Exploring Ziggurat Graphs

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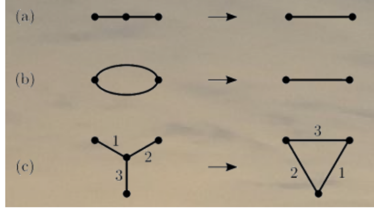
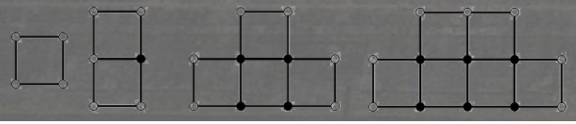
Massless Landau equations are invariant to Y-Δ transform

Landau equations of the first type determine the possible loci of singularities of Feynman diagrams when pinching of the contour does not happen at infinity (these are the so called 2nd type singularities). There are two distinct types of Landau equations, the on-shell conditions, $\alpha_i q_i^2 = 0$, where it was assumed that the theory is massless, and the "Kirchhoff equations", $\Sigma \alpha_i q_i = 0$, where the sum goes over all edges in an internal face of the graph. The parameters α are the corresponding Feynman parameters, while quantities q represent the 4-momentum flowing through a given edge. The Kirchhoff equations are obviously invariant under transformations known from circuit analysis: series reduction, parallel reduction and Y-Δ reduction, (a), (b) and (c) on the following figure, respectively. It can be shown that the on-shell conditions are invariant as well, and thus all of these transformations preserve the set of possible 1st type singularities in any massless theory. We also present an example use of the operations, relevant for the case n=6.



Any n-point graph is reducible to $[(n-2)^2/4]$ loop ziggurat graph

Simplifying graphs using the described circuit operations is a well studied problem in mathematics, called Y-Δ reducibility. If a given vertex isn't allowed to be transformed, it is called a terminal (hollow circle in figures). On the level of Landau graphs, terminals are vertices connected to external legs. The Y-Δ reducibility also implies allowing the following operations: deletion of a tadpole, deletion of a "hanging propagator", and the "FP assignment," contraction of an edge connected to a terminal of degree one. The last operation does not technically leave Landau equations invariant, but it only removes an otherwise uninteresting bubble singularity. I. Gitler has shown that any two-connected 2-connected m-terminal planar graph can be Y-Δ reduced to a m-terminal "ziggurat" graph, presented in figure, or it's minors (edge contractions and deletions). As all non-2-connected Landau graphs decouple into disjoint equations, this implies that all 1st type singularities at any loop order of a massless planar theory must be found in Landau singularities of the ziggurat graphs (edge contraction corresponds to α=0, edge deletion to q=0).



Ziggurat graph for 6 particles

For n=6, the corresponding ziggurat diagram further reduces to a three-loop "wheel" graph, see the following figure. Exactly solving the Landau equations of this diagram give the list of possible 1st type singularities of six point amplitudes in any planar massless theory. In a theory with dual conformal symmetry such as *N = 4 SYM*, 2nd type singularities are absent and this calculation gives us the full set of singularities, at all loops.

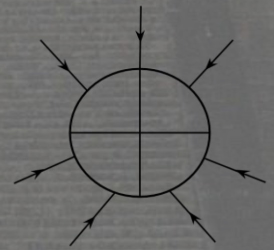
$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}},\ v = \frac{s_{23}s_{56}}{s_{234}s_{123}},\ w = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

$$S = \{u, v, w, 1-u, 1-v, 1-w, \tfrac{1}{u}, \tfrac{1}{v}, \tfrac{1}{w}\}$$

Ziggurat graph for 7 particles

The 7 – terminal ziggurat graph can be reduced to a four-loop form shown below.

This set of Landau equations can be studied semi-numerically in *Mathematica*, by starting with numerical values for all momenta, adding a free parameter to one of them, and solving the Landau equations. Such tests indicate that singularities of the n = 7 ziggurat graph occur when the set of letters familiar from *heptagon bootstrap* vanishes. This calculation is most conveniently done in momentum twistor space, where our analysis indicates that solutions appear when any Plucker coordinates $< i, j, k, l >$, as well as the following vanish:



$< 1\,(2,3)(4,5)(6,7) >$ + cyclic permutations

$< 2\,(1,3)(4,5)(6,7) >$ + cyclic permutations

where $< i, j, k, l > = \epsilon_{abcd} Z_i^a Z_j^b Z_k^c Z_l^d$

$< a\,(b,c)(d,e)(f,g) > = < a,b,c,f >< a,d,e,g > - < a,b,c,g >< a,d,e,f >$

Symbol alphabets for n = 6,7 can be understood in terms of certain *cluster algebras*. The corresponding cluster algebra for n = 8 becomes infinite, in which case the set of all-loop singularities is shrouded in mystery.