Analytic Form of Planar Two-Loop Five-Parton QCD Amplitudes from Numerical Unitarity

S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, B. Page and V. Sotnikov

Abstract

We present the calculation of all two-loop five-parton helicity amplitudes [1] required for the computation of NNLO QCD corrections to the production of three jets at hadron colliders in the leading-color approximation. We obtain the analytic form of the amplitudes from exact numerical evaluations over finite fields. Their systematic simplification using multivariate partial-fraction decomposition leads to a particularly compact form.

Motivation

After a first phase of the LHC which culminated in the discovery of the Higgs boson by the ATLAS and CMS collaborations, the experiments have moved towards a phase of high-precision measurements that is probing the Standard Model at the percent level for several observables. Maximizing the impact of the new data requires theoretical predictions of similar precision. In practice, this means that NNLO QCD results are required. A crucial ingredient in obtaining theory predictions is the evaluation of two-loop amplitudes.

process	known	desired	
$pp \to 2\,\text{jets}$	$\text{N}^2\text{LO}_{\text{QCD}}$ $\text{NLO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$		
$pp \to 3\,\text{jets}$	NLO_{QCD}	$\text{N}^2\text{LO}_{\text{QCD}}$	α_s measurements
⋮	⋮	⋮	
$pp \to H + 2j$	$\text{NLO}_{\text{HEFT}}\otimes\text{LO}_{\text{QCD}}$ $\text{N}^3\text{LO}^{(\text{VBF}^*)}_{\text{QCD}}$ (incl.) $\text{N}^2\text{LO}^{(\text{VBF}^*)}_{\text{QCD}}$ $\text{NLO}^{(\text{VBF})}_{\text{EW}}$	$\text{N}^2\text{LO}_{\text{HEFT}}\otimes\text{NLO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$ $\text{N}^2\text{LO}_{\text{QCD}}+\text{NLO}^{(\text{VBF})}_{\text{EW}}$	VBF studies
⋮	⋮	⋮	
$pp \to V + j$	$\text{N}^2\text{LO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$	hadronic decays	
$pp \to V + 2j$	$\text{NLO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$ NLO_{EW}	$\text{N}^2\text{LO}_{\text{QCD}}$	QCD precision physics
$pp \to V + b\bar{b}$	NLO_{QCD}	$\text{N}^2\text{LO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$	$H \to b\bar{b}$ decays
⋮	⋮	⋮	
$pp \to \gamma\gamma + j$	NLO_{QCD}	$\text{N}^2\text{LO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$	H p_T spectrum

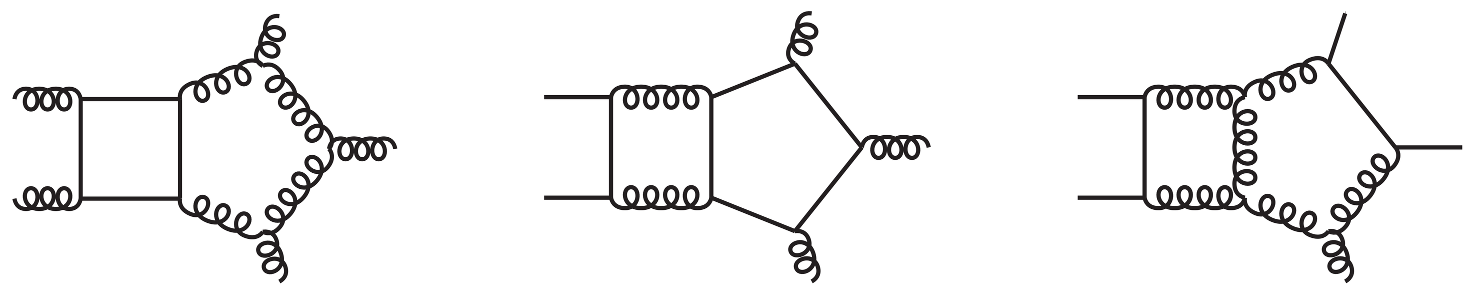
Les Houches wishlist 2017 [arXiv:1803.07977]

Two-loop Amplitudes

We consider all five-parton amplitudes at leading-color. They can be decomposed into partial amplitudes $\mathcal{A}$,

$$A(1,2,3,4,5)\big|_{\text{leading color}} = \sum_{\sigma\in S} C_\sigma\, \mathcal{A}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5))\,,$$

where all properties of each particle (parton type, momentum, helicity, etc.) are kept implicit. S denotes the symmetry group of the color decomposition for each parton type, and C_σ is a color factor.



Feynman diagrams

Tensor reduction [Passarino, Veltman '79]
Integration-by-parts identities
[Tkachov, Chetyrkin '81]

Sum of master integrals

$$\mathcal{A}^{(2)} = \sum_{\Gamma}\sum_{i\in M_\Gamma} c_{\Gamma,i}(\epsilon)\,\mathcal{I}_{\Gamma,i}(\epsilon)$$

Master integrals
direct computation, differential equations, numerical integration, etc.

Integrated amplitude

The partial amplitudes have a perturbative expansion in the QCD coupling α_s, and we denote by $\mathcal{A}^{(k)}$ a k-loop partial amplitude.

Computing the amplitude $\mathcal{A}^{(2)}$ means determining its decomposition into a linear combination of master integrals $\mathcal{I}_{\Gamma,i}$. The complete set of master integrals is determined by the kinematics only, and thus common to all five-parton amplitudes. For five-point massless amplitudes, all master integrals are known [2, 3] and the main task is to determine the coefficients $c_{\Gamma,i}$.

Difficulties of the standard approach:

- Very sensitive to the number of variables on which the amplitude depends
- Intermediate expressions become very large, mainly due to the complexity of the integration-by-parts (IBP) identities
- Obscures the simplicity of the final expressions

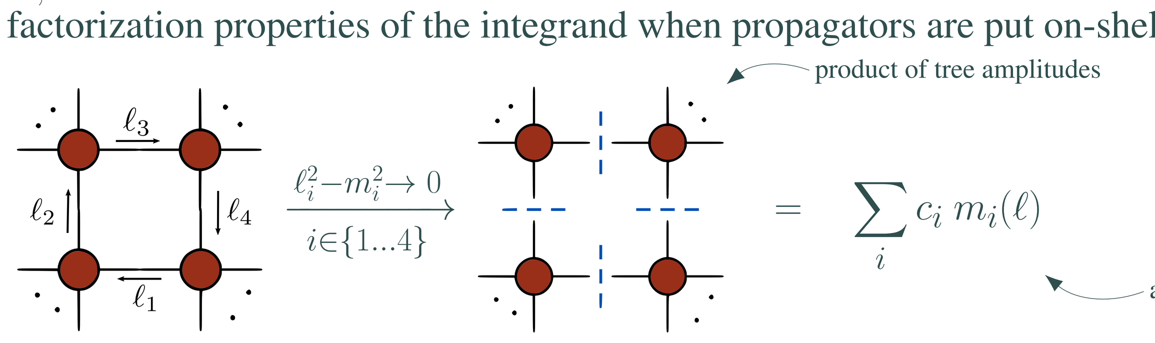
The amplitudes we are interested in depend on four dimensionless variables and the complexity of the required IBP tables makes this approach too inefficient for their calculation.

Two-Loop Numerical Unitarity

To reduce amplitudes to master integrals we employ the *two-loop numerical unitarity* method [4, 5]. First one constructs an ansatz for the integrand of the amplitude in terms of a set of tensor insertions [4], which are chosen so that they are separated into terms that integrate to master integrals ($i \in M_\Gamma$) and *surface terms* which integrate to zero ($i \in S_\Gamma$)

$$\mathcal{A}^{(2)}(\ell_l) = \sum_{\Gamma\in\Delta}\sum_{i\in M_\Gamma\cup S_\Gamma} c_{\Gamma,i}\frac{m_{\Gamma,i}(\ell_l)}{\prod_{j\in P_\Gamma}\rho_j}$$

The coefficients $c_{\Gamma,i}$ (for $i \in M_\Gamma$) are determined by solving a system of linear equations constructed by exploring the factorization properties of the integrand when propagators are put on-shell.



The required D-dimensional tree amplitudes can be evaluated efficiently through off-shell recursion. The numerical evaluations are performed using *finite-field arithmetic*, allowing to efficiently obtain exact results for rational phase-space points. This is key for the task of *functional reconstruction*.

Dimensional Regularization in a Numerical Framework

We regularize divergences of loop amplitudes by analytic continuation of the dimensionality of space-time D. Tensor representation are then formally infinite, which is in conflict with numerical evaluations.

Regularization
Compute in $D = 4 - 2\epsilon$ dimensions including *tensor* and *spinor* representation.

Numerical framework
Explicit *finite-dimensional* representations of all objects.

Finite-dimensional formulation
Understand how relevant information can be *extracted* from (a set of) computations in *integer* dimensions

1. Identify and separate the sources of D-dependence: D — loop momenta, enters through IBPs; D_s — tensor and spinor representations
2. Exploit the fact that infinite-dimensional representations must agree with finite when D_s is integer

$$\mathcal{M}_n(D_s) = \sum_{i=0}^{2}\mathcal{K}_{n,i}\,D_s^i, \qquad \mathcal{A} = \sum_n v_n\,\mathcal{M}_n$$

only 4-dim objects (definite *helicity*)
open $(D_s - 4)$ indices

3. To evaluate $\mathcal{M}(D_s)$ external states must be embedded into a higher-dimensional space. This is trivial for vector particles. For fermions however there is an ambiguity, and we make use of a *tensor decomposition* in a carefully constructed basis to identify relevant tensor structures.

Reconstruction of Analytics from Numerical Samples

Any multivariate rational function $f(\mathbf{x}) = \frac{\sum_\alpha n_\alpha \mathbf{x}^\alpha}{\sum_\beta d_\beta \mathbf{x}^\beta}$ can be reconstructed from its numerical evaluations [6]. Numerical unitarity provides the means to evaluate the master-integral coefficients directly, which are rational functions of external kinematics (and ϵ). Thus it is possible to obtain the analytic results for the master-integral coefficients by evaluating them numerically on a sufficient number of points. The whole reconstruction procedure can be carried out in a finite field leading to major performance boost.

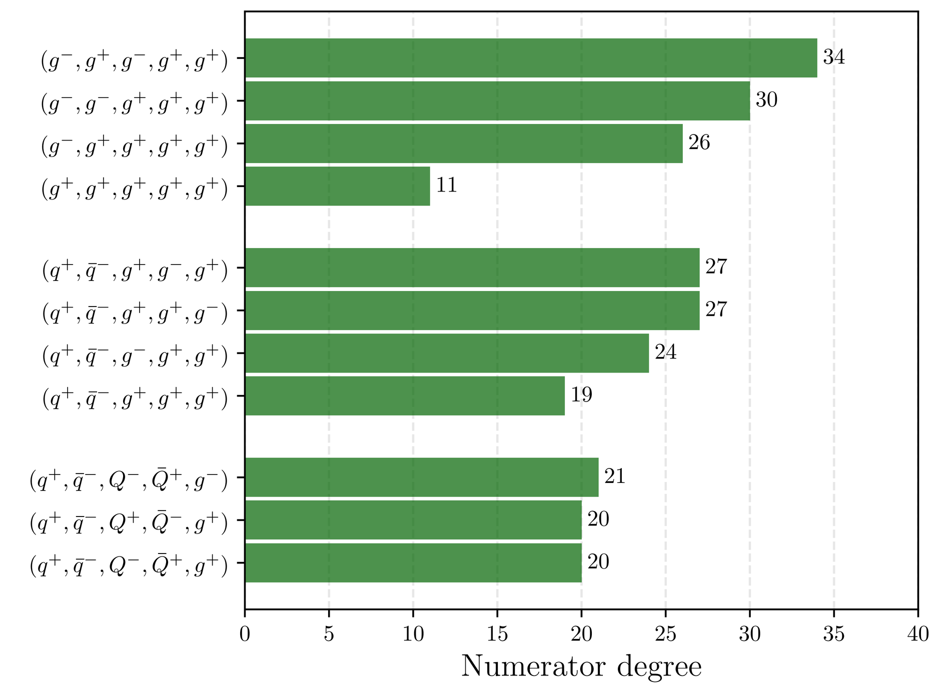
The number of points required for reconstruction is given by $\sim \binom{R+n}{n}$, with the total degree R and the number of variables n. For two-loop five-parton amplitudes, the degree of coefficients $c_{\Gamma,i}$ is in general too high and makes reconstruction challenging. However a number of physically-motivated ideas can help to find much simpler objects to reconstruct.

- Expand the amplitude around $\epsilon = 0$ and choose a "good" basis $h_i \in B$ of special functions [3]

$$\mathcal{A}^{(2)} = \sum_{i\in B}\sum_{k=-4}^{0}\epsilon^k\,\tilde{c}_{k,i}h_i + \mathcal{O}(\epsilon)$$

- Remove redundant information from lower loops, consider a *finite remainder*

$$\mathcal{R}^{(2)} \equiv \mathcal{A}^{(2)} - \mathbf{I}^{(1)}\mathcal{A}^{(1)} - \mathbf{I}^{(2)}\mathcal{A}^{(0)} = \sum_{i\in B} r_i\,h_i$$



- Denominators are connected to properties of integral functions: can be determined from a reconstruction on a one-dimensional curve, which requires only a few evaluations

$$r_i(\mathbf{x}) = \frac{n_i(\mathbf{x})}{W(\mathbf{x})^{\vec{q}_i}},$$

where $W(\mathbf{x})$ are *letters* of the alphabet, corresponding to branch points of integral functions.

- The choice of variables has a large impact [1].

Results

The two-loop D-dimensional unitarity methods and analytic reconstruction algorithms are implemented in a C++ library `Caravel`, which supports both floating point and finite-field evaluations.

- We obtained analytical expressions through reconstruction for **33 amplitudes**.
- For the most complex amplitude ~ 95000 phase-space points are used, with average evaluation time of ~ 4.5 min/point.
- Systematic application of multivariate partial fractioning to simplify the obtained expressions leads to *very compact* expressions: *total size* ~ 10 *Mb* (uncompressed)

The analytic expressions for all amplitudes are publicly available as ancillary files of [1].

References

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