Cluster Adjacency of Scattering Amplitudes

Jack Foster

University of Southampton

Analytics of Amplitudes

The analytics of scattering amplitudes have been studied in great detail for several years leading to new developments in calculating amplitudes. One such development is the **Steinmann relations**. These relations dictate the order in which one can perform branch cuts of loop amplitudes.

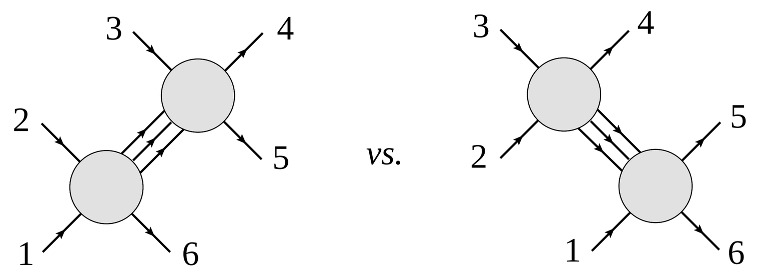

Figure 1: A cut in s_{345} followed by one is s_{234} is forbidden as they overlap [1].

This feature demonstrates the factorisation of loops into products of trees on poles but more importantly it provides a condition to impose when calculating amplitudes. It is conceivable that understanding more properties, such as the Steinmann relations, could allow us to write down amplitudes without calculation.

Cluster Algebras

Cluster algebras were first introduced in [2] and connected to amplitudes in [3]. It was observed that the poles of n-point amplitudes in planar $\mathcal{N}=4$ SYM are given by cluster $\mathcal{A}$-coordinates on the kinematic configuration space $\mathrm{Conf}_n(\mathbb{P}^3)$. A cluster algebra consists of different clusters which one can move between via an operation called *mutation*. Each cluster can be represented by a quiver diagram where each node or $\mathcal{A}$-coordinate is connected to at least one other node by arrows which encode how each node mutates.

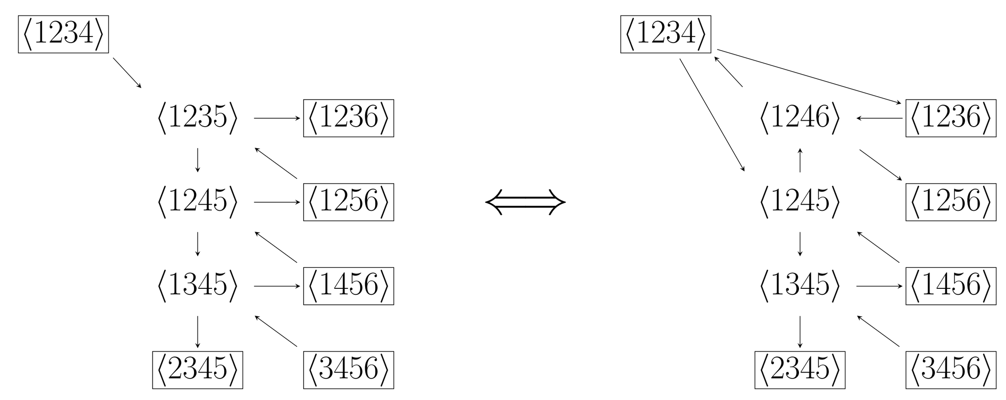

Figure 2: A mutation on ⟨1235⟩ in the $\mathrm{Conf}_6(\mathbb{P}^3)$ initial cluster.

Cluster algebras have a geometric interpretation in their associahedra; clusters correspond to vertices, mutations between them are given by edges, and the nodes are faces.

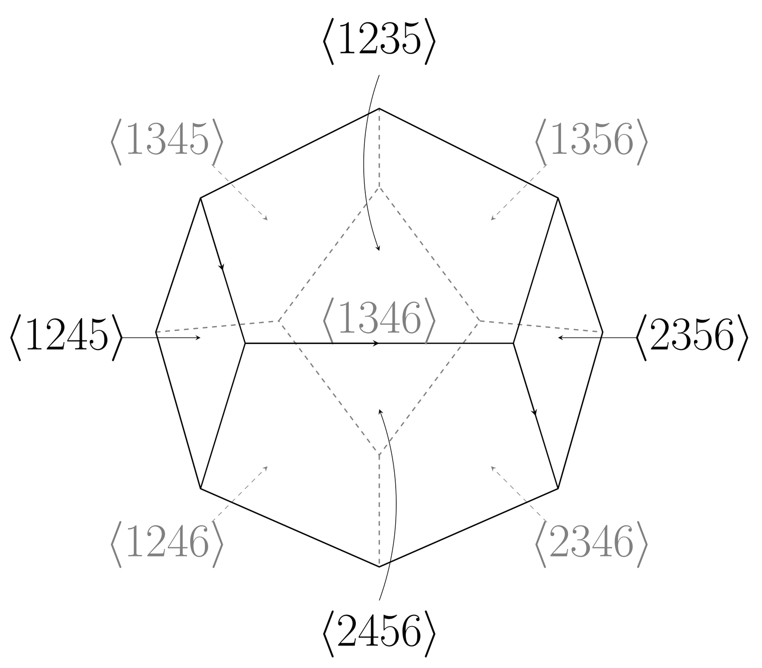

Figure 3: The $\mathrm{Conf}_6(\mathbb{P}^3)$ associahedron.

We can construct *neighbour sets* for each unfrozen node, being the set of all nodes which appear in at least one cluster with a given node. We then define **cluster adjacency** [4]:

> Consecutive residues/branch cuts can only be taken around poles which appear in a cluster together i.e. their faces meet at an edge in the associahedron.

We analysed all available six and seven-point amplitudes in SYM and found no counterexamples. However certain pairs allowed by cluster adjacency did not appear in any available seven-point amplitudes.

Adjacency Table for Heptagon Alphabet

	a_{1i}	a_{2i}	a_{3i}	a_{4i}	a_{5i}	a_{6i}
a_{11}	● ○ ○ ◆ ◆ ○ ○	◆ ◆ ○ ● ◆ ● ○	◆ ○ ● ◆ ● ○ ◆	● ○ ◆ ○ ○ ◆ ○	● ○ ◆ ○ ○ ◆ ○	◇ ◆ ○ ○ ○ ○ ◆
a_{21}	◆ ○ ● ◆ ● ○ ◆	● ○ ● ◆ ◆ ● ○	◆ ○ ◆ ◆ ○ ◆ ●	◆ ○ ◆ ○ ● ◆ ○	○ ◆ ● ○ ◆ ○ ◆	○ ◆ ○ ● ○ ◆ ○
a_{31}	◆ ◆ ○ ● ◆ ● ○	◆ ● ◆ ○ ◆ ◆ ○	● ○ ● ◆ ◆ ● ○	◆ ○ ◆ ● ○ ◆ ○	○ ◆ ○ ◆ ○ ● ◆	○ ○ ◆ ○ ● ○ ◆
a_{41}	● ○ ◆ ○ ○ ◆ ○	◆ ○ ◆ ● ○ ◆ ○	◆ ○ ◆ ○ ● ◆ ○	● ◇ ◆ ○ ○ ◆ ◇	● ○ ○ ○ ○ ○ ○	◇ ◆ ◇ ○ ○ ◇ ◆
a_{51}	● ○ ◆ ○ ○ ◆ ○	○ ◆ ○ ◆ ○ ● ◆	○ ◆ ● ○ ◆ ○ ◆	● ○ ○ ○ ○ ○ ○	● ◇ ◆ ○ ○ ◆ ◇	◇ ◆ ◇ ○ ○ ◇ ◆
a_{61}	◇ ◆ ○ ○ ○ ○ ◆	○ ○ ◆ ○ ● ○ ◆	○ ◆ ○ ● ○ ◆ ○	◇ ◆ ◇ ○ ○ ◇ ◆	◇ ◆ ◇ ○ ○ ◇ ◆	● ◇ ○ ◇ ◇ ○ ◇

◆: There are clusters where the coordinates appear together connected by an arrow.
●: There are clusters where the coordinates appear together but they are never connected.
○: The coordinates never appear in the same cluster but there is a mutation that links them.
◇: The coordinates do not appear in the same cluster nor there is a mutation that links them.

Loop Integrals

In [4] we calculate the symbol of a seven-point, three-loop, massless pentaladder in SYM using the Steinmann cluster bootstrap [5] along with differential equations [6].

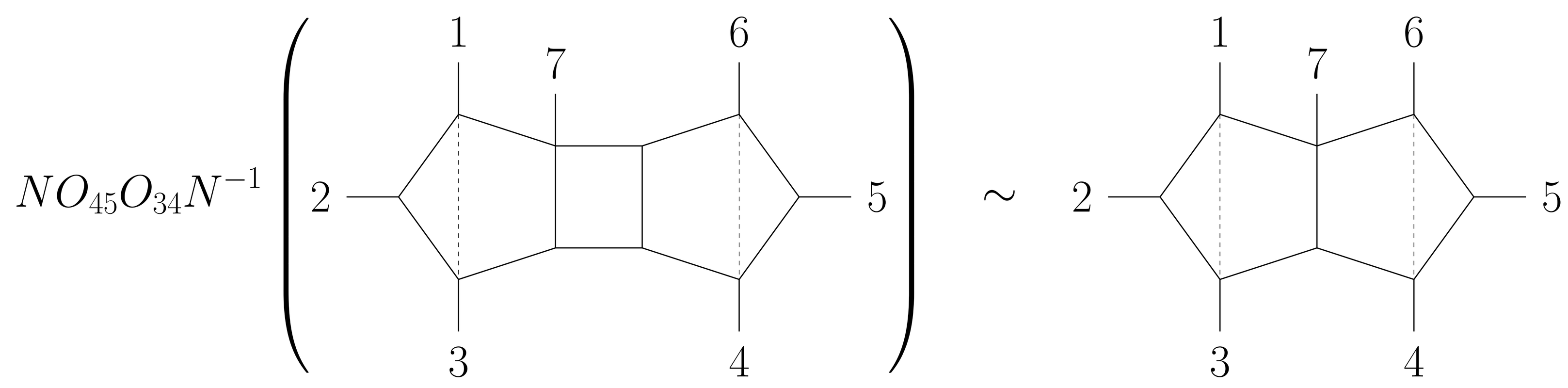

Figure 4: A differential relation relating seven-point, three and two-loop integrals.

where $O_{ij} = Z_i \cdot \frac{\partial}{\partial Z_j}$ and N is a numerator to ensure the three-loop integral is finite and has unit leading singularity.
Analysing the symbol of the three-loop integral, we found one of the missing pairs mentioned above thus supporting our cluster adjacency conjecture.

BCFW and NMHV

Cluster adjacency was first defined in terms of branch cuts of loop amplitudes however it is also applicable to trees. The BCFW expansion of the n-point NMHV tree-amplitude of SYM (divided by the MHV tree) is given by

$$\mathcal{A}^{tree}_{n,1} = \sum_{1<i<j<n} [1 i i+1 j j+1]$$

where the R-invariant $[ijklm]$ is given by

$$[ijklm] \sim \frac{1}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle}.$$

The poles of R-invariants are given by cluster $\mathcal{A}$-coordinates and in [7] we prove that *all* R-invariants are cluster adjacent i.e. all poles of a given R-invariant can be found together within a cluster. Hence we identify R-invariants with clusters in $\mathrm{Conf}_6(\mathbb{P}^3)$ and subalgebras in $\mathrm{Conf}_n(\mathbb{P}^3)$ for $n>6$. An example of [12345] in $\mathrm{Conf}_7(\mathbb{P}^3)$ is given below.

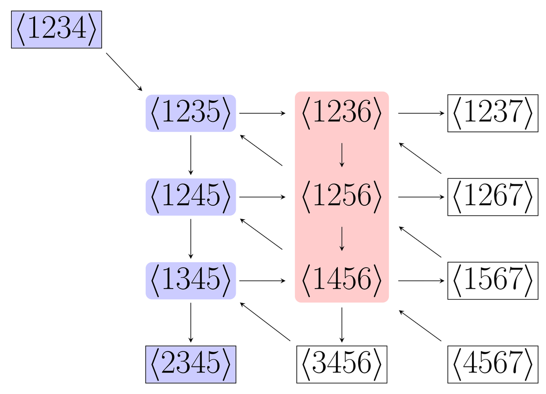

Figure 5: A cluster containing the poles (blue) of [12345] in $\mathrm{Conf}_7(\mathbb{P}^3)$. The nodes highlighted in red generate an A_3 subalgebra.

We also conjecture that *all* BCFW terms are cluster adjacent by analysing the BCFW expansion of tree-amplitudes up to eight-point N^{2}MHV finding no counterexample.
R-invariants are a type of Yangian invariant and recently it was conjectured, through use of the Sklyanin bracket, that every Yangian invariant in SYM is cluster adjacent [8].

Seven-Point, Four-Loop, NMHV Amplitude in $\mathcal{N}=4$ SYM

In [9] we construct a manifestly cluster adjacent form of the seven-point, four-loop planar NMHV amplitude in SYM

$$\mathcal{E}^{(4)}_7 = e^{(4)}_{12}(12) + e^{(4)}_{13}(13) + e^{(4)}_{14}(14) + \text{cyclic}$$

where $(12) = [34567]$ and e_{ij} are cluster adjacent symbols such that their final entries are cluster adjacent with all the poles of the R-invariant they multiply. These symbols, however, are only integrable on identities that the R-invariants satisfy

$$(12) - (13) + (14) - (15) + (16) - (17) = 0 \quad \& \text{ cyclic.}$$

On these identities the amplitude takes the form given in (2.13) of [9].
Cluster adjacency along with integrability on the identities, no spurious poles, and good collinear limits fixed our ansatz coefficients completely. Writing the amplitude in a manifestly cluster adjacent, albeit non-integrable, form also produced it in a more compact way than when manifestly integrable.

Conclusion

Amplitudes are rich in analytic structure and understanding this structure is key to understanding amplitudes. Cluster adjacency controls this analytic structure and allows us to calculate previously uncalculated amplitudes. It also provides algebraic and geometric interpretations of poles of amplitudes and how they talk to each other.

References

[1] S. Caron-Huot, L. J. Dixon, A. McLeod and M. von Hippel, *Bootstrapping a Five-Loop Amplitude Using Steinmann Relations*, *Phys. Rev. Lett.* **117** (2016) 241601 [1609.00669].
[2] S. Fomin and A. Zelevinsky, *Cluster algebras i: Foundations*, *JOURNAL OF THE AMERICAN MATHEMATICAL SOCIETY* (2001) 497.
[3] J. Golden, A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, *Motivic Amplitudes and Cluster Coordinates*, *JHEP* **01** (2014) 091 [1305.1617].
[4] J. Drummond, J. Foster and Ö. Gürdoğan, *Cluster Adjacency Properties of Scattering Amplitudes in N = 4 Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **120** (2018) 161601 [1710.10953].
[5] L. J. Dixon, J. Drummond, T. Harrington, A. J. McLeod, G. Papathanasiou and M. Spradlin, *Heptagons from the Steinmann Cluster Bootstrap*, *JHEP* **02** (2017) 137 [1612.08976].
[6] J. M. Drummond, J. M. Henn and J. Trnka, *New differential equations for on-shell loop integrals*, *JHEP* **04** (2011) 083 [1010.3679].
[7] J. Drummond, J. Foster and Ö. Gürdoğan, *Cluster adjacency beyond MHV*, *JHEP* **03** (2019) 086 [1810.08149].
[8] J. Mago, A. Schreiber, M. Spradlin and A. Volovich, *Yangian Invariants and Cluster Adjacency in N=4 Yang-Mills*, 1906.10682.
[9] J. Drummond, J. Foster, Ö. Gürdoğan and G. Papathanasiou, *Cluster adjacency and the four-loop NMHV heptagon*, *JHEP* **03** (2019) 087 [1812.04640].

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