

The set π may contain an index twice, this is the case if both half-edges of a cut edge belong to T_1.

R. Runkel, Z. Szőr, J.P. Vesga, S.W., '19

Example

Two-loop eight-point graph.

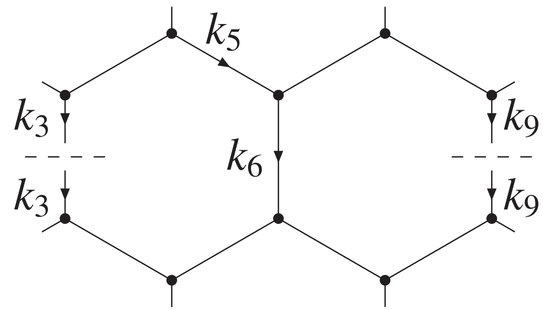


Consider the cut $\sigma = (3,9)$.

Then

$$s_5(\sigma) = \frac{E_3 + E_5}{E_3}$$

$$s_6(\sigma) = \frac{E_3E_6 + E_3E_9 + E_6E_9}{E_3E_9}$$



Absence of higher poles in the on-shell scheme

Self-energy insertion on internal lines lead to higher poles.
Have also UV-counterterms.

Some cuts are unproblematic, some other cuts correspond to residues of higher poles:

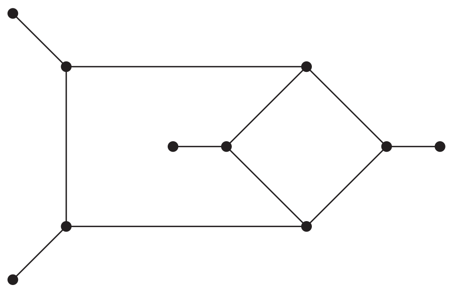
In the on-shell scheme we may choose an integral representation for the UV-counterterm sucht that the problematic residues are zero.

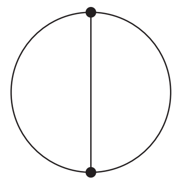
R. Baumeister, D. Mediger, J. Pečovnik, S.W. '19

Chain graphs

Two propagators belong to the same chain, if their momenta differ only by a linear combination of the external momenta.

Chain graph: delete all external lines and choose one propagator for each chain as a representative.

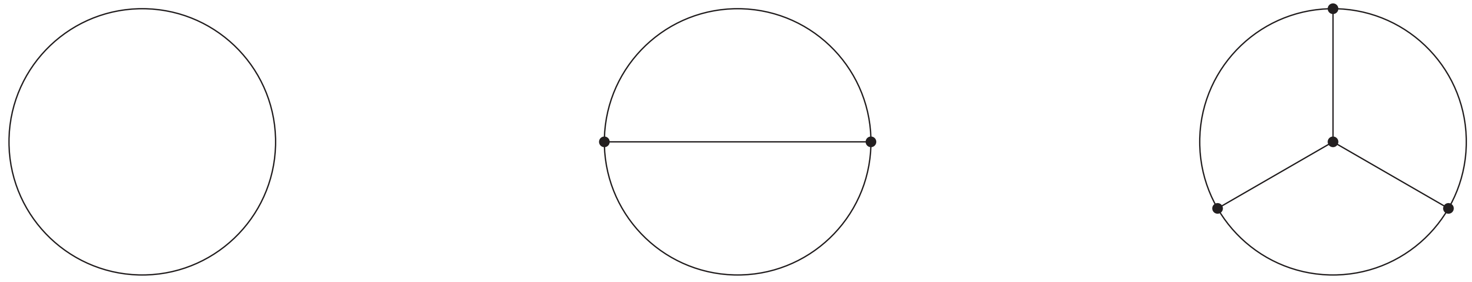




Kinoshita '62

Chain graphs

Up to three loops, all chain graphs are (sub-) topologies of



Combinatorial factors

Γ a graph with l loops and n external legs, $I_{l,n}$ the corresponding Feynman integral. Take l-fold residues:

$$I_{l,n} = \sum_{\sigma \in \mathcal{C}_\Gamma} \sum_{\alpha=1}^{2^l} c_{\sigma\alpha} \, \mathrm{Cut}\,(\sigma, \alpha)$$

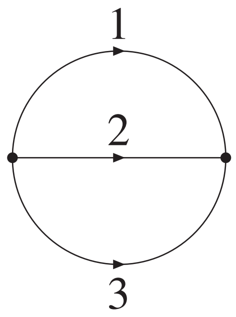
for some coefficients $c_{\sigma\alpha}$.

Recall:

- $\mathcal{C}_\Gamma$ set of all spanning trees / cut trees.
- $\sigma = (\sigma_1, ..., \sigma_l) \in \mathcal{C}_\Gamma$: indices of the cut edges
- $\alpha = (\alpha_1, ..., \alpha_l) \in \{1, -1\}^l$: energy signs

Remark: The representation in terms of cuts is not unique. The sum of all residues in any subloop equals zero.

The pitfall



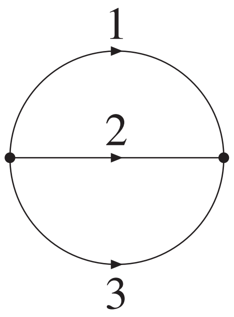
$$=$$

$$c_1 \mathrm{Cut}\,(1^+, 2^+) \quad + \quad c_2 \mathrm{Cut}\,(1^+, 2^-) \quad + \quad c_3 \mathrm{Cut}\,(1^-, 2^+) \quad + \quad c_4 \mathrm{Cut}\,(1^-, 2^-)$$

$$+ \quad c_5 \mathrm{Cut}\,(1^+, 3^+) \quad + \quad c_6 \mathrm{Cut}\,(1^+, 3^-) \quad + \quad c_7 \mathrm{Cut}\,(1^-, 3^+) \quad + \quad c_8 \mathrm{Cut}\,(1^-, 3^-)$$

$$+ \quad c_9 \mathrm{Cut}\,(2^+, 3^+) \quad + \quad c_{10} \mathrm{Cut}\,(2^+, 3^-) \quad + \quad c_{11} \mathrm{Cut}\,(2^-, 3^+) \quad + \quad c_{12} \mathrm{Cut}\,(2^-, 3^-)$$

The pitfall

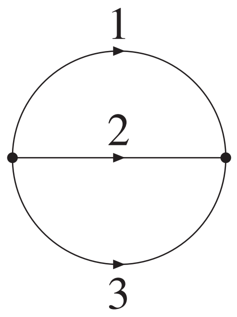


$$= \frac{1}{4}\mathrm{Cut}\,(1^+,2^+) + \frac{1}{4}\mathrm{Cut}\,(1^+,2^-) + \frac{1}{4}\mathrm{Cut}\,(1^-,2^+) + \frac{1}{4}\mathrm{Cut}\,(1^-,2^-)$$

$$+ c_5\mathrm{Cut}\,(1^+,3^+) + c_6\mathrm{Cut}\,(1^+,3^-) + c_7\mathrm{Cut}\,(1^-,3^+) + c_8\mathrm{Cut}\,(1^-,3^-)$$

$$+ c_9\mathrm{Cut}\,(2^+,3^+) + c_{10}\mathrm{Cut}\,(2^+,3^-) + c_{11}\mathrm{Cut}\,(2^-,3^+) + c_{12}\mathrm{Cut}\,(2^-,3^-)$$

The pitfall



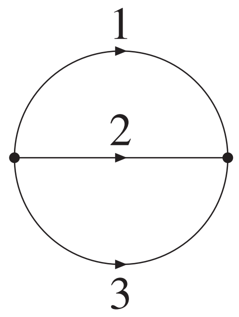
$$=$$

$$\frac{1}{4}\mathrm{Cut}\,(1^+,2^+) + \frac{1}{4}\mathrm{Cut}\,(1^+,2^-) + \frac{1}{4}\mathrm{Cut}\,(1^-,2^+) + \frac{1}{4}\mathrm{Cut}\,(1^-,2^-)$$

$$+ \frac{1}{4}\mathrm{Cut}\,(1^+,3^+) + \frac{1}{4}\mathrm{Cut}\,(1^+,3^-) + \frac{1}{4}\mathrm{Cut}\,(1^-,3^+) + \frac{1}{4}\mathrm{Cut}\,(1^-,3^-)$$

$$+ c_9\mathrm{Cut}\,(2^+,3^+) + c_{10}\mathrm{Cut}\,(2^+,3^-) + c_{11}\mathrm{Cut}\,(2^-,3^+) + c_{12}\mathrm{Cut}\,(2^-,3^-)$$

The pitfall

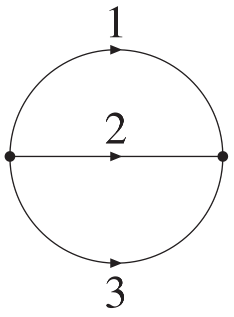


$$= \quad \tfrac{1}{4}\mathrm{Cut}\,(1^+,2^+) + \tfrac{1}{4}\mathrm{Cut}\,(1^+,2^-) + \tfrac{1}{4}\mathrm{Cut}\,(1^-,2^+) + \tfrac{1}{4}\mathrm{Cut}\,(1^-,2^-)$$

$$+ \tfrac{1}{4}\mathrm{Cut}\,(1^+,3^+) + \tfrac{1}{4}\mathrm{Cut}\,(1^+,3^-) + \tfrac{1}{4}\mathrm{Cut}\,(1^-,3^+) + \tfrac{1}{4}\mathrm{Cut}\,(1^-,3^-)$$

$$+ \tfrac{1}{2}\mathrm{Cut}\,(2^+,3^+) \qquad + \tfrac{1}{2}\mathrm{Cut}\,(2^-,3^-)$$

More symmetric



$$= \frac{1}{3}\mathrm{Cut}\,(1^+,2^+) + \frac{1}{6}\mathrm{Cut}\,(1^+,2^-) + \frac{1}{6}\mathrm{Cut}\,(1^-,2^+) + \frac{1}{3}\mathrm{Cut}\,(1^-,2^-)$$

$$+ \frac{1}{3}\mathrm{Cut}\,(1^+,3^+) + \frac{1}{6}\mathrm{Cut}\,(1^+,3^-) + \frac{1}{6}\mathrm{Cut}\,(1^-,3^+) + \frac{1}{3}\mathrm{Cut}\,(1^-,3^-)$$

$$+ \frac{1}{3}\mathrm{Cut}\,(2^+,3^+) + \frac{1}{6}\mathrm{Cut}\,(2^+,3^-) + \frac{1}{6}\mathrm{Cut}\,(2^-,3^+) + \frac{1}{3}\mathrm{Cut}\,(2^-,3^-)$$

S. Caron-Huot, '10

The correct procedure

$$I_{l,n} = \sum_{\sigma \in \mathcal{C}_\Gamma} \sum_{\pi \in S_l} \sum_{\alpha=1}^{2^l} C_{\sigma\pi\alpha}^{\tilde{\sigma}\tilde{\pi}\tilde{\alpha}} \, \mathrm{Cut}\,(\sigma, \alpha)$$

- $\tilde{\sigma} = (\tilde{\sigma}_1, ..., \tilde{\sigma}_l) \in \mathcal{C}_\Gamma$: indices of the chosen independent loop momenta
- $\tilde{\pi} = (\tilde{\pi}_1, ..., \tilde{\pi}_l) \in S_l$: order in which the integration are carried out
- $\tilde{\alpha} = (\tilde{\alpha}_1, ..., \tilde{\alpha}_l) \in \{1, -1\}^l$: specifications whether the contour is closed below or above

- $\sigma = (\sigma_1, ..., \sigma_l) \in \mathcal{C}_\Gamma$: indices of the cut edges
- $\pi = (\pi_1, ..., \pi_l) \in S_l$: order in which the residues are picked up
- $\alpha = (\alpha_1, ..., \alpha_l) \in \{1, -1\}^l$: energy signs

Z. Capatti, V. Hirschi, D. Kermanschah, B. Ruijl, '19

Averaging

Sum over π and average over $\tilde{\sigma}$, $\tilde{\pi}$, $\tilde{\alpha}$. For a chain graph:

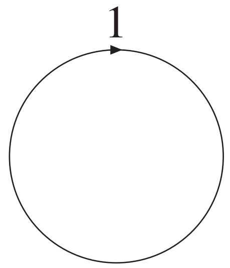
$$S_{\sigma\alpha} = \frac{1}{2^l l! \, |\mathcal{C}_\Gamma|} \sum_{\pi \in S_l} \sum_{\tilde{\sigma} \in \mathcal{C}_\Gamma} \sum_{\tilde{\pi} \in S_l} \sum_{\tilde{\alpha} \in \{1,-1\}^l} C_{\sigma\pi\alpha}^{\tilde{\sigma}\tilde{\pi}\tilde{\alpha}}$$

Then

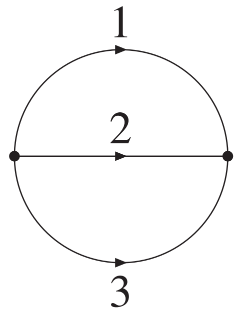
$$I_{l,n} = \sum_{\sigma \in \mathcal{C}_\Gamma} \sum_{\alpha=1}^{2^l} S_{\sigma\alpha} \, \mathrm{Cut}\,(\sigma, \alpha)$$

with combinatorial factor $S_{\sigma\alpha}$.

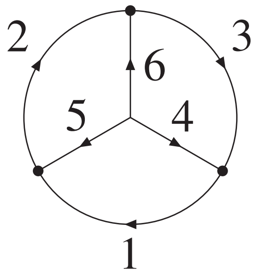
Examples



Cut	(1^+)	(1^-)
$S_{\sigma\alpha}$	$\frac{1}{2}$	$\frac{1}{2}$



Cut	$(1^+,2^+)$	$(1^+,2^-)$	$(1^-,2^+)$	$(1^-,2^-)$
$S_{\sigma\alpha}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$



Cut	$(1^+,2^+,3^+)$	$(1^+,2^+,3^-)$	$(1^+,2^-,3^+)$	$(1^+,2^-,3^-)$
$S_{\sigma\alpha}$	$\frac{3}{64}$	$\frac{29}{192}$	$\frac{29}{192}$	$\frac{29}{192}$

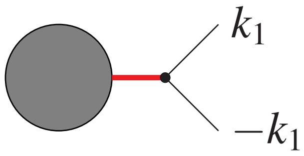
Cut	$(1^+,2^+,4^+)$	$(1^+,2^+,4^-)$	$(1^+,2^-,4^+)$	$(1^+,2^-,4^-)$
$S_{\sigma\alpha}$	$\frac{5}{96}$	$\frac{19}{192}$	$\frac{19}{192}$	$\frac{1}{4}$

From graphs to amplitude-like objects

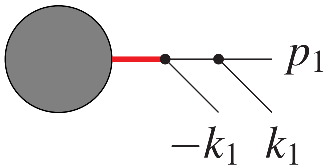
- UV-subtracted
- Regularised forward limit
- Minus signs for closed fermion loops
- Combinatorial factors

Regularised forward limit

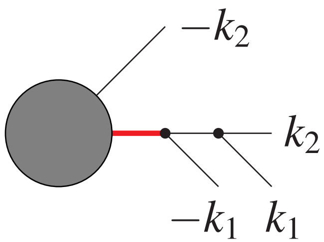
l-fold forward limit of tree-amplitude like objects: Exclude singular contributions.



$\Rightarrow$ Tadpole



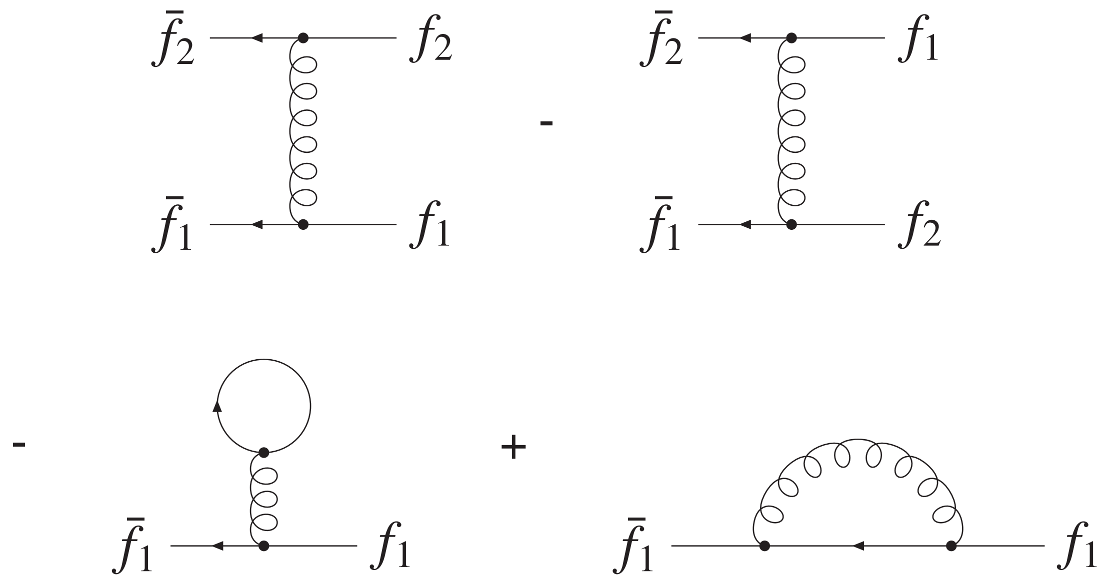
$\Rightarrow$ Self-energy insertion on an external line



$\Rightarrow$ Self-energy insertion on an internal line

Minus signs for closed fermion loops from the forward limit of tree amplitudes

Solution: Include a minus sign for every forward limit of a fermion-antifermion pair.



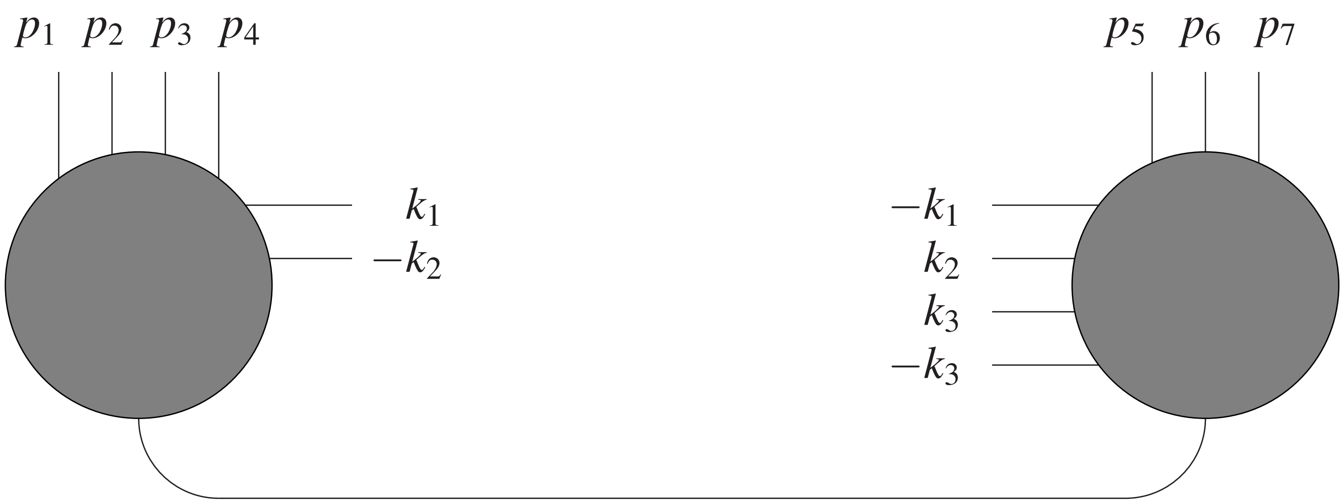
Combinatorial factors

- Combinatorial factor $S_{\sigma\alpha}$ independent of σ and α: One-loop
- Combinatorial factor $S_{\sigma\alpha}$ independent of σ, but dependent on α: Two-loop
- Combinatorial factor $S_{\sigma\alpha}$ dependent on σ and α: Three-loop and beyond

General case

Replace sum over spanning trees by a sum over spanning two-forests and sew one leg. For a theory with three-valent vertices:

$$\sum_{\mathcal{C}_\Gamma} = \frac{1}{n+2l-3} \sum_{(T_1,T_2)} \sum_{\text{sew}}$$



Summary and outlook

- Modified causal $i\delta$-prescription
- Absence of higher poles in the on-shell scheme
- Combinatorial factors
- Tree amplitude-like objects (sewed product of two off-shell currents)

Next steps:

- Infrared limits of these objects

Literature on loop-tree duality

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