The black hole S-matrix and classical potential

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Background & Aim

- For practical purposes, a black hole (BH) can be considered as a stable point particle.
- BHs in asymptotically flat (3+1)d are specified by mass, charge, and spin. These set of quantum numbers also specify point particles in QFT.
- Classical objects have continuous internal degrees of freedom (spin), while point particles in QFT have discrete internal degrees of freedom.
- Smaller degrees of freedom implies manipulation of point particles in QFT is easier than that of classical objects.
- Can BHs (and other classical objects) be mapped to point particles of QFT?
- What are QFT observables (amplitudes) of mapped classical objects?
- Can QFT methods be used to produce classical quantities?

Set-up

- Approximate BHs as stable point particles; no emission or absorption.
- Use massive spinor-helicity variables [1,2] as kinematical variables.
- Match with one-particle effective theory to fix three-point amplitudes.
- Construct rational part of graviton Compton amplitude by gluing two three-point amplitudes.
- Use HCL [3] to compute leading PN order classical potential (Fourier transform of non-relativistic COM amplitude).

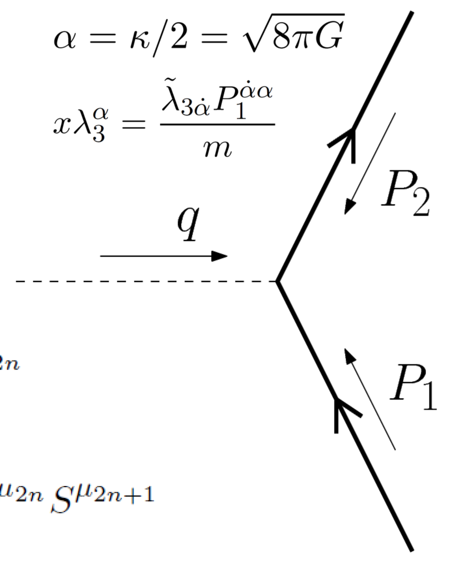
Amplitudes

Three-point amplitudes

The one-particle EFT Lagrangian is [4]

$$S = \int d\sigma \left\{ -m\sqrt{u^2} - \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu} + L_{SI}\left[u^\mu, S_{\mu\nu}, g_{\mu\nu}(y^\mu)\right] \right\}$$

$$\alpha = \kappa/2 = \sqrt{8\pi G} \qquad x\lambda_3^\alpha = \frac{\tilde\lambda_{3\dot\alpha} P_1^{\dot\alpha\alpha}}{m}$$



where (C_{S^n} are Wilson coefficients)

$$L_{SI} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{ES^{2n}}}{m^{2n-1}} D_{\mu_{2n}} \cdots D_{\mu_3} \frac{E_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \cdots S^{\mu_{2n-1}} S^{\mu_{2n}}$$
$$+ \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{BS^{2n+1}}}{m^{2n}} D_{\mu_{2n+1}} \cdots D_{\mu_3} \frac{B_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \cdots S^{\mu_{2n}} S^{\mu_{2n+1}}$$

Insert polarisation tensors to obtain three-point amplitude ($C_{S^0} = C_{S^1} = 1$)

$$M_s^{2\eta} = \epsilon^*(\mathbf{2}) \left[\sum_{n=0}^{\infty} \frac{\kappa m x^{2\eta}}{2} \frac{C_{S^n}}{n!} \left(-\eta \frac{q \cdot S}{m} \right)^n \right] \epsilon(\mathbf{1}) \qquad n_{a,b}^s \equiv \frac{1}{m^{2s}} \binom{s}{a} \binom{s}{b}$$

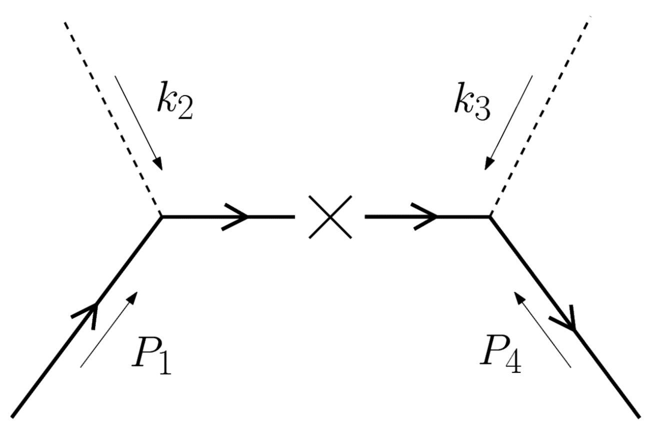
$$= \sum_{a+b \leq s} \frac{\kappa m x^{2\eta}}{2} C_{S^{a+b}} n_{a,b}^s \langle \mathbf{21} \rangle^{s-a} \left(-\eta \frac{x \langle \mathbf{2} q \rangle \langle q \mathbf{1} \rangle}{2m} \right)^a [\mathbf{21}]^{s-b} \left(\eta \frac{[\mathbf{2} q][q \mathbf{1}]}{2mx} \right)^b$$

Matching this to minimal coupling yields $C_{S^n} = 1 + \mathcal{O}(1/s)$. Kerr BHs have unity Wilson coefficients $C_{S^n} = 1$ [4], i.e. minimal coupling corresponds to Kerr BHs in $s \to \infty$ limit.

$$M^{+2}_{s,min} = \frac{\kappa m x^2}{2} \frac{\langle \mathbf{21} \rangle^{2s}}{m^{2s}}, \quad M^{-2}_{s,min} = \frac{\kappa m x^{-2}}{2} \frac{[\mathbf{21}]^{2s}}{m^{2s}} \Longrightarrow C_{S^n} = 1 + \mathcal{O}(1/s)$$

Four-point amplitudes

The four-point amplitude is obtained by gluing two minimal coupling three-point amplitudes [1,2]. Consider the Compton amplitude $M(\mathbf{1}^s, 2^{+2}, 3^{-2}, \mathbf{4}^s)$



$$-\frac{\langle 3|p_1|2]^{4-2s}}{(s-m^2)(u-m^2)tM_{Pl}^2} (\langle \mathbf{43} \rangle [\mathbf{12}] + \langle \mathbf{13} \rangle [\mathbf{42}])^{2s}$$

The expression has spurious poles for $s > 2$, which must be cancelled [1,2]. Resolution of spurious poles yield the following expression [1].

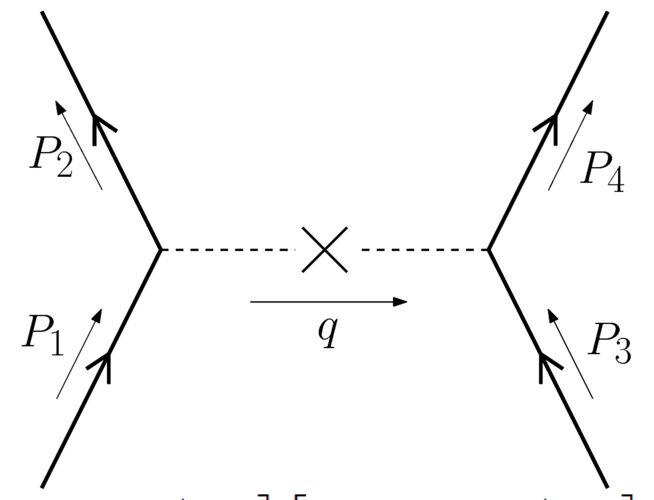
$$-\frac{\langle 3|p_1|2]^4}{(s-m^2)(u-m^2)tM_{pl}^2} \mathcal{F}^{2s}$$
$$+ \frac{2s\langle 3|p_1|2]^3}{t(s-m^2)} \frac{\langle \mathbf{34} \rangle [\mathbf{21}]}{2m^2 M_{pl}^2} \mathcal{F}^{2s-1} - \frac{2s\langle 3|p_4|2]^3}{t(u-m^2)} \frac{\langle \mathbf{31} \rangle [\mathbf{24}]}{2m^2 M_{pl}^2} \mathcal{F}^{2s-1}$$
$$+ \left\{ \frac{\langle 3|p_1|2]^2 \langle \mathbf{34} \rangle^2 [\mathbf{21}]^2}{4m^4(s-m^2)M_{pl}^2} \left[\sum_{r=2}^{2s} \binom{2s}{r} \mathcal{F}^{2s-r} \left(\frac{-\langle \mathbf{43} \rangle [32] \langle \mathbf{21} \rangle}{4m^3} - \frac{[\mathbf{43}] \langle 32 \rangle [\mathbf{21}]}{4m^3} \right)^{r-2} \right] \right.$$
$$\left. + \frac{\langle 3|p_1|2]^2 \langle \mathbf{31} \rangle^2 [\mathbf{24}]^2}{4m^4(u-m^2)M_{pl}^2} \left[\sum_{r=2}^{2s} \binom{2s}{r} (-1)^r \mathcal{F}^{2s-r} \left(\frac{-\langle \mathbf{13} \rangle [32] \langle \mathbf{24} \rangle}{4m^3} - \frac{[\mathbf{13}] \langle 32 \rangle [\mathbf{24}]}{4m^3} \right)^{r-2} \right] \right\}$$
$$- \frac{Poly + Poly_{[23]} + Poly_{\langle 23 \rangle}}{tM_{pl}^2}$$

For definitions of the variables, consult [1]. Similar construction exists for non-minimal couplings (non-BHs) [1].

Classical potential

1 PM (G^1) leading PN ($^v/_c$) all orders in spin

Computing 1 PM leading PN all orders in spin classical potential is simplified in the *holomorphic classical limit* (HCL) [1,3,5] as t-channel residue computation. Glue two three-point amplitudes to find the residue.



$$\text{Res}_t = A^+_{3a} A^-_{3b} + A^-_{3a} A^+_{3b}$$
$$= (-1)^{s_a+s_b} \sum_{i=0}^{2s_a} \sum_{j=0}^{2s_b} \alpha^2 m_a^2 m_b^2 \frac{C_{S_a^i} C_{S_b^j}}{i!j!} \left((x_1 \bar{x}_3)^2 \left[\epsilon^*(\mathbf{2}) \left(\frac{q \cdot S_a}{m_a} \right)^i \epsilon(\mathbf{1}) \right] \left[\epsilon^*(\mathbf{4}) \left(\frac{q \cdot S_b}{m_b} \right)^j \epsilon(\mathbf{3}) \right] \right.$$
$$\left. + (\bar{x}_1 x_3)^2 \left[\epsilon^*(\mathbf{2}) \left(-\frac{q \cdot S_a}{m_a} \right)^i \epsilon(\mathbf{1}) \right] \left[\epsilon^*(\mathbf{4}) \left(-\frac{q \cdot S_b}{m_b} \right)^j \epsilon(\mathbf{3}) \right] \right)$$

Particles 1 and 2 do not live in the same Hilbert space, so additional terms are generated from matching the Hilbert spaces.

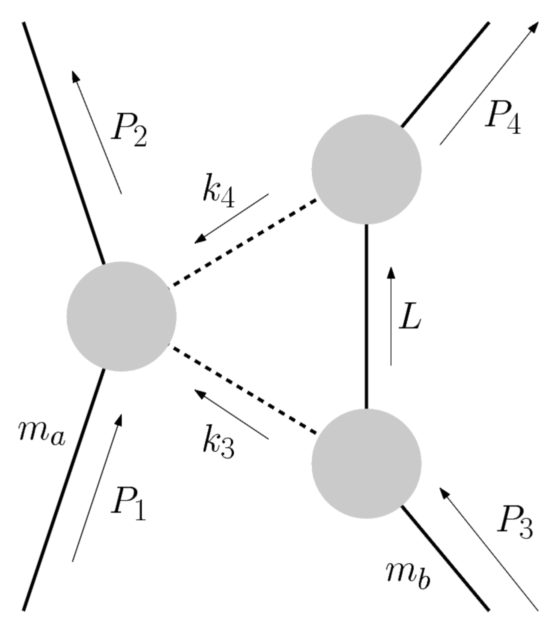
$$\epsilon^*(P_2)\epsilon(P_1) = \epsilon^*(P_a) \left[\mathbb{1} - \left(\frac{\vec{p}_a}{m_a} \times \frac{\vec{S}_a}{m_a} \right) \cdot \frac{-i\vec{q}}{2} + \cdots \right] \epsilon(P_a) \quad \Big| \quad \epsilon^*(P_4)\epsilon(P_3) = \epsilon^*(P_b) \left[\mathbb{1} + \left(\frac{\vec{p}_b}{m_b} \times \frac{\vec{S}_b}{m_b} \right) \cdot \frac{-i\vec{q}}{2} + \cdots \right] \epsilon(P_b)$$

Adding up the contributions [5],

$$V_{cl} = -\sum_{m,n=0}^{\infty} \frac{(-1)^n C_{S_a^{2n-m}} C_{S_b^m}}{(2n-m)!m!} \left(\frac{\vec{S}_a}{m_a} \cdot \vec{\nabla} \right)^{2n-m} \left(\frac{\vec{S}_b}{m_b} \cdot \vec{\nabla} \right)^m \frac{Gm_a m_b}{r}$$
$$- \sum_{m,n=0}^{\infty} \frac{2(-1)^{m+n} C_{S_a^{2m+1}} C_{S_b^{2n}}}{(2m+1)!(2n)!} \left[\left(\frac{\vec{p}_a}{m_a} - \frac{\vec{p}_b}{m_b} \right) \times \frac{\vec{S}_a}{m_a} \cdot \vec{\nabla} \right] \left(\frac{\vec{S}_a}{m_a} \cdot \vec{\nabla} \right)^{2m} \left(\frac{\vec{S}_b}{m_b} \cdot \vec{\nabla} \right)^{2n} \frac{Gm_a m_b}{r}$$
$$- \sum_{m,n=0}^{\infty} \frac{2(-1)^{m+n} C_{S_a^{2m}} C_{S_b^{2n+1}}}{(2m)!(2n+1)!} \left[\left(\frac{\vec{p}_a}{m_a} - \frac{\vec{p}_b}{m_b} \right) \times \frac{\vec{S}_b}{m_b} \cdot \vec{\nabla} \right] \left(\frac{\vec{S}_a}{m_a} \cdot \vec{\nabla} \right)^{2m} \left(\frac{\vec{S}_b}{m_b} \cdot \vec{\nabla} \right)^{2n} \frac{Gm_a m_b}{r}$$
$$+ \sum_{m,n=0}^{\infty} \frac{(-1)^n C_{S_a^{2n-m}} C_{S_b^m}}{2(2n-m)!m!} \left(\left[\frac{\vec{p}_a}{m_a} \times \frac{\vec{S}_a}{m_a} - \frac{\vec{p}_b}{m_b} \times \frac{\vec{S}_b}{m_b} \right] \cdot \vec{\nabla} \right) \left(\frac{\vec{S}_a}{m_a} \cdot \vec{\nabla} \right)^{2n-m} \left(\frac{\vec{S}_b}{m_b} \cdot \vec{\nabla} \right)^m \frac{Gm_a m_b}{r}$$

Construction for 2 PM (G^2)

Only the triangle topology contributes to the classical potential [1,3]. Computation of 1-loop can be simplified by working in the anti-chiral basis [3], where the following pattern can be observed in the HCL [1].



$$\sum_{i=0}^{2s_a} \sum_{j=0}^{2s_b} N_{i,j}^{s_a,s_b} \tilde{A}_{i,j} (Sp_a)^i (Sp_b)^j$$

$$N_{i,j}^{s_a,s_b} = \frac{(2s_a)!}{(2s_a-i)!} \frac{(2s_b)!}{(2s_b-j)!} \qquad Sp_a = \frac{|\hat\lambda][\hat\lambda|}{m_a}, \quad Sp_b = \frac{|\lambda][\lambda|}{m_b}$$

The $\tilde{A}_{i,j}$ coefficients are independent of s_a and s_b, so they can be used to construct classical spin $s \to \infty$ answers. Above answer can be mapped to polarisation tensor basis, the basis classical potential is written in [1,5].

$$\sum_{i=0}^{2s_a} \sum_{j=0}^{2s_b} B_{i,j} \left[\epsilon^*(\mathbf{2}) \left(\frac{q \cdot S_a}{m_a} \right)^i \epsilon(\mathbf{1}) \right] \left[\epsilon^*(\mathbf{4}) \left(-\frac{q \cdot S_b}{m_b} \right)^j \epsilon(\mathbf{3}) \right] \quad \Big| \quad \sum_{i,j=0}^{\infty} \frac{B_{i,j}}{2^{i+j}} x^i y^j \overset{(s_a, s_b \to \infty \text{ limit})}{=} e^{-x/2-y/2} \sum_{i,j=0}^{\infty} \tilde{A}_{i,j} x^i y^j$$

This can be generalised to non-minimal couplings (non-BHs) [5].

References

[1] M-Z. Chung, Y-T. Huang, J-W. Kim, S. Lee, *The simplest massive S-matrix: from minimal coupling to Black Holes*, JHEP 1904 (2019) 156 [arXiv:1812.08752]

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[4] M. Levi, *Effective Field Theories of Post-Newtonian Gravity: A comprehensive review*, [arXiv:1807.01699]

[5] M-Z. Chung, Y-T. Huang, J-W. Kim, *From one-particle EFT to Gravitational S-matrix*, to appear