On the Poincaré dual of Feynman integrals

Simon Caron-Huot and Andrzej Pokraka*

McGill University

Abstract

Recently, it has been shown that the intersection numbers of twisted differential forms are equivalent to tree-level string theory and CHY scattering amplitudes [1]. Intersections have also been used to algebraically generate integration by parts identities and differential equations for maximal cut Feynman integrals [2]. We extend these ideas to non-maximally cut Feynman integrals by introducing the Poincaré dual of a Feynman integral.

Feynman integrals

- We define a Feynman integral in $4-2\epsilon$ dimensions to be

$$G(\{D_i\},\{\nu_i\}) = \int u(\ell_1,\dots,\ell_n)\phi(\{D_i\},\{\nu_i\}) \quad (1)$$

$$\phi(\{D_i\},\{\nu_i\}) := \frac{d^4\ell_1\wedge\cdots\wedge d\ell_{1,\perp}^2\wedge d(\ell_{1,\perp}\cdot\ell_{2,\perp})\wedge\cdots}{D_1^{\nu_1}\cdots D_n^{\nu_n}} \quad (2)$$

where u is a multi-valued 0-form. At 1-loop, ϕ is a 5-form with $u=(\ell_\perp^2)^{-1-\epsilon}$, and at 2-loops, ϕ is a 11-form with $u=(\ell_{1,\perp}^2\ell_{2,\perp}^2-\ell_{1,\perp}\cdot\ell_{2,\perp})^{-1-\epsilon}$. (While u is multi-valued, ϕ is single valued.)
- Similar to Baikov representation, but applies simultaneously to all topologies.

Twisted and relative cohomology

- In a nutshell, the homology (H_n) and cohomology (H^n) of a manifold X classify the possible integration contours and integrands

Figure: Integration contours in $H_1(\mathrm{CP}^1\backslash\{a,b,c\},\{d,e\})$ and $H_1(\mathrm{CP}^1\backslash\{a,b,c\},\{d,e\};u)$, where the red points $\{a,b,c\}$, branch points of u, have been removed from the manifold and the blue points $\{d,e\}$ are allowed boundaries.

- We are interested in Feynman integrals modulo integration by parts (IBP)

$$\int d(u\phi)=0 \implies \phi\simeq\phi+\nabla_\omega\xi \quad (3)$$

where $\nabla_\omega=d+\omega\wedge$ is the covariant derivative and $\omega=d\log u$. That is, Feynman integrals live in the twisted cohomology

$$H^n(X\backslash\{uD=0\};u)=\frac{\{\phi\in\{n\text{-forms on }X\backslash\{uD=0\}\}|\nabla_\omega\phi=0\}}{\nabla_\omega\{(n-1)\text{-forms on }X\backslash\{uD=0\}\}}. \quad (4)$$

Exhibits **branch cuts** at $u=0$ and propagator **poles** at $D=0$.
- n-forms are dual to integration cycles, but also to n-forms via (simpler!) intersection pairing
- Example: let $u=x^s(1-x)^t$ and $\phi=\frac{dx}{x(1-x)}\in H^1(\mathrm{CP}^1\backslash\{0,1,\infty\};u)$, see [1]. Intersection is:

$$\langle\phi^\vee|\phi\rangle\equiv\int(u^{-1}\phi^\vee)\wedge(u\phi_{\text{reg}}) \quad (5)$$

where $|_{\text{reg}}$ maps ϕ (or $\phi^\vee$) to its compactly supported version

$$\phi_{\text{reg}}\equiv\phi-\nabla_\omega\Big(\sum_{\text{singular points}}\theta(\varepsilon^2-|x-x_i|^2)\nabla_\omega^{-1}\phi\Big). \quad (6)$$

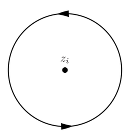
$\phi\wedge\phi_{\text{reg}}\neq0$ only because $d\theta\supset d\bar{x}$. Intersections are **algebraic**: $u^{-1}\langle\frac{dx}{x(1-x)}|\frac{dx}{x(1-x)}\rangle_u=\frac{1}{s}+\frac{1}{t}$.

- Poincaré duals of Feynman integrals live in $4+2\epsilon$ dimension (ϵ is flipped to cancel branch cuts in $\phi^\vee\wedge\phi$), and must have zeros (*relative cohomology*) to cancel propagator poles [3]:

$$\phi^\vee\in H^n(X\backslash\{u=0\},D=0;u^{-1}). \quad (7)$$

Idea: intersection pairing between the two H^n's is nondegenerate, thus physical equivalence.
- Dual forms **localize on cuts**, because dual H^n would be trivial without $D=0$ boundaries:

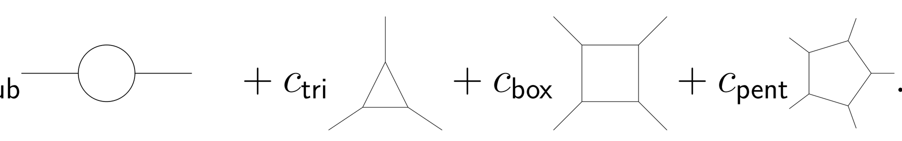
$$d\theta(\varepsilon^2-|z-z_i|^2)\simeq$$



Application to generalized unitarity

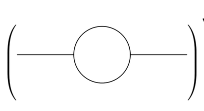
- **Dual** forms automatically extract integral coefficients mod IBP
- Example: well-known that bubbles, triangles, boxes and pentagons form a basis for all 1-loop integrals (say for massless propagators and 4D external momenta):

$$I=c_{\text{bub}}\;(\ldots)+c_{\text{tri}}\;(\ldots)+c_{\text{box}}\;(\ldots)+c_{\text{pent}}\;(\ldots). \quad (8)$$

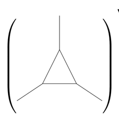


By definition, $c_{\text{bub}}=\langle\phi_{\text{bub}}^\vee|\phi_I\rangle$, etc.

- Dual forms rigorously live on cuts. Schematically:

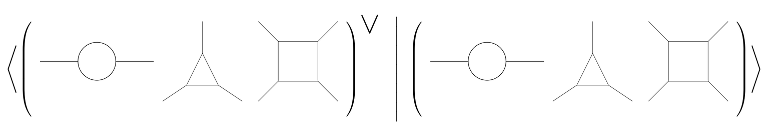


$$(\ldots)^\vee=\frac{d\theta(D_1)\wedge d\theta(D_2)\wedge d\ell_2\wedge d\ell_3\wedge d\ell_4}{(R^2-\ell_2^2-\ell_3^2-\ell_4^2)^{2-\epsilon}} \quad (9)$$



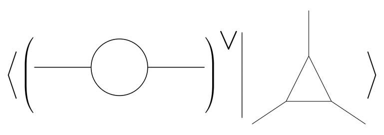
$$(\ldots)^\vee=\frac{d\theta(D_1)\wedge d\theta(D_2)\wedge d\theta(D_3)\wedge d\ell_3\wedge d\ell_4}{(R^2-\ell_3^2-\ell_4^2)^{1-\epsilon}},\quad\text{etc.} \quad (10)$$

- Straightforward to show that dual forms are orthonormal to scalar basis:



$$\langle(\ldots)^\vee|(\ldots)\rangle=1_{3\times3}. \quad (11)$$

For example, intersecting the bubble dual with the triangle we find



$$\langle(\ldots)^\vee|(\ldots)\rangle\sim\Big\langle\frac{d\ell_2\wedge d\ell_3\wedge d\ell_4}{(R^2-\ell_2^2-\ell_3^2-\ell_4^2)^{2-\epsilon}}\Big|\frac{d\ell_2\wedge d\ell_3\wedge d\ell_4}{D_3\,(R^2-\ell_2^2-\ell_3^2-\ell_4^2)^{1+\epsilon}}\Big\rangle \quad (12)$$

$$\sim\Big\langle\frac{d\ell_2}{(R^2-\ell_2^2)^{1-\epsilon}}\Big|\frac{d\ell_2}{D_3(\ell_2)\,(R^2-\ell_2^2)^{\epsilon}}\Big\rangle=0 \quad (13)$$

where we have used *fiber integration* to trivially integrate out perpendicular components ℓ_3,ℓ_4. Since bubble dual is regular where triangle has simple poles and vice versa, intersection vanishes.

Generalized Unitarity: features

- At finite ϵ, this computes integral coefficients in terms of Laurent series of **cut amplitudes** near $\ell_\perp\to0$ (and possibly $\ell_\perp=\infty$).
- Smooth $\epsilon\to0$ limit: reproduces formulas for bubble, triangle and box coefficients in $d=4$.
- At 2-loops, we checked (by pen and paper!) that they automatically yield the master contours of [4] for double box heptacut.
- **Thus, dual forms provide a framework for generalized unitarity above 1-loop and also for sub-topologies.**

Preview: Differential equations for the massless 2-loop double box

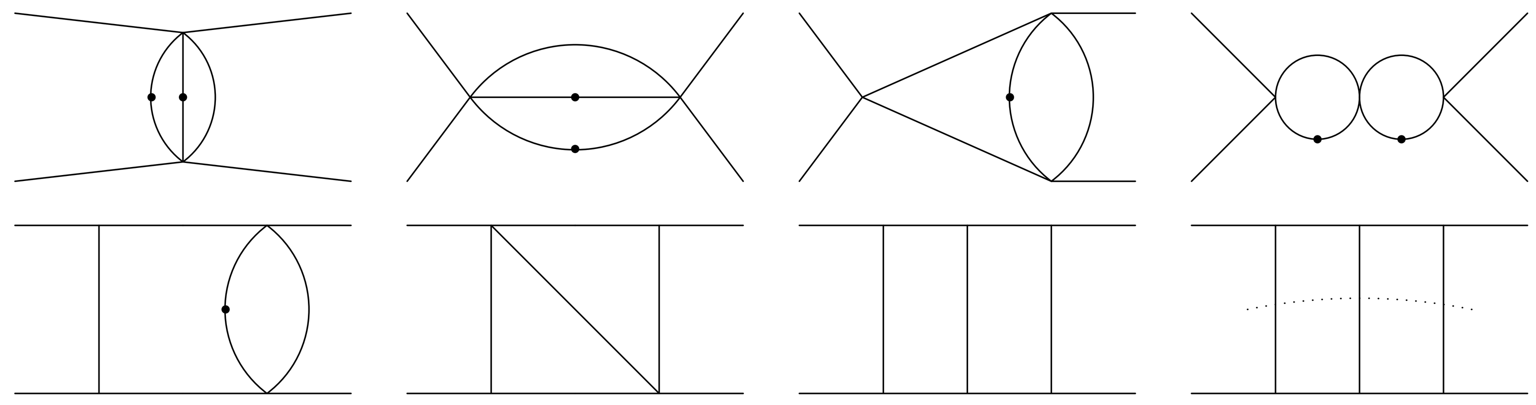


Figure: Double box master integrals from [5]. We focus here on last three forms ϕ_6, ϕ_7, ϕ_8.

- To "solve" scattering amplitudes: need integral coefficients, and to do the integrals. Can one apply differential equation method directly to dual forms?
- For integrals, differentiating can't add propagators, leading to **block triangular structure**:

$$\nabla_x\begin{pmatrix}\vdots\\\phi_6\\\phi_7\\\phi_8\end{pmatrix}=\epsilon\left[\frac{dx}{x}\begin{pmatrix}\cdots&0&0&0\\\cdots&-2&0&0\\\cdots&12&-2&0\\\cdots&-18&1&1\end{pmatrix}+\frac{dx}{1+x}\begin{pmatrix}\cdots&0&0&0\\\cdots&2&0&0\\\cdots&-12&2&2\\\cdots&18&-1&-1\end{pmatrix}\right]\wedge\begin{pmatrix}\vdots\\\phi_6\\\phi_7\\\phi_8\end{pmatrix} \quad (14)$$

where $x=t/s$, $\nabla_x=d+\omega_x\wedge$ and $\omega_x=d_x\log u$.

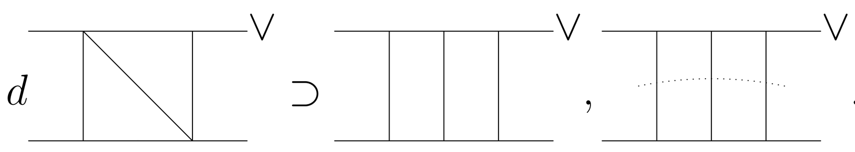
- For dual forms, information flows in the opposite direction: expect **upper triangular**
- Example: start from $\phi_6^\vee$. It is a 11-form, with 5 $d\theta$'s, two trivial (perpendicular) integrals, and a nontrivial 4-form that lives strictly on the 5-cut:

$$\phi_6^\vee=d\theta(D_1)\wedge\cdots\wedge d\theta(D_5)\wedge\tilde{\phi}_6^\vee \quad (15)$$

where $\tilde{\phi}_6^\vee\in H^4((5\text{-cut})\backslash\{u=0\},\{D_6=0,D_7=0\};u)$.
- Were it not for the boundaries, $\dim H^4=1$. Thus we can find $a(x)$ and IBP vector ξ such that:

$$\nabla_x\phi_6^\vee=a(x)\phi_6^\vee+\nabla_\omega\xi. \quad (16)$$

To recover off-diagonal terms, multiply IBP vector ξ by step functions $\theta(|D_i|^2>\epsilon)$ and pick up $d\theta'$s. (In fact, since $\dim H^3=0$, do twice to pick two extra $d\theta$'s localizing on heptacut):



$$d\,(\ldots)^\vee\supset(\ldots)^\vee,\ (\ldots)^\vee. \quad (17)$$

or more explicitly

$$\nabla_x\begin{pmatrix}\phi_6^\vee\\\phi_7^\vee\\\phi_8^\vee\end{pmatrix}=\epsilon\left[\frac{dx}{x}\begin{pmatrix}2&-6&0\\0&2&-1\\0&0&-1\end{pmatrix}+\frac{dx}{1+x}\begin{pmatrix}-2&0&0\\0&-2&1\\0&-2&1\end{pmatrix}\right]\wedge\begin{pmatrix}\phi_6^\vee\\\phi_7^\vee\\\phi_8^\vee\end{pmatrix}. \quad (18)$$

Note that (18) is consistent with (14) transposed, up to a (computable) basis change – "natural" dual forms pick a different basis, in fact, simplifying the form of the differential equations. Differential equations for the remaining dual master integrals are in progress.

- To summarize, dual forms live on cuts: only IBPs on cuts ever needed. Evidence that off-diagonal terms obtained simply by multiplying IBPs by step functions, and collecting $d\theta$.

Conclusions

We have introduced the Poincaré dual forms for Feynman integrals and motivated their utility through several examples. We argue that working with the dual forms is advantageous for the following reasons:

- The dual forms are supported only on cuts.
- They have nice $\epsilon\to0$ limits, where they are dual to the master contours that extract integral coefficients, providing a framework for understanding generalized unitarity (mod IBPs) for higher-loop integrals and sub-topologies
- Play a similar role as integration cycles, but easier to work with since all operations algebraic.

References

[1] S. Mizera, Aspects of Scattering Amplitudes and Moduli Space Localization, arxiv:1906.02099. S. Mizera, Scattering Amplitudes from Intersection Theory, Phys. Rev. Lett. **120**, 141602 (2018)

[2] M. Pierpaolo, S. Mizera, Feynman integrals and intersection theory, JHEP **2**, 139 (2019). H. Frellesvig, F.Gasparotto, S. Laporta, M. Mandal, P. Mastrolia, L.Mattiazzi, S. Mizera, Decomposition of Feynman Integrals on the Maximal Cut by Intersection Numbers, JHEP **05**, 153 (2019)

[3] K. Matsumoto, Relative twisted homology and cohomology groups associated with Lauricella's F_D, arXiv:1804.00366.

[4] S. Caron-Huot, K. J. Larsen, Uniqueness of two-loop master contours, JHEP **10**, 026 (2012).

[5] J. M. Henn, Multiloop Integrals in Dimensional Regularization Made Simple, Phys. Rev. Lett. **110**, 251601 (2013)