UNIVERSITY OF COPENHAGEN

Scattering Equations and Factorization of Amplitudes

N. Emil J. Bjerrum-Bohr, Humberto Gomez & Andreas Helset

Niels Bohr Institute

based on **arXiv:1811.06024** & **arXiv:1902.02633**

Contact Information:
Blegdamsvej 17,
2100 Copenhagen Ø,
Niels Bohr Institute

Abstract

We obtain novel factorization identities for non-linear sigma model amplitudes using a new integrand in the CHY double-cover prescription. Using the factorization relations recursively, any non-linear sigma model amplitude can be expressed in terms of off-shell three-point amplitudes. The resulting expression is purely algebraic, and we do not have to solve any scattering equations. We also discuss boundary terms in BCFW recursion.

Double Cover Formalism

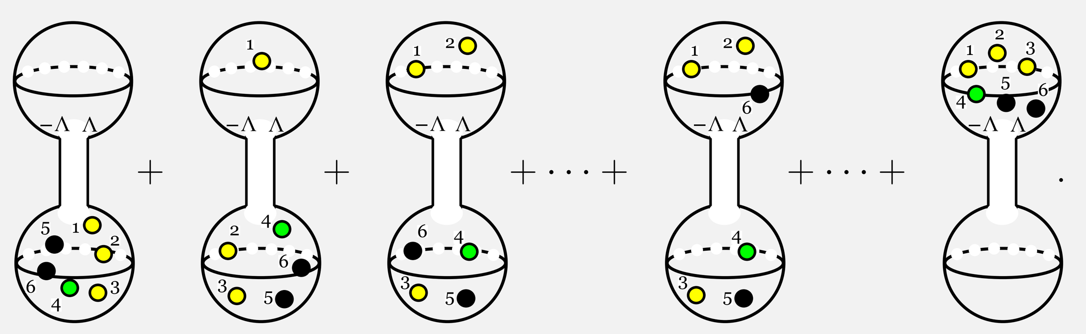
Cachazo, He and Yuan formulated a new approach to calculating S-matrix elements for various theories, called the CHY-formalism. Recently, the CHY-formalism was formulated by H. Gomez using a double cover. In the new formulation, the basic variables live in $\mathbb{CP}^2$, and not $\mathbb{CP}^1$ as in the original CHY-formalism. One advantage to introducing this extra machinery is that the double-cover scattering amplitudes naturally factorize into smaller $\mathbb{CP}^1$ pieces.

Single Cover	Double Cover
• Symmetry: SL(2,C)	• Symmetry: GL(2,C)
• Variables live in $\mathbb{CP}^1$	• Variables live on a curve in $\mathbb{CP}^2$
• Three punctures fixed	• Four punctures fixed
	• Naturally leads to factorization relations

The different configurations of punctures on the two Riemann sheets are summed over.

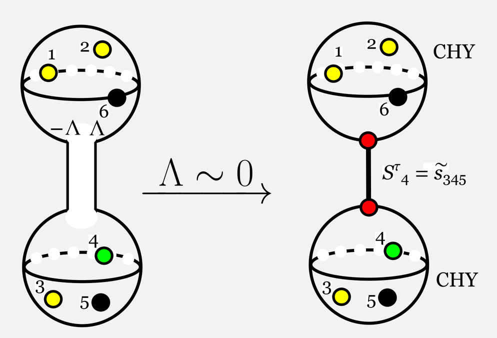
Double cover prescription

$A_6(1,2,3,4,5,6) =$



The double cover formalism naturally factorizes amplitudes into sub-amplitudes.

Factorization



(1)

A new CHY prescription for NLSM amplitudes

The flavor-ordered partial $U(N)$ NLSM amplitude in the scattering equation framework is given by

$$A_n(\alpha) = \int d\mu_n\, \mathrm{PT}(\alpha) \times (\mathrm{Pf}'\mathsf{A})^2, \quad \text{with} \quad d\mu_n \equiv (z_{ij}\, z_{jk}\, z_{ki})^2 \prod_{\substack{a=1\\ a\neq\{i,j,k\}}}^{n} \frac{dz_a}{S_a}, \tag{2}$$

$$\mathrm{PT}(\alpha) \equiv \frac{1}{z_{\alpha(1)\alpha(2)} z_{\alpha(2)\alpha(3)} \cdots z_{\alpha(n)\alpha(1)}}, \qquad \mathrm{Pf}'\mathsf{A} \equiv \frac{(-1)^{i+j}}{z_{ij}} \mathrm{Pf}[(\mathsf{A})^{ij}_{ij}],$$

where $(\alpha) = (\alpha(1), ..., \alpha(n))$ is a partial ordering. The $n \times n$ anti-symmetric matrix, A, is defined as,

$$\mathsf{A}_{ab} \equiv \begin{cases} \frac{s_{ab}}{z_{ab}} & \text{for } a \neq b, \\ 0 & \text{for } a = b. \end{cases} \tag{3}$$

By removing the same rows and columns in the half-integrand, the product of Pfaffians turns into a determinant. However, we proved that on the support of the scattering equations and massless condition, the following relation holds.

$$\mathrm{Pf}[(A)^{mp}_{mp}] \times \mathrm{Pf}[(A)^{pq}_{pq}] = \det[(A)^{mp}_{pq}], \tag{4}$$

with $m \neq p \neq q$. The determinant vanishes when the number of particles n is odd. Using the non-antisymmetric matrix, $(\mathsf{A})^{ij}_{jk}$, we define (with $i < j < k$)

CHY Integrand

$$A'_n(\alpha) = \int d\mu_n\, \mathrm{PT}(\alpha) \frac{(-1)^{i+k}}{z_{ij}\, z_{jk}} \det[(\mathsf{A})^{ij}_{jk}]. \tag{5}$$

By the previous identities,

$$A'_n(\alpha) = A_n(\alpha) \tag{6}$$

when all particles are on-shell. However, when there are off-shell particles, the identity only holds if the number of particles is even. When the number of particles is odd and there are off-shell particles, one has $A_n(\alpha) = 0$ while $A'_n(\alpha) \neq 0$.

Novel recursion relation for NLSM amplitudes

The double cover formalism naturally produces factorization relations. Matrix identities need to be applied to convert the factorized parts into lower-point NLSM amplitudes. An amplitude with an even number of external points factorizes into products of two amplitudes, where the sub-amplitudes have both an even or odd number of points.

Factorization Relation 1

$$\begin{aligned} A'_{2n}(P_1, P_2, P_3, 4, ..., 2n) = & \sum_{i=3}^{n} \frac{A'_{2(n-i+2)}(P_1, P_2, P_{3:2i-1}, 2i, ..., 2n) \times A'_{2(i-1)}(P_{2i:2}, P_3, 4, ..., 2i-1)}{s_{3:2i-1}} \\ & + \sum_{i=3}^{n+1} \frac{A'_{2(n-i+2)+1}(P_1, P_2, P_{3:2i-2}, 2i-1, ..., 2n) \times A'_{2(i-1)-1}(P_{2i-1:2}, P_3, 4, ..., 2i-2)}{P_1^2 - P_2^2 + P^2_{3:2i-2}} \\ & + (-1) \frac{A'_3(P_{4:1}, P_2, P_3) \times A'_{2n-1}(P_1, P_{23}, 4, \ldots, 2n)}{P^2_{4:1} - P_2^2 + P_3^2}. \end{aligned} \tag{7}$$

An amplitude with an odd number of external points factorizes into products of an even lower-point amplitude and an odd lower-point amplitude.

Factorization Relation 2

$$\begin{aligned} A'_{2n+1}(P_1, P_2, P_3, 4, ..., 2n+1) = & \left(P_1^2 - P_2^2 + P_3^2\right) \times \Bigg[\sum_{i=3}^{n+1} \left(\frac{1}{P_1^2 - P_2^2 + P^2_{3:2i-1}} \right) \\ & \times \frac{A'_{2(n-i+2)+1}(P_1, P_2, P_{3:2i-1}, 2i, ..., 2n+1) \times A'_{2(i-1)}(P_{2i:2}, P_3, 4, ..., 2i-1)}{s_{3:2i-1}} \\ & + \left(\frac{1}{P^2_{4:1} - P_2^2 + P_3^2} \right) \times \frac{A'_3(P_{4:1}, P_2, P_3) \times A'_{2n}(P_1, P_{23}, 4, \ldots, 2n+1)}{s_{4:1}} \Bigg]. \end{aligned} \tag{8}$$

Here P_i can be off-shell ($P_i^2 \neq 0$), with $P_{i:j} \equiv (k_i + \cdots + k_j)$ and $s_{i_1:i_p} \equiv \sum_{a\neq b, a,b,=1}^{p} k_{i_a} \cdot k_{i_b}$. The factorization relations give us a novel off-shell recursion relation, which have been checked against known results for up to twelve points (nineteen points for odd amplitudes). The main advantage with this relation is that it is purely algebraic, as any non-linear sigma model amplitude can be decomposed to off-shell three-point amplitudes (without solving any scattering equations).

BCFW recursion

We compare the new factorization relation with Britto-Cachazo-Feng-Witten (BCFW) recursion:

$$A_n(\mathbb{I}) = \sum_{i=3}^{n/2} \underset{P^2_{2i:2}(z)=0}{\mathrm{Res}} \left[A'_{n-2i+4}(1, 2, P_{3:2i-1}, 2i, ..., n) \frac{A'_{2i-2}(P_{2i:2}, 3, ..., 2i-1)}{z\, P^2_{2i:2}(z)} \right] - \underset{z=\infty}{\mathrm{Res}} \left[\frac{A'_n(\mathbb{I})(z)}{z} \right] \tag{9}$$

Only the even amplitudes, namely A'_{2q}, contribute to the physical residues. The residues, $P^2_{2i:2}(z) = 0$, in eq. (9) give extra nonphysical contributions, which cancel out when the same subamplitudes are evaluated at $\mathrm{Res}_{z=\infty}$. Therefore, the effective boundary contribution is just given by the subamplitudes with an odd number of particles, *i.e.*

Boundary Term

$$\begin{aligned} \underset{z=\infty}{\mathrm{Res}} \left[\frac{A'_n(\mathbb{I})(z)}{z} \right] = \partial_{\frac{1}{z}} \Bigg[& \sum_{i=3}^{n/2+1} A'_{n-2i+5}(1, 2, P_{3:2i-2}, 2i-1, ..., n) \times \frac{A'_{2i-3}(P_{2i-1:2}, 3, ..., 2i-2)}{z\, P^2_{2i-1:2}(z)} \\ & + \frac{A'_3(P_{4:1}, 2, 3) \times A'_{n-1}(P_{23}, 4, \ldots, n, 1)}{z\, P^2_{23}} \Bigg]_{z=\infty}. \end{aligned} \tag{10}$$

Conclusions

First of all, we have proposed a new CHY prescription for the $U(N)$ NLSM. In this new prescription, the kinematic matrix, $(\mathsf{A})^{ij}_{jk}$, is no longer anti-symmetric, which implies we can see a different inner structure of the model. Thus, using the double-cover representation found by H. Gomez, we have obtained new factorization relations. The factorization relations can be applied recursively, such that any non-linear sigma model amplitude can be decomposed in terms of off-shell three-point amplitudes, without solving any scattering equations.

Since the two new factorization formulas can be split among even and odd subamplitudes, for example $A'_{2q} \times A'_{2m}$ and $A'_{2q+1} \times A'_{2m+1}$ respectively, then, by using the BCFW method, we were are able to give a physical meaning to the odd subamplitudes as boundary contributions.

References

[1] N. E. J. Bjerrum-Bohr, H. Gomez, and A. Helset, Phys. Rev. **D99**, 045009 (2019).

[2] H. Gomez and A. Helset, JHEP **05**, 129 (2019).

Acknowledgements

This work was supported in part by the Danish National Research Foundation (DNRF91) and the Carlsberg Foundation.