Kinematic Algebra for the Next-to-MHV Sector of Yang-Mills Theory

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Based on 1906.10683 and work in progress with G. Chen, H. Johansson and F. Teng

Introduction

The duality between color and kinematics, also known as the Bern-Carrasco-Johansson (BCJ) duality, controls the perturbative structures of various gauge theories and some scalar effective theories.

- The color-kinematics duality at tree level in pure Yang-Mills theory refers to the decomposition of the n-point amplitude

$$\mathcal{A}_n^{\text{tree}} = g^{n-2} \sum_{i=1}^{(2n-5)!!} \frac{n_i c_i}{D_i}$$

where the color factor and the kinematic numerator obey the same algebraic relations

$$c_i + c_j + c_k = 0 \quad \Rightarrow \quad n_i + n_j + n_k = 0$$

The color-kinematics duality suggests that there exists a hidden Lie algebra that computes the kinematic numerators.

- The existence of a Lie algebra may imply new symmetries.

  In this case, however, not all presumed generators might correspond to symmetries.
- The kinematic algebra remains enigmatic in general.

  An exception where the generators and the structure constants of the kinematic algebra are known is the self-dual Yang-Mills.

  Other attempts at characterizing the kinematic algebra have been piloted.

  It is expected to be an infinite-dimensional Lie algebra involving continuous parameters. Auxiliary degrees of freedom may be necessary to make the Feynman rules both cubic and obey the duality.

Our proposal: introduce tensor currents and realize the kinematic algebra in the NMHV sector of Yang-Mills

Review of Kinematic Algebra in Self-Dual Yang-Mills

In the self-dual sector, the kinematic algebra is studied by Monteiro and O'Connell, where the spectrum contains only the plus helicity gauge field.

- The generators of the kinematic algebra are labeled by momenta $L_p = e^{-ip\cdot x}(p_u \partial_w - p_w \partial_u)$
- The commutator is given by

$$[L_{p_1}, L_{p_2}] = iX(p_1, p_2) L_{p_1+p_2}$$

where the structure constant is captured by the truncated interaction $X(p_1, p_2) = p_{1w}p_{2u} - p_{2w}p_{1u}$

- All numerators in the self-dual sector are given by the nested commutators.

e.g. the four-point s-channel numerator:

$$n_s = \operatorname{Tr}\left([[L_{p_1}, L_{p_2}], L_{p_3}] L_{p_4}\right) = X(p_1, p_2) X(p_1+p_2, p_3) \delta^4(p_1+p_2+p_3+p_4)$$

The Jacobi relations of the s-, t-, and u-channels is automatically obeyed.

- Beyond the self-dual sector, labeling the generators with only momenta is insufficient.

Set-up and Building Blocks

- Polarization power is defined as the power of $(\varepsilon_i \cdot \varepsilon_j)$

  We choose the reference momenta of the polarization vectors such that only $(\varepsilon_i^+ \cdot \varepsilon_j^-)$ is nonzero.

  Hence the maximum number of the polarization power is the helicity degree.

$$\begin{array}{lcll} \text{polarization power one:} & (\varepsilon_i\cdot\varepsilon_j)\prod(\varepsilon_k\cdot p_l) & \longrightarrow & \text{MHV} \\ \text{polarization power two:} & (\varepsilon_{i_1}\cdot\varepsilon_{j_1})(\varepsilon_{i_2}\cdot\varepsilon_{j_2})(p_{i_3}\cdot p_{j_3})\prod(\varepsilon_k\cdot p_l) & \longrightarrow & \text{NMHV} \end{array}$$

  Polarization power two $\longleftrightarrow$ first order in the Mandelstam variables

- Abstract tensor current denoted as $J^{(w)}_{\mathfrak{a}_1\otimes\mathfrak{a}_2\otimes\cdots\otimes\mathfrak{a}_m}(p)$

Labels carried by a tensor current

- Tensor label: $\mathfrak{a}_1 \otimes \mathfrak{a}_2 \otimes \cdots \otimes \mathfrak{a}_m$, constructed from polarization vectors and momenta
- Momentum: p
- Discrete label: w, allowing the existence of multiple types of tensor currents with identical tensor labels and momenta

Rank = the number of components in the tensor label (rank-1 current is called vector current)

Dimension = 1+ the number of momenta in the tensor label

- The tensor current is a generalization of the tensorial contraction $\bar{v}(q)\not{\mathfrak{a}}_1\not{\mathfrak{a}}_2\cdots\not{\mathfrak{a}}_m u(p)$, where q is soft.

  When p is on-shell ($p^2 = 0\,, \not{p}u = 0$), it can be replaced by

$$J^{(w)}_{\mathfrak{a}_1\otimes\mathfrak{a}_2\otimes\cdots\otimes\mathfrak{a}_m}(p) \to \bar{v}(q)\not{\mathfrak{a}}_1\not{\mathfrak{a}}_2\cdots\not{\mathfrak{a}}_m u(p)$$

Assumed properties

- Linearity: $J^{(w)}_{\cdots\otimes(x\mathfrak{a}_i+y\mathfrak{a}'_i)\otimes\cdots}(p) = xJ^{(w)}_{\cdots\otimes\mathfrak{a}_i\otimes\cdots}(p) + yJ^{(w)}_{\cdots\otimes\mathfrak{a}'_i\otimes\cdots}(p)$
- "Clifford algebra": $J^{(w)}_{\cdots\mathfrak{a}_i\otimes\mathfrak{a}_j\otimes\mathfrak{a}_k\otimes\mathfrak{a}_l\otimes\cdots}(p) + J^{(w)}_{\cdots\mathfrak{a}_i\otimes\mathfrak{a}_k\otimes\mathfrak{a}_j\otimes\mathfrak{a}_l\otimes\cdots}(p) = (2\mathfrak{a}_j\cdot\mathfrak{a}_k)J^{(w)}_{\cdots\otimes\mathfrak{a}_i\otimes\mathfrak{a}_l\otimes\cdots}(p)$

- The tensor currents can be viewed as the generators of the algebra.
- Fusion Product describes the interaction of tensor currents.

  We consider cubic interactions only, namely two-to-one fusion products denoted as $J \star J$.

  Fusion products are a certain kind of multiplication of the generators.

Bi-linearity

$$(aJ_1 + bJ_2)\star(cJ_3) = acJ_1\star J_3 + bcJ_2\star J_3$$

Framework for Realizing the Algebra

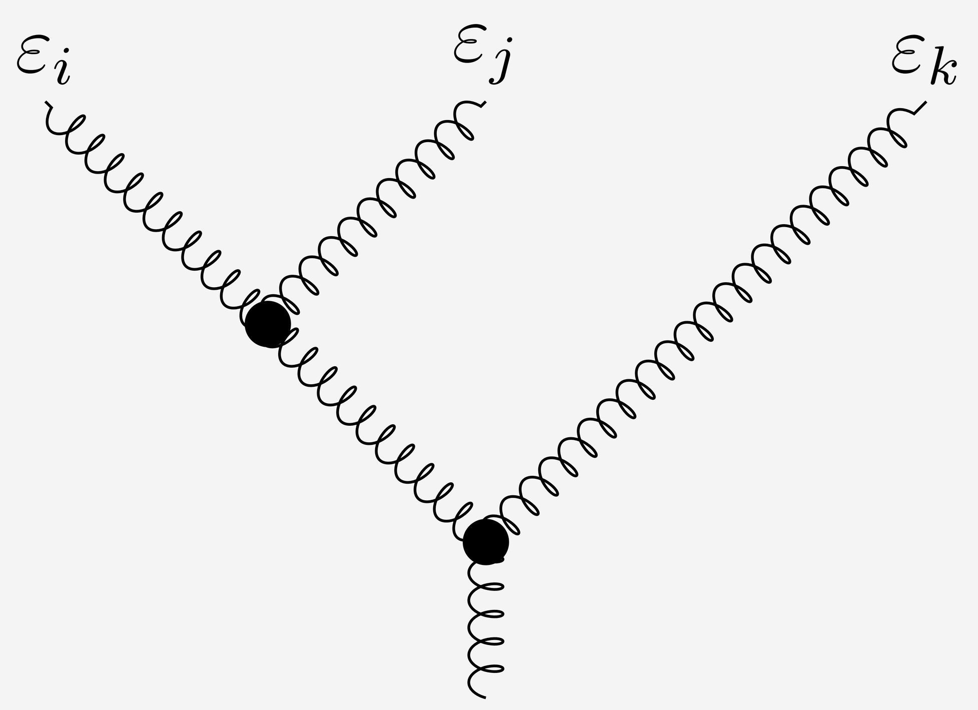
Color factor given by $[[T_i, T_j], T_k]$ controlled by the SU(N) Lie algebra

Color-kinematics duality suggests:

Kinematic numerator also given by nested commutators $[[J_{\varepsilon_i}, J_{\varepsilon_j}], J_{\varepsilon_k}]$

Goal of our construction:

To realize an algebra at the level of commutators of tensor currents which computes the kinematic numerators and manifests the Jacobi identities



- Commutator in terms of fusion products

$$[J_1, J_2] = J_1\star J_2 - J_2\star J_1$$

- Kinematic numerator can be constructed from nested commutator in the following sense.

  The nest commutator yields

$$N(12\cdots n) = \left[\cdots[[[J_{\varepsilon_1}, J_{\varepsilon_2}], J_{\varepsilon_3}]\cdots], J_{\varepsilon_{n-1}}\right] = N^{\mathsf{T}}_{\mathsf{b}}(12\cdots n) + N^{\mathsf{V}}_{\mathsf{b}}(12\cdots n)$$

  where we separate the result into a tensor part and a vector part. The subscript indicates that the expressions depend on a basis choice for the tensor currents.

The tensor and vector parts above must satisfy the following conditions

- Null space condition: $\displaystyle\sum_{\sigma\in S_{n-2}} m(\rho|\sigma)N^{\mathsf{T}}_{\mathsf{b}}(\sigma) = 0$
- Gauge invariance condition: $\displaystyle\sum_{\sigma\in S_{n-2}} m(\rho|\sigma)N^{\mathsf{V}}_{\mathsf{b}}(\sigma)\big|_{\varepsilon_i\to p_i} = 0$

where $m(\rho|\sigma)$ denotes the propagator matrix.

When the two conditions are satisfied, we can drop the tensor part and obtain the standard BCJ numerators by $n_{\mathsf{b}}(\sigma) = N^{\mathsf{T}}_{\mathsf{b}}|_{J_a\to a\cdot\varepsilon_n}$

- In the bi-scalar NMHV subsector, where we only consider terms proportional to $\varepsilon_1\cdot\varepsilon_n$, the commutators reduce to ordered multiplications. For details, please see 1906.10683.

Construction of Commutators

Associativity Assumption

We assume that all the fusion products satisfy the associative condition

$$J_i\star(J_j\star J_k) = (J_i\star J_j)\star J_k$$

- The associativity assumption allows us to write the commutators in terms of ordered fusion products.

  e.g.

$$[[J_{\varepsilon_1}, J_{\varepsilon_2}], J_{\varepsilon_3}] = J_{\varepsilon_1}\star J_{\varepsilon_2}\star J_{\varepsilon_3} - J_{\varepsilon_2}\star J_{\varepsilon_1}\star J_{\varepsilon_3} - J_{\varepsilon_3}\star J_{\varepsilon_1}\star J_{\varepsilon_2} + J_{\varepsilon_3}\star J_{\varepsilon_2}\star J_{\varepsilon_1}$$

  Crossing symmetry: the commutator reduces to one ordered fusion products and index permutations.

  Hence, we only need to consider multiplications from the right with a vector current corresponding to a physical external field.

  Solving the null space and gauge invariance, now with the numerators given by the commutators, we realize an algebra that closes at eight points.

- The algebra is an extension of the one given in 1906.10683 and computes the amplitudes in the NMHV sector.
- The crossing symmetry in leg 1 is restored, whereas the last leg remains special (on-shell).
- The kinematic Jacobi relations are made manifest.

I'm happy to show the algebra to the interested, which is slightly too technical to fit in the poster.

Discussions

- The validity of the associativity assumption:

  The kinematic algebra (if exists at all) is expected to be local and associative, like its color counterpart.

  At the moment, we have only preliminary evidences that it is possible to define the fusion products involving two tensor currents and multiplications from the left that are consistent with the assumption.

  Further investigations are needed in this direction.
- Higher polarization power (N^{k}MHV):

  To explore these sectors, amplitudes beyond multiplicity-eight are in principle needed.

  Alternatively, one might consider three- and four-point amplitudes between (un)physical states.
- Lagrangian description:

  It is possible that the commutators can be lifted to cubic interactions of off-shell fields.

I'm happy to discuss anything related!

Selected References:

[1] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. D78 (2008) 085011, [0805.3993]

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[3] M. Chiodaroli, M. Gunaydin, H. Johansson and R. Roiban, JHEP 07 (2017) 002, [1703.00421]

[4] Z. Bern and T. Dennen, Phys. Rev. Lett.107 (2011) 081601, [1103.0312]