The Analytic Structure of Five-Parton Amplitudes for LHC Phenomenology at NNLO

Ben Page

IPhT CEA Saclay

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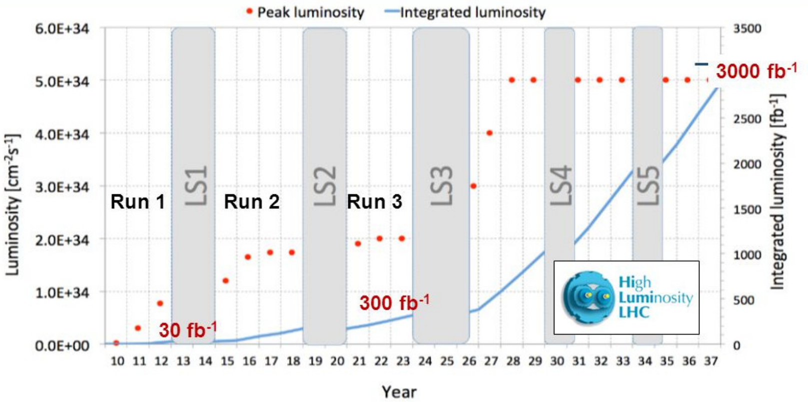
In collaboration with S. Abreu, J. Dormans, F. Febres Cordero, H. Ita and V. Sotnikov

based on [1812.04586], [1904.00945]

The LHC Potential: Precision Era



- LHC will be operating for two more decades
- ATLAS and CMS collect large proton collision data sets

Les Houches Precision Wish List [arXiv:1803.07977]

State of the art (NNLO)

- Many $2 \to 2$ QCD processes available
- Next frontier is $2 \to 3$

Calculational challenges

- Multi-scale two-loop integrals.
- Complex rational coefficients.

process	known	desired
$pp \to 2\,\text{jets}$	$\text{N}^2\text{LO}_{\text{QCD}}$ $\text{NLO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$	
$pp \to 3\,\text{jets}$	NLO_{QCD}	$\text{N}^2\text{LO}_{\text{QCD}}$
⋮	⋮	⋮
$pp \to H + 2j$	$\text{NLO}_{\text{HEFT}}\otimes\text{LO}_{\text{QCD}}$ $\text{N}^3\text{LO}^{(\text{VBF}^*)}_{\text{QCD}}$ (incl.) $\text{N}^2\text{LO}^{(\text{VBF}^*)}_{\text{QCD}}$ $\text{NLO}^{(\text{VBF})}_{\text{EW}}$	$\text{N}^2\text{LO}_{\text{HEFT}}\otimes\text{NLO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$ $\text{N}^2\text{LO}_{\text{QCD}}+\text{NLO}^{(\text{VBF})}_{\text{EW}}$
⋮	⋮	⋮
$pp \to V + j$	$\text{N}^2\text{LO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$	hadronic decays
$pp \to V + 2j$	$\text{NLO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$ NLO_{EW}	$\text{N}^2\text{LO}_{\text{QCD}}$
$pp \to V + b\bar{b}$	NLO_{QCD}	$\text{N}^2\text{LO}_{\text{QCD}}$ $+\text{NLO}_{\text{EW}}$
⋮	⋮	⋮
$pp \to \gamma\gamma + j$	NLO_{QCD}	$\text{N}^2\text{LO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$

A Flurry of Five-Point Two-Loop Amplitudes

- Analytic planar all-plus QCD [Gehrmann, Henn, lo Presti 15]

Numeric Planar QCD:

- 5-gluon [Badger, Brønnum-Hansen, Bayu Hartanto, Peraro 17] [Abreu, Ita, Febres Cordero, BP, Sotnikov, Zeng 17]
- 5-parton [Badger, Brønnum-Hansen, Bayu Hartanto, Gehrmann, Henn, lo Presti, Peraro 18] [Abreu, Ita, Febres Cordero, BP, Sotnikov, Zeng 18]

Analytic:

- Non-planar $\mathcal{N} = 4/8$ symbols [Abreu, Dixon, Herrmann, BP, Zeng 18], [Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia 18]
- Planar QCD -++++ [Badger, Brønnum-Hansen, Bayu Hartanto, Peraro 18]
- Planar QCD 5-parton [Abreu, Dormans, Febres Cordero, Ita, BP, Sotnikov 19]
- Non-planar QCD +++++ [Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia 19] [Johannes' talk]

The Standard Approach to General Two-loop Amplitudes

Integrand

Tensor reduction
[Passarino, Veltman '79]
IBPs
[Tkachov, Chetyrkin '81; Laporta '01]
↓

Sum of master integrals

$$\sum_{\text{Topologies}\,\Gamma} \sum_{i \in M_\Gamma} c_{\Gamma,i}(\vec{p}) I_{\Gamma,i}(\vec{p}).$$

Differential equations
[Kotikov '91; Remiddi '97; Gehrmann, Remiddi '01; Henn '13]
↓

ϵ-expansion

$$\sum_{i \in B} \sum_{k=-4}^{0} \epsilon^k \, \tilde{c}_{k,i}(\vec{p}) h_i(\vec{p}) + \mathcal{O}(\epsilon).$$

General procedure, but:

- Large intermediate expressions.
- GBs of integral relations.
- Simplifying final result is computationally demanding.
- Cluster parallelization is non-trivial.

Alternatively: can we use numerical methods?

Ansätze and Numerical Methods

- Write ansatz for result.
- Sample ansatz with numerical unitarity. [Abreu et al '17/'18]
- No IBP tables or large intermediate expressions.
- Trivially parallelizable.
- Practical tools: finite fields $\mathbb{F}_p$ and functional reconstruction. [Schabinger, von Manteuffel '14], [Peraro '16]

$$c(\vec{p}) = \sum_{i=0}^{n} a_i g_i(\vec{p})$$

$$\begin{pmatrix} c(\vec{p}_0) \\ \vdots \\ c(\vec{p}_n) \end{pmatrix} = \begin{pmatrix} g_0(\vec{p}_0) & \cdots & g_n(\vec{p}_0) \\ \vdots & \cdots & \vdots \\ g_0(\vec{p}_n) & \cdots & g_n(\vec{p}_n) \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}$$

$$\mathbb{F}_p = \{0, \ldots, p-1\}$$

Numerical Unitarity @ Two Loops

[Abreu, Ita, Jaquier, Febres Cordero, BP '17]

- Start with amplitude ansatz: combination of master integrals.

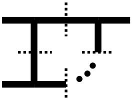
$$\bar{A}(\vec{p}) = \sum_{\text{Topologies}\,\Gamma} \sum_{i \in M_\Gamma} c_{\Gamma,i}(\vec{p}) I_{\Gamma,i}(\vec{p}).$$

- Lift to ansatz for integrand, requiring surface (S_Γ) integrands.

$$\bar{\mathcal{A}}(\ell_l, \vec{p}) = \sum_{\text{Topologies}\,\Gamma} \sum_{i \in M_\Gamma \cup S_\Gamma} \frac{c_{\Gamma,i}(\vec{p})\, m_{\Gamma,i}(\ell_l)}{\prod_{\text{props}\, j} \rho_j}.$$

[Ita '15]

- By unitarity, integrand residues factorize into product of trees.



$$= \sum_{\substack{\Gamma' \geq \Gamma,\\ i \in M_{\Gamma'} \cup S_{\Gamma'}}} \frac{c_{\Gamma',i}(\vec{p})\, m_{\Gamma',i}(\ell_l^\Gamma)}{\prod_{\text{props}\, j} \rho_j}.$$

[BDDK '94, '95]

- Numerically evaluate tree amplitudes to fix master coefficients.

Finite Field Evaluation

- Algorithm can be built over finite fields [Peraro '16], [Abreu, Febres Cordero, Ita, B.P, Zeng '17].
- Map from $\mathbb{F}_p \to \mathbb{Q}$ with "Rational Reconstruction". [Wang, '81]
- $\mathbb{Q}$-point in twistor space corresponds to $\mathbb{Q}$-momenta. [Hodges '09]
- Coefficients evaluations suitable input for ansatz.

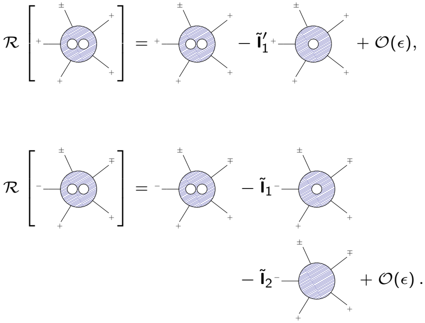
$$\mathbb{F}_p = \{0, \ldots, p-1\}$$

$$Z = \begin{pmatrix} 1 & 0 & 1 & 1+\frac{1}{x} & 1+\frac{1}{x}+\frac{x-s_{23}+s_{45}}{x s_{51}} \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{s_{23}}{x} & 1 \\ 0 & 0 & 1 & 1 & 1-\frac{s_{45}}{s_{23}} \end{pmatrix}$$

$$\bar{A}(\vec{p}) = \sum_{\Gamma\in\Delta} \sum_{i\in M_\Gamma} c_{\Gamma,i}(\vec{p}) I_{\Gamma,i}(\vec{p}).$$

Targeting the Finite Remainder

- Should target simpler, easier to ansatz objects.
- Integral coefficients contain information for general D.
- UV/IR poles are universal [Catani 98]. Finite remainder is new, $4D$ information.
- Subtraction of one-loop to $O(\epsilon^2)$ introduces finite shift.
- See also $\pm++++$ [Gehrmann et al '15], [Badger et al '18, '19]



Pentagon Functions

- Master integrals are linear combinations of multiple polylogarithms.

$$I(\vec{p}) = \sum_i \sum_{k=-4}^{0} c_{i,k}\epsilon^k G_i(\vec{p}).$$

- Naive basis of GPLs obscures physical properties, complicating coefficients.
- Express in basis B of ~ 400 "pentagon functions" h_i. [Gehrmann, Henn, lo Presti '18]

$$\bar{\mathcal{A}}^{(2)} = \sum_{i\in B} \sum_{k=-4}^{0} \epsilon^k \, c_{k,i}(\vec{s}) h_i(\vec{s}) + \mathcal{O}(\epsilon),$$

$$\mathcal{R}^{(2)} = \sum_{i\in B} r_i(\vec{s}) \, h_i(\vec{s}) \, .$$

- Representation free of spurious branch points.

Mandelstam Invariant Ansatz [Abreu, Dormans, Febres Cordero, Ita, B.P., Sotnikov '19]

- Normalize amplitude to be Lorentz invariant.
- Fewer ansatz parameters in terms of invariants.
- Coefficients split into parity-odd and parity-even.
- Denominator is product of even symbol letters W_j^+.
- Parameterize twistor space with mostly mandelstams.

$$\vec{s} = \{s_{23}, s_{34}, s_{45}, s_{51}\}, \quad s_{12} = 1.$$

$$tr_5 = 4i\varepsilon(p_1, p_2, p_3, p_4).$$

$$r_i = r_i^{+}(\vec{s}) + \mathrm{tr}_5 \cdot r_i^{-}(\vec{s}),$$

$$r_i^{\pm}(\vec{s}) = \frac{n_i^{\pm}(\vec{s})}{W^{\vec{q}_i}(\vec{s})}.$$

$$(s_{23}, s_{45}, s_{51}, x)$$

$$s_{34} = \frac{(s_{45}-1)s_{51}x + s_{23}(s_{45}+s_{51}-s_{23}+x)}{x(s_{45}-s_{23}+x)}$$

Denominator Determination

[Abreu, Dormans, Febres Cordero, Ita, B.P., Sotnikov '19]

- Looking for exponents in denominator ansatz.
- Determine on special univariate slice.
- On slice, compare factored denominator with alphabet.
- Simultaneously find numerator total degree.

$$x = c_0$$

$$s_{23} = c_1 + d_1 t, \qquad s_{45} = c_2 + d_2 t$$

$$s_{51} = c_3(c_0 - c_1 + c_2 + [d_2 - d_1]t)$$

$$\Rightarrow s_{34} \text{ linear in } t$$

$$W_j(\vec{s}(t)) = w_j \cdot (\omega_j + t)$$

$$r_i^{\pm}(\vec{s}(t)) = \frac{n_i^{\pm}(\vec{s}(t))}{\prod_{j \in A} w_j \cdot (\omega_j + t)^{q_{ij}}}.$$

The Problem of Large Ansätze

- Function numerators are polynomials in s_{ij}.

$$n_i^{\pm}(\vec{s}) = \sum_{N_{\min} \leq |\vec{\alpha}| \leq N_{\max}} a_{\vec{\alpha}} \vec{s}^{\vec{\alpha}}$$

- Degree $N_{\max}$, d variables. Many unknowns.

$$\#\ \text{terms} = \binom{N_{\max} + d}{d}$$

- $N_{\max} = 50, d = 4$ $\Rightarrow \sim 320000$ terms.

- Dense system too large for random sampling.

$$\begin{pmatrix} n(\vec{s}_0) \\ \vdots \\ n(\vec{s}_m) \end{pmatrix} = \begin{pmatrix} g_0(\vec{s}_0) & \cdots & g_n(\vec{s}_0) \\ \vdots & \cdots & \vdots \\ g_0(\vec{s}_m) & \cdots & g_n(\vec{s}_m) \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_m \end{pmatrix}$$

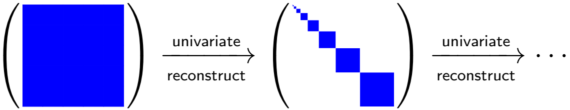
- Any sampling strategy is feasible with finite fields.

Sampling Algorithm

- Idea: univariate reconstruct to block diagonalize. [Peraro '16]

$$n_i^{\pm}(\vec{s}) = \sum_{j=N_{\text{min}}}^{N_{\text{max}}} n_{ij}^{\pm}(s_{23}, s_{45}, s_{51}) s_{34}(x)^j$$

- For Mandelstam ansatz s_{34} is a function of x.
  [Abreu, Dormans, Febres Cordero, Ita, B.P., Sotnikov '19]
- Blocks now simpler functions of fewer variables $\Rightarrow$ recurse.



- Multivariate Newton interpolation to compute $n_{i,j}^{\pm}$. [Peraro '16]
- Cluster capable: can predetermine evaluation points.

Partial Fractions

[Abreu, Dormans, Febres Cordero, Ita, B.P., Sotnikov '19]

- To avoid multiple $\mathbb{F}_p$, desire "small" rational numbers.
- Factorization controls residues of poles. $\Rightarrow$ "Leĭnartas" decomposition. [Leĭnartas '83]

$$r_i^{\pm}(\vec{s}) = \frac{n_i^{\pm}}{W^{\vec{q}_i}} = \sum_{\vec{\alpha}\in D_i} \frac{n_{i,\vec{\alpha}}^{\pm}}{W^{\vec{\alpha}}}.$$

- Write $r_i^{\pm}(\vec{s})$ as polynomial subject to constraints C_j.
- Leĭnartas is unique remainder Δ from Gröbner basis division.

$$C_j = W_j(\vec{s})\, Q_j - 1.$$

$$r_i^{\pm}(\vec{s}) = n_i^{\pm}(\vec{s})\, Q^{\vec{q}_i}\Big|_{C=0},$$
$$= B_{ij} C_j + \Delta_i\big|_{C=0}.$$

See also: [Zhang '12], [Mastrolia et al '12], [Smirnov, Smirnov '05]

Implementation and Results

[Abreu, Dormans, Ita, Febres Cordero, BP '18]

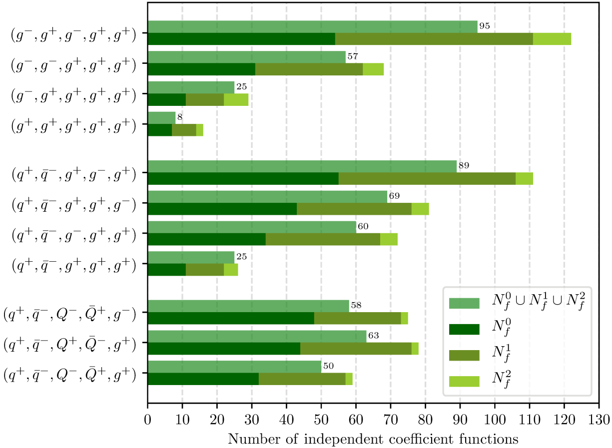
Calculation:

- Implemented in C++ framework - "Caravel".
- For $-+-++$: $\sim$ 3mins/eval, $\sim$100k required points.
- Calculation over single finite field of cardinality $O(2^{31})$.

Analytic expressions:

- One-loop amplitudes as linear combination of master integrals.
- Pentagon functions (euclidean) for two-loop remainders.
- All 5-parton amplitudes 10MB in total. (See arXiv ancillary.)

Coefficients Have Many Linear Dependencies



Conclusions

- We have computed the analytic form of the leading-colour 5-parton two-loop amplitudes.
- Alternative method for analytic computations: reconstruction from numerical samples over finite fields.
- Fruitful philosophy: simplify ansatz with physical principles.
- Exciting outlook for future computations.

Finite Fields Crash Course (I)

- Take integers $\mathbb{F}_p = \{0, \ldots, p-1\}$, where p is prime.
- Perform multiplication/addition/subtraction modulo p,

$$5+7 \mod 11 = 1 \qquad 5\times 7 \mod 11 = 2 \qquad 5-7 \mod 11 = 9.$$

- Every $a \in \mathbb{F}_p$ has a multiplicative inverse, a^{-1} so $\mathbb{F}_p$ is a field,

$$5^{-1} \mod 11 = 9.$$

- All rational operations possible, but (e.g.) no square roots.

Exact numerics with speed of integer operations.

Finite Fields Crash Course (II)

- From $\mathbb{Q}$ to $\mathbb{F}_p$

$$a = \frac{r}{s} \in \mathbb{Q} \quad \rightarrow \quad a \bmod p \equiv r \cdot (s^{-1} \bmod p) \bmod p$$

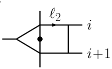
- Is there an inverse map?
- Consider $a = \frac{r}{s}$ where $r, s < \sqrt{n}$ and we know $a \mod n$.
- Find r and s, using a "Rational Reconstruction" algorithm. [Wang, '81]
- p not big enough? Use Chinese Remainder Theorem.

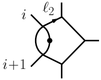
$$\{a \mod n_1,\ a \mod n_2,\ \ldots\} \xRightarrow{CRT} a \mod (n_1 \cdot n_2 \cdots)$$

Structure of Results

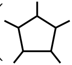
[Abreu, Dormans, Ita, Febres Cordero, BP '18]

- All-plus and single-minus remainders, no weight 3, 4.
- Revealed interesting cross-order identity.

$$2\sum_{i=1}^{5}\Bigg($$


$$(\mu_{22}) -$$


$$(\mu_{22})\Bigg)$$

$$= \sum_{i=1}^{5} \frac{(-s_{i,i+1})^{-\epsilon}}{\epsilon^2}\Bigg($$

$$(\mu^2)\Bigg) + O(\epsilon)$$

Compact irreducible weight 4:

- Part of $--+++$ with $\varepsilon(p_1, p_2, p_3, p_4)$ letters.
- Only even functions.
- Also simple in $-+-++$.

$$\mathcal{R}_{\varepsilon,4}^{--+++} = -2x_0x_1\left(f^1_{4,11} + f^5_{4,11}\right)C,$$

$$C = \frac{6+x_1(6(2+x_1)+5x_0(3+(3+2x_0)x_1))}{3(1+x_1)^3}.$$

Master/Surface Decomposition

[Abreu, Febres Cordero, Ita, Jaquier, B.P., Zeng '17]

- Surface terms vanish upon integration.

$$0 = \int \prod_{l=1,2} d^D \ell_l \frac{m_{\Gamma,i}}{\prod_{\text{props } k} \rho_k}, \quad \forall i \in Surf(\Gamma).$$

- Natural starting point given by IBPs.

$$0 = \int \prod_{l=1,2} d^D \ell_l \sum_{i=1,2} \frac{\partial}{\partial \ell_i^\nu} \left[\frac{u_i^\nu}{\prod_{\text{props } k} \rho_k} \right].$$

- Account for mismatch by controlling powers of ρ_j.
- (u, f) are numerators, so must be polynomial in ℓ_l.
  $\Rightarrow$ Algebraic geometry.

$$\sum_{i=1,2} u_i^\nu \frac{\partial}{\partial \ell_i^\nu} \rho_j = f_j \rho_j.$$

[Gluza, Kajda and Kosower '11]

Surface Terms and Syzygys

[Abreu, Febres Cordero, Ita, Jaquier, B.P., Zeng '17]

- Expand u_i^ν in ℓ_l and p_a.

$$u_i^\nu = u_{ia}^{\text{loop}}(\alpha,\rho)\ell_a^\nu + u_{ib}^{\text{ext}}(\alpha,\rho)p_b^\nu$$

- Use inverse propagators ρ and ISPs α as variables.

$$\ell_l^\mu \to (\rho_i, \alpha_j)$$

- Syzygy: takes form of polynomial null space.
- Use SINGULAR to compute generating set.

$$\begin{pmatrix} \ell_1^\nu \partial_\nu^1 \rho_1 & \cdots & p_1^\nu \partial_\nu^1 \rho_1 & \cdots & \rho_1 & 0 & \cdots \\ \ell_1^\nu \partial_\nu^1 \rho_2 & \cdots & p_1^\nu \partial_\nu^1 \rho_2 & \cdots & 0 & \rho_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} u_{11}^{\text{loop}} \\ \vdots \\ u_{11}^{\text{ext}} \\ \vdots \\ f_1 \\ \vdots \end{pmatrix} = 0$$

- Surface terms from polynomial combinations of generating vectors.

$$0 = \int \prod_{l=1,2} d^D \ell_l \frac{\sum_{i=1,2} \partial_\nu^i u_i^\nu - \sum_j f_j}{\prod_{\text{props } k} \rho_k}$$

Rational Kinematics

- Need to generate set of on-shell momenta which sum to zero.
- Twistor variables rationally parameterize spinor components:

[Hodges '09, Badger et al '13, Peraro '16]

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} \frac{1}{x_1} \\ 1 \end{pmatrix}, \quad \cdots$$

$$|1] = \begin{pmatrix} 1 \\ \frac{x_4 - x_5}{x_4} \end{pmatrix}, \quad |2] = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \quad |3] = \begin{pmatrix} x_1 x_4 \\ -x_1 \end{pmatrix}, \quad \cdots$$

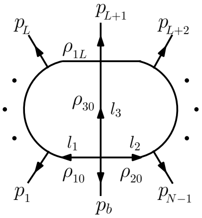
- Rational massless momenta, $p^\mu = \frac{1}{2} \langle p | \sigma^\mu | p]$.
- Using the $\mathrm{diag}(+,-,+,-)$ metric, all momenta and polarization vectors are real, reducing computational cost.

On-Shell Phase Spaces and Finite Fields

- Loop-momenta live on "on-shell variety".
- Would like parameterization for all topologies (general).
- Difficulty - need to solve quadratic equations over $\mathbb{F}_p$.

  Avoid with a trick.

- Strictly only need ℓ_l^μ to satisfy constraints during computation.



$$\ell_1^2 = (\ell_1 - p_1)^2 \quad = \cdots = 0$$
$$\ell_2^2 = (\ell_2 - p_{N-1})^2 = \cdots = 0$$
$$\ell_3^2 = 0.$$

Related work: [Peraro '16]

Algebraic Loop Momenta with $\mathbb{F}_p$

[Abreu, Febres-Cordero, Ita, B.P., Zeng '17]

- Aim: Algebraic momenta in controlled fashion.
- Use adapted coordinates.
- 3 quadratic constraints.
- Choose α independent, μ_{ij} dependent $\Rightarrow$ algebraic μ_k.

$$\ell_l^\mu \to (\rho_i, \alpha_j, \mu_{ij}),$$

$$\ell_l = \sum_{j\in\text{phys}} v_l^j r_l^j(\rho_l) + \sum_{j\in\text{trans}} \frac{n_l^j}{(n_l^j)^2}\alpha_l^j + \mu_l,$$

$$(\mu_l)^2 = \rho_{l0} - \sum_{v=0}^{3} \ell_l^\nu \ell_{l\,\nu}.$$

- Take μ_k as basis vectors - coefficients non-algebraic.
- Affects scalar product and state sums.

$$\ell_l^{(D-4)} = w_{l,1}\mu_1 + w_{l,2}\mu_2.$$

$$\ell_r \cdot \ell_s = \ell_r^4 \cdot \ell_s^4 - \sum_{i,j=1}^{2} w_i^r w_j^s \mu_{ij}.$$

$$\sum_{i=1}^{D_s-2} \epsilon_i^\mu \epsilon^{*\nu}_i \to \sum_{i=1}^{D_s} a_i^\mu a^{*\nu}_i$$

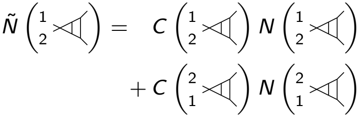
On-Shell Data - Colour Decomposition

[B.P., Ochirov '16]

- Starting point: coloured numerator ansatz.
- Constrain through coloured product of trees.
- Use KK independent tree colour decompositions.
- All numerators can be built from colour ordered trees.
- $\Rightarrow$ Colour decomposition of cuts projects to integrand.
- Natural generalization for all particle content.

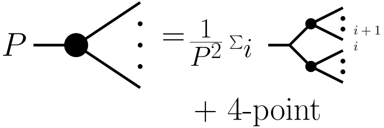
$$\mathcal{A}^l(\ell_l) = \sum_{\text{Topologies }\Gamma} \frac{1}{S_\Gamma} \frac{\tilde{N}_\Gamma(\ell_l)}{\prod_{i\in\Gamma} \rho_i}$$

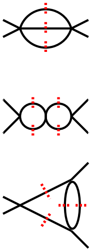
$$\mathcal{A}^0(1,\ldots,n) = \sum_{\sigma\in P_{n-2}} C(1,\sigma,n) A(1,\sigma,n).$$

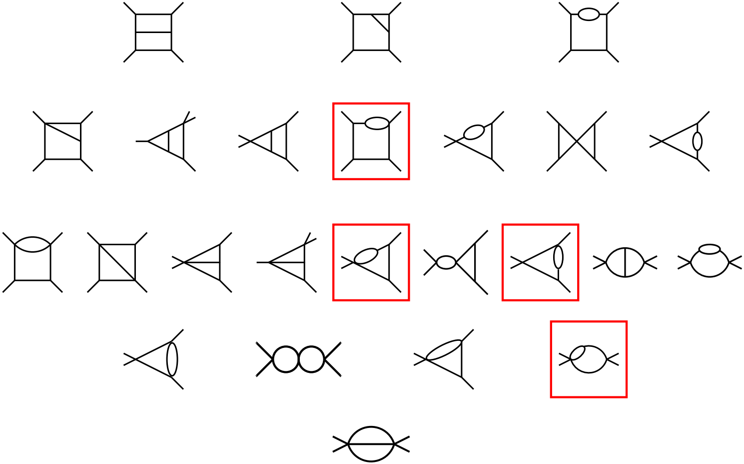


On-Shell Data - Integrand Residues

- Products of trees from colour-ordered off-shell recursion. [Berends, Giele '87]
- Works well with finite fields.
- Eases DimReg as dimension agnostic.
- Performant as can cache common sub-currents.
- Different spectra from different Feynman rules.





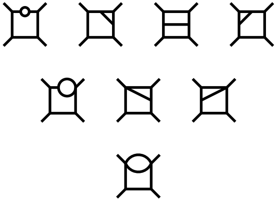


Some topologies have no associated product of trees.

Sub-leading Poles

[Abreu, Febres Cordero, Ita, Jaquier, B.P. '17]

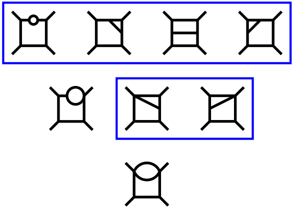
- Beyond one loop multiple poles can be associated to a given factorization limit.



Sub-leading Poles

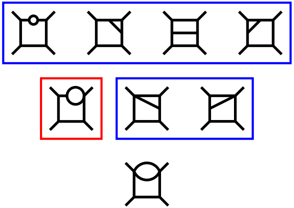
[Abreu, Febres Cordero, Ita, Jaquier, B.P. '17]

- Beyond one loop multiple poles can be associated to a given factorization limit.
- Visible leading poles given by product of trees.



Sub-leading Poles

[Abreu, Febres Cordero, Ita, Jaquier, B.P. '17]



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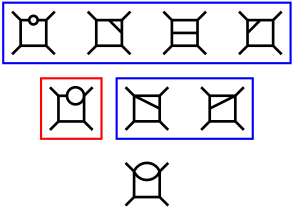
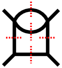


- Some numerators lack associated cut equation.
- Numerators determined from descendant cut equations.

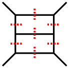
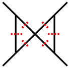


$$- \sum_{\substack{\Gamma \in \Delta \setminus \widetilde{\Delta} \\ \Gamma > \Gamma'}} \frac{N(\Gamma, \ell_l^{\Gamma'})}{\prod_{k \in P_\Gamma \setminus P_{\Gamma'}} \rho_k(\ell_l^{\Gamma'})} \, .$$

$$= N\Big($$  $$\Big) + \frac{1}{\rho} N\Big($$  $$\Big)$$

Regulator Dependence of Coefficients

- Integrand quadratic in $(D_s - 2)$, the number of spin states.
- One power of D_s from each separable component e.g.

 is linear in D_s  is quadratic in D_s

- Interpolate from evaluation at three D_s values. [Giele et al '08]
- As we perform IBP reduction, resulting coefficients are rational functions of D, i.e.

$$c(D) = \frac{P(D, s_{ij})}{Q(D)} = \frac{p_0(s_{ij}) + p_1(s_{ij})D + ... + p_i(s_{ij})D^i}{q_0 + q_1 D + ... + q_{j-1}D^{j-1} + D^j}.$$

- $Q(D)$ free of kinematics $\Rightarrow$ can determine once and for all.

Finding a Good Basis

- How to pick basis R_i?
- Start from independent set ordered by total degree.

$$r_i(\vec{s}) = \sum_{j \in \text{basis}} M_{ij} R_j(\vec{s}) \,.$$

- R_i denominators factorize.
- Coefficients share residues on denominator poles.
- Search for combinations of R_i where N_k factorizes w_j.
- Work on univariate slice $\vec{s(t)}$.

$$\sum_{i \in B} a_{i,k} R_i(\vec{s}) = \frac{N_k(\vec{s}, a_{i,k})}{\prod_{j \in A} w_j(\vec{s})^{q'_{kj}}} \,,$$

$$a_{i,k} \;:\; N_k(\vec{s}(t), a_{i,k}) \mid w_j(\vec{s}(t))$$

With all improvements, worst total degree drops to ~ 30.