Full solution of the medium-induced radiation spectrum

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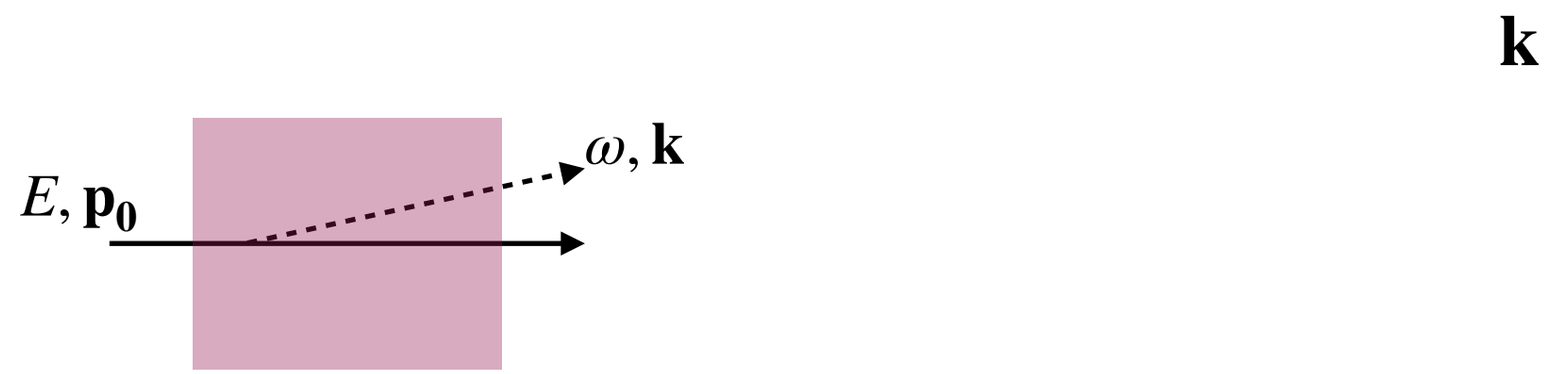
CA, Liliana Apolinario, Fabio Dominguez: arXiv: 2002.01517

Jefferson Lab
EXPLORING THE NATURE OF MATTER

Medium-induced gluon spectrum

- **Emission** off a parton with energy E of a **soft gluon** $(\omega \ll E)$ BDMPS-Z

$$\omega \frac{dI}{d\omega d^2\boldsymbol{k}} = \frac{2\alpha_s C_R}{(2\pi)^2\omega^2} \mathrm{Re} \int_0^\infty dt' \int_0^{t'} dt \int_{\boldsymbol{pq}} \boldsymbol{p}\cdot\boldsymbol{q}\, \widetilde{\mathcal{K}}(t',\boldsymbol{q};t,\boldsymbol{p})\mathcal{P}(\infty,\boldsymbol{k};t',\boldsymbol{q})$$



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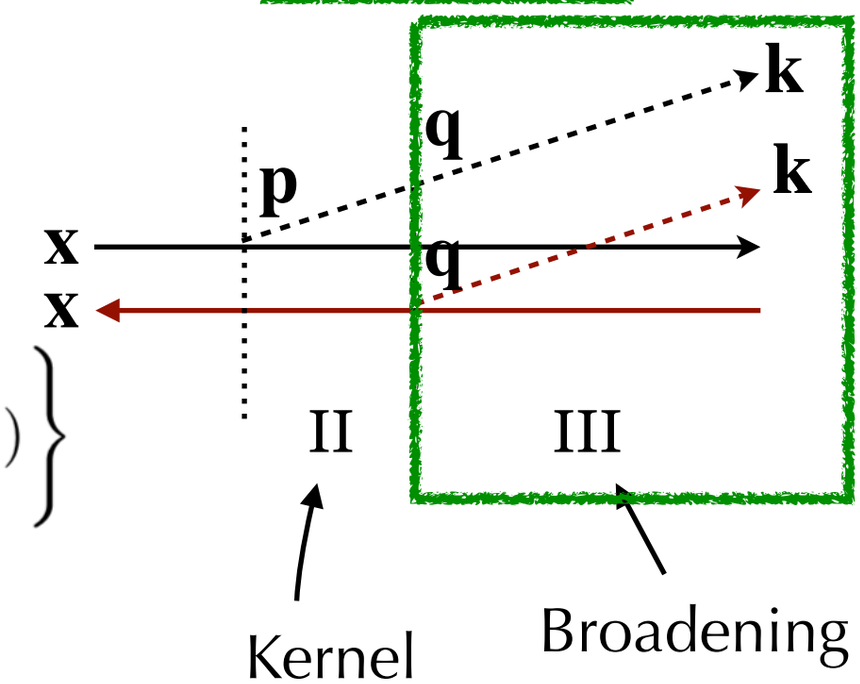

- Broadening

$$\mathcal{P}(t'',\boldsymbol{k};t',\boldsymbol{q}) \equiv \int d^2\boldsymbol{z}\, e^{-i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{z}} \exp\left\{-\frac{1}{2}\int_{t'}^{t''} ds\, n(s)\,\sigma(\boldsymbol{z})\right\}$$

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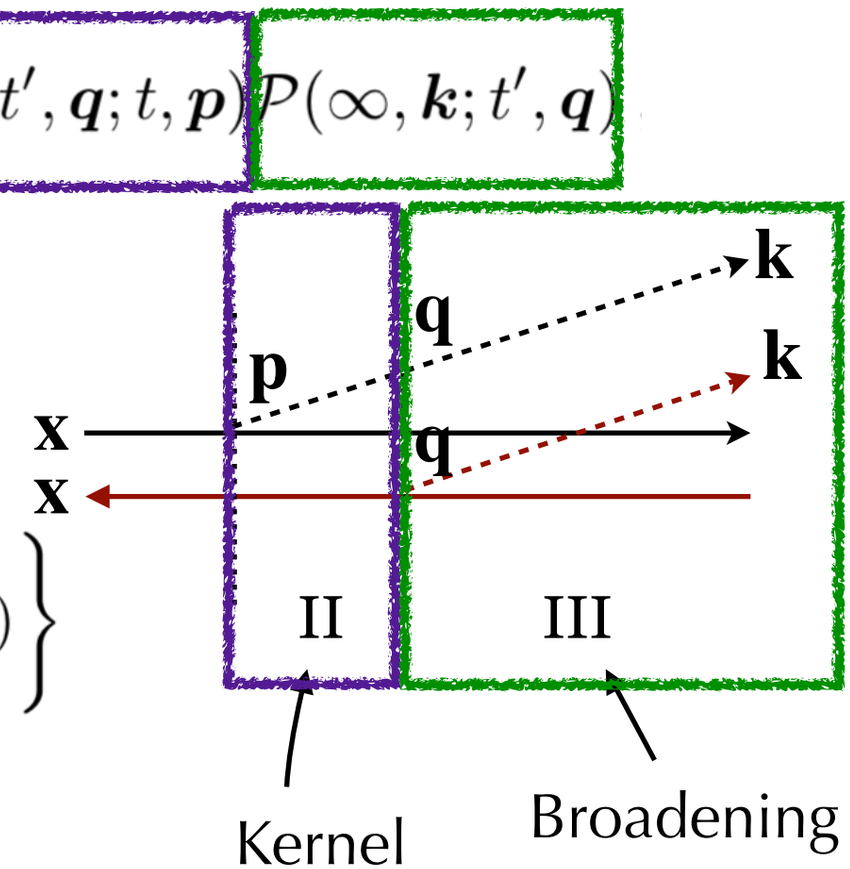

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- Emission Kernel

$$\mathcal{K}(t',\boldsymbol{z};t,\boldsymbol{y}) \equiv \int_{\boldsymbol{pq}} e^{i(\boldsymbol{q}\cdot\boldsymbol{z}-\boldsymbol{p}\cdot\boldsymbol{y})}\, \widetilde{\mathcal{K}}(t',\boldsymbol{q};t,\boldsymbol{p})$$

$$= \int_{\boldsymbol{r}(t)=\boldsymbol{y}}^{\boldsymbol{r}(t')=\boldsymbol{z}} \mathcal{D}\boldsymbol{r} \exp\left[\int_t^{t'} ds \left(\frac{i\omega}{2}\dot{\boldsymbol{r}}^2 - \frac{1}{2}n(s)\sigma(\boldsymbol{r})\right)\right]$$

Medium-induced gluon spectrum

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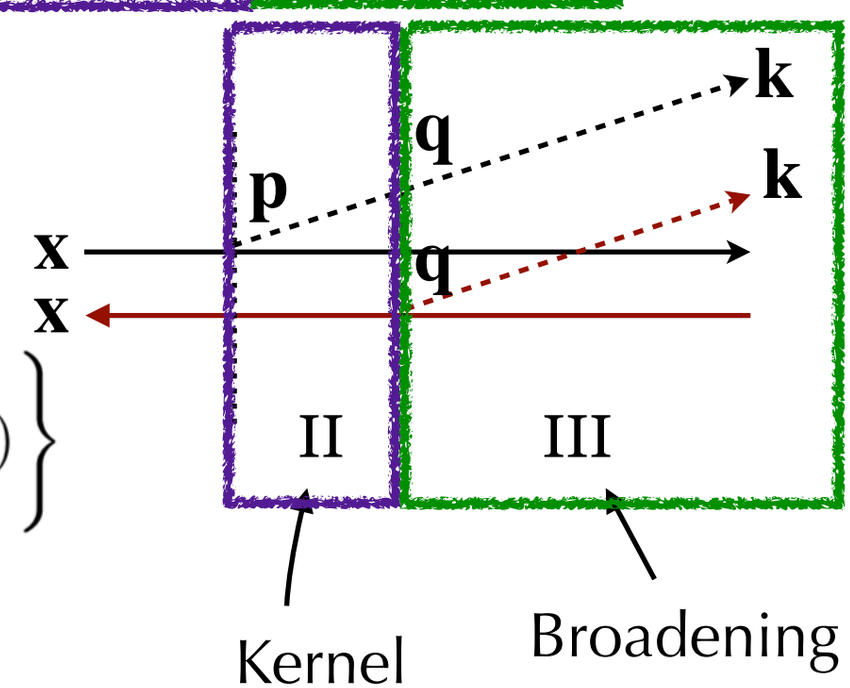

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Medium information

Difficult to solve numerically for realistic $\sigma(\mathbf{r})$

Analytic approximations

- **<u>Harmonic oscillator</u>** approximation

Perturbative tails neglected

$$n(s)\sigma(\mathbf{r}) \approx \frac{1}{2}\hat{q}(s)\mathbf{r}^2 + \mathcal{O}(\mathbf{r^2}\ln\mathbf{r^2})$$

The Kernel can be analytically computed for a static medium

Parameters: $\hat{q}\,,\,L$

- **<u>Opacity expansion</u>**

$\sigma(\mathbf{r})$ in terms of the full Yukawa collision rate $\sigma(\mathbf{r}) \equiv \int_{\mathbf{q}} V(\mathbf{q})\left(1 - e^{i\mathbf{q}\mathbf{r}}\right)$ $V(\mathbf{q}) = \dfrac{8\pi\mu^2}{\left(\mathbf{q}^2 + \mu^2\right)^2}$

The integrand in the Kernel is expanded in powers of $\left(n(s)\sigma(\mathbf{r})\right)^N$

$N = 1$: **First opacity** or **GLV** approximation

Latest developments

- Full resummation of all scatterings within a MC approach

  Feal and Vazquez, arXiv:1810.02645

- Analytical expansion around the HO

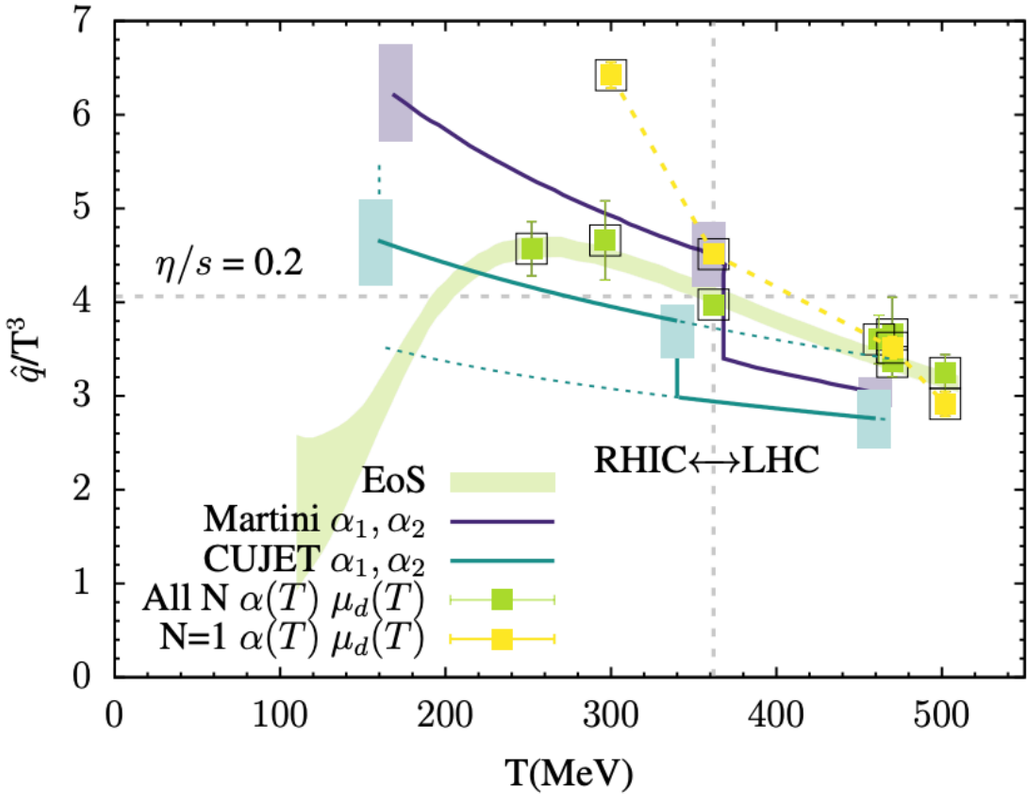
  Mehtar-Tani and Tywoniuk, arXiv:1910.02032

  Mehtar-Tani, arXiv:1903.00506

- Centrality/energy puzzle seems to disappear

  One dimensional expansion only

  Feal, Salgado, Vazquez, arXiv:1911.01309



Formalism

Solving the spectrum

- The spectrum can be also written as (vacuum contribution subtracted)

$$\omega \frac{dI}{d\omega d^2\boldsymbol{k}} = \frac{2\alpha_s C_R}{(2\pi)^2\omega} \mathrm{Re} \int_0^L ds\, n(s) \int_0^s dt \int_{\boldsymbol{pql}} i\boldsymbol{p} \cdot \left(\frac{\boldsymbol{l}}{\boldsymbol{l}^2} - \frac{\boldsymbol{q}}{\boldsymbol{q}^2} \right) \boxed{\sigma(\boldsymbol{l}-\boldsymbol{q})} \widetilde{\mathcal{K}}(s,\boldsymbol{q};t,\boldsymbol{p}) \mathcal{P}(L,\boldsymbol{k};s,\boldsymbol{l})$$

Guarantees convergence of $\boldsymbol{q}$ and $\boldsymbol{l}$ integrals as $\sigma(\mathbf{q}) \sim 1/\mathbf{q}^4$ for large $\boldsymbol{q}$

- The propagators satisfy the following differential equations:

$$\partial_\tau \mathcal{P}(\tau,\boldsymbol{k};s,\boldsymbol{l}) = -\frac{1}{2} n(\tau) \int_{\boldsymbol{k}'} \sigma(\boldsymbol{k}-\boldsymbol{k}') \mathcal{P}(\tau,\boldsymbol{k}';s,\boldsymbol{l})$$

$$\partial_t \widetilde{\mathcal{K}}(s,\boldsymbol{q};t,\boldsymbol{p}) = \frac{i\boldsymbol{p}^2}{2\omega} \widetilde{\mathcal{K}}(s,\boldsymbol{q};t,\boldsymbol{p}) + \frac{1}{2} n(t) \int_{\boldsymbol{k}'} \sigma(\boldsymbol{k}'-\boldsymbol{p}) \widetilde{\mathcal{K}}(s,\boldsymbol{q};t,\boldsymbol{k}')$$

Solving the spectrum II

$$\omega \frac{dI}{d\omega d^2\boldsymbol{k}} = \frac{2\alpha_s C_R}{(2\pi)^2\omega} \mathrm{Re} \int_0^L ds\, n(s) \int_0^s dt \int_{\boldsymbol{pql}} i\boldsymbol{p} \cdot \left(\frac{\boldsymbol{l}}{\boldsymbol{l}^2} - \frac{\boldsymbol{q}}{\boldsymbol{q}^2} \right) \sigma(\boldsymbol{l}-\boldsymbol{q}) \widetilde{\mathcal{K}}(s,\boldsymbol{q};t,\boldsymbol{p}) \mathcal{P}(L,\boldsymbol{k};s,\boldsymbol{l})$$

- We write the spectrum as a convolution

$$\omega \frac{dI}{d\omega d^2\mathbf{k}} = \frac{2\alpha_s C_R}{(2\pi)^2\omega} \mathrm{Re} \int_0^L ds\, n(s) \int_0^s dt \int_{\mathbf{p}} i\,\mathbf{p} \cdot \psi(s,\mathbf{k};t,\mathbf{p})$$

$$\boldsymbol{\psi}(s,\boldsymbol{k};t,\boldsymbol{p}) = \int_{\boldsymbol{q}} \boldsymbol{\phi}(L,\boldsymbol{k};s,\boldsymbol{q})\, \widetilde{\mathcal{K}}(s,\boldsymbol{q};t,\boldsymbol{p}) \qquad \boldsymbol{\phi}(\tau,\boldsymbol{k};s,\boldsymbol{q}) = n(s) \int_{\boldsymbol{l}} \left(\frac{\boldsymbol{l}}{\boldsymbol{l}^2} - \frac{\boldsymbol{q}}{\boldsymbol{q}^2} \right) \sigma(\boldsymbol{l}-\boldsymbol{q}) \mathcal{P}(\tau,\boldsymbol{k};s,\boldsymbol{l})$$

- We switch to the interaction picture

$$\boldsymbol{\psi}_I(s,\boldsymbol{k};t,\boldsymbol{p}) = e^{\frac{i\boldsymbol{p}^2}{2\omega}(s-t)} \boldsymbol{\psi}(s,\boldsymbol{k};t,\boldsymbol{p})$$

Caron-Huot and Gale, arXiv:1006.2379

Final equations

1. Solve:

$$\partial_\tau \boldsymbol{\phi}(\tau, \boldsymbol{k}; s, \boldsymbol{q}) = -\frac{1}{2} n(\tau) \int_{\boldsymbol{k}'} \sigma(\boldsymbol{k} - \boldsymbol{k}') \boldsymbol{\phi}(\tau, \boldsymbol{k}'; s, \boldsymbol{q})$$

   With initial condition

$$\boldsymbol{\phi}(s, \boldsymbol{k}; s, \boldsymbol{q}) = n(s) \left(\frac{\boldsymbol{k}}{\boldsymbol{k}^2} - \frac{\boldsymbol{q}}{\boldsymbol{q}^2} \right) \sigma(\boldsymbol{k} - \boldsymbol{q})$$

2. Solve:

$$\partial_t \boldsymbol{\psi}_I(s, \boldsymbol{k}; t, \boldsymbol{p}) = \frac{1}{2} n(t) \int_{\boldsymbol{k}'} e^{\frac{i\boldsymbol{p}^2}{2\omega}(s-t)} \sigma(\boldsymbol{k}' - \boldsymbol{p}) e^{-\frac{i\boldsymbol{k}'^2}{2\omega}(s-t)} \boldsymbol{\psi}_I(s, \boldsymbol{k}; t, \boldsymbol{k}')$$

   with initial condition

$$\boldsymbol{\psi}_I(s, \boldsymbol{k}; s, \boldsymbol{p}) = \boldsymbol{\phi}(L, \boldsymbol{k}; s, \boldsymbol{p})$$

3. Compute:

$$\omega \frac{dI}{d\omega d^2\mathbf{k}} = \frac{2\alpha_s C_R}{(2\pi)^2 \omega} \mathrm{Re} \int_0^L ds \int_0^s dt \int_{\mathbf{p}} i\, e^{-i\frac{\mathbf{p}^2}{2\omega}(s-t)} \mathbf{p} \cdot \psi_I(s, \mathbf{k}; t, \mathbf{p})$$

Energy distribution

- The energy distribution is obtained by integrating the k_T-differential spectrum with the constraint: $k < \omega$

$$\omega \frac{dI}{d\omega} = \int_0^{\omega^2} \omega \frac{dI}{d\omega d^2\mathbf{k}^2} d\mathbf{k}^2$$

This is the $\bar{R}$ finite case

- We can also remove this kinematic constraint:

$$\omega \frac{dI}{d\omega} = \int_0^{\infty} \omega \frac{dI}{d\omega d^2\mathbf{k}^2} d\mathbf{k}^2$$

This is also known as $\bar{R} \to \infty$ limit

Results

The set-up

- For simplicity : $n(s) = n_0$

- Two types of Parton-medium interactions considered: $\sigma(\mathbf{r}) = \int_{\mathbf{q}} V(\mathbf{q})\left(1 - e^{i\mathbf{q}\mathbf{r}}\right)$

  - Yukawa

$$V(\mathbf{q}) = \frac{8\pi\mu^2}{\left(\mathbf{q}^2 + \mu^2\right)^2}$$

Parameters: $n_0\,,\ L\,,\ \mu$

  - Hard thermal loop

$$\frac{1}{2} n\, V(\mathbf{q}) = \frac{g_s^2 N_c\, m_D^2 T}{\mathbf{q}^2\left(\mathbf{q}^2 + m_D^2\right)}$$

Parameters: $T,\ L\,,\ m_D$

- Can we match them?

$$n_0 \mu^2 = \alpha_s m_D^2 N_c T$$

For small dipole sizes:

$$m_D^2 = e\mu^2$$

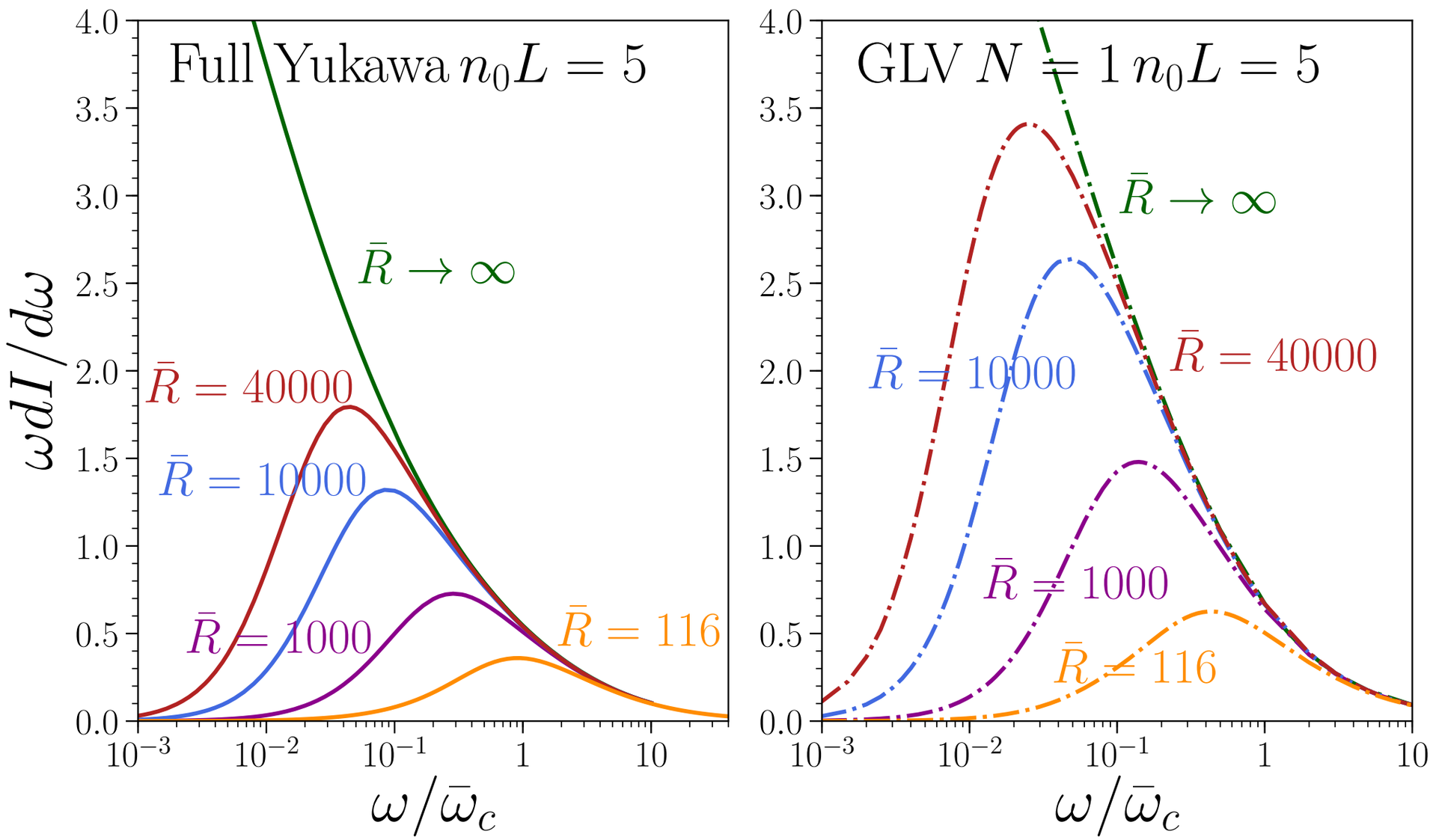
Barata and Mehtar-Tani, arXiv:2004.02323

Yukawa-type interaction

Energy distribution

$$\bar{R} = \bar{\omega}_c L$$

$$\bar{\omega}_c = \frac{1}{2}\mu^2 L$$



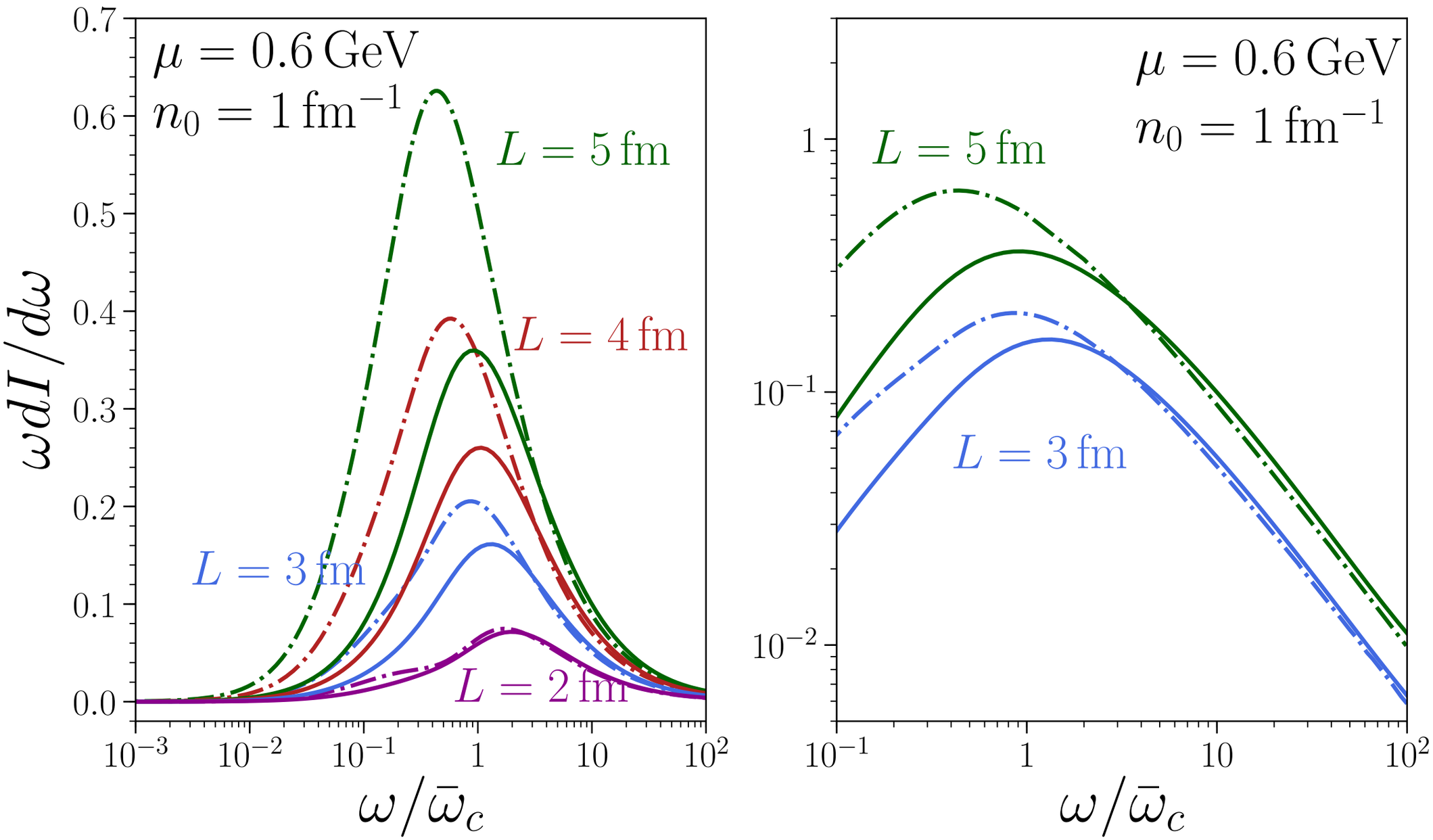
The GLV approximation overpredicts the spectrum

Energy distribution

GLV

Full spectrum

Discrepancies larger the **larger** the value of $n_0 L$



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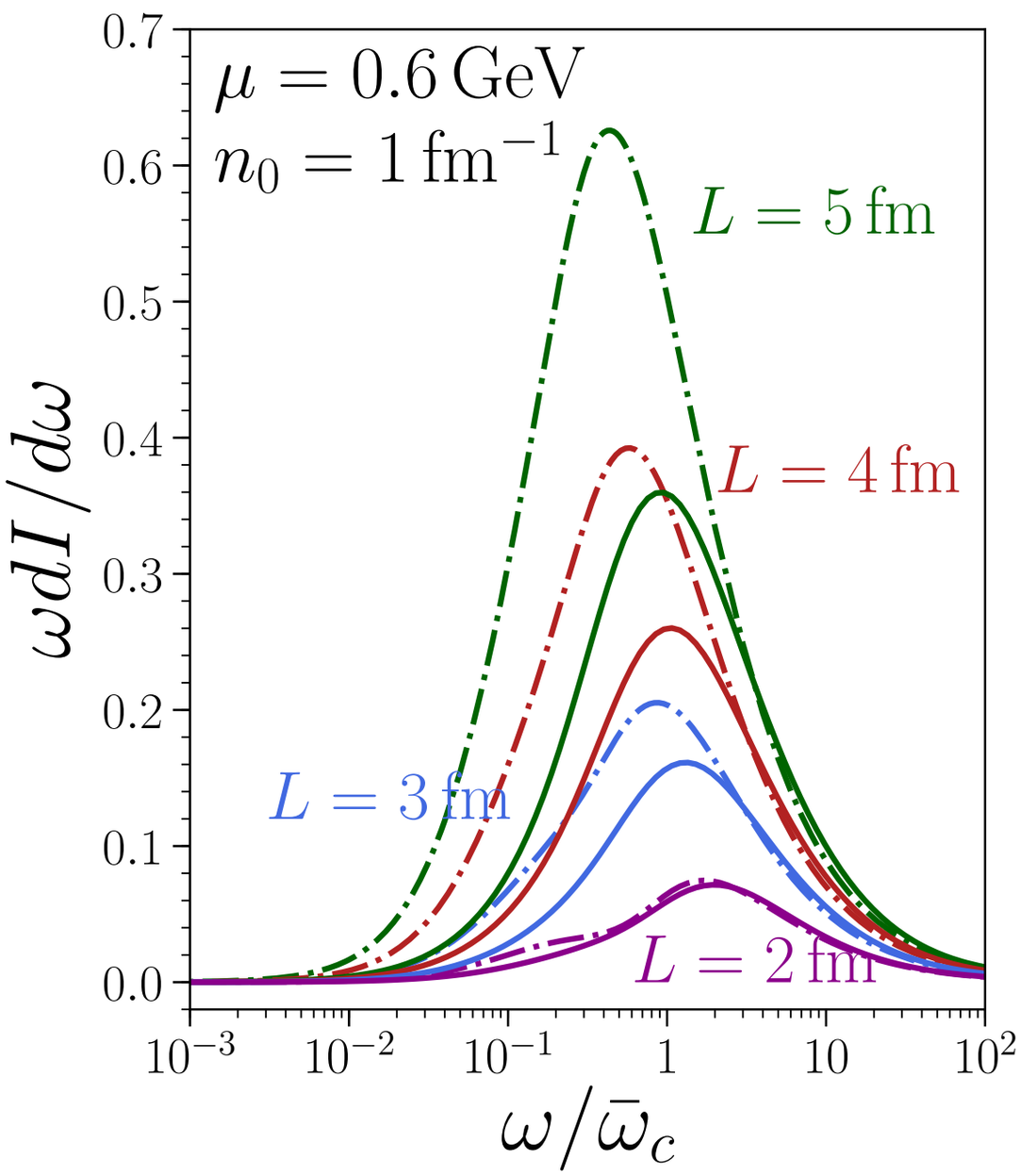
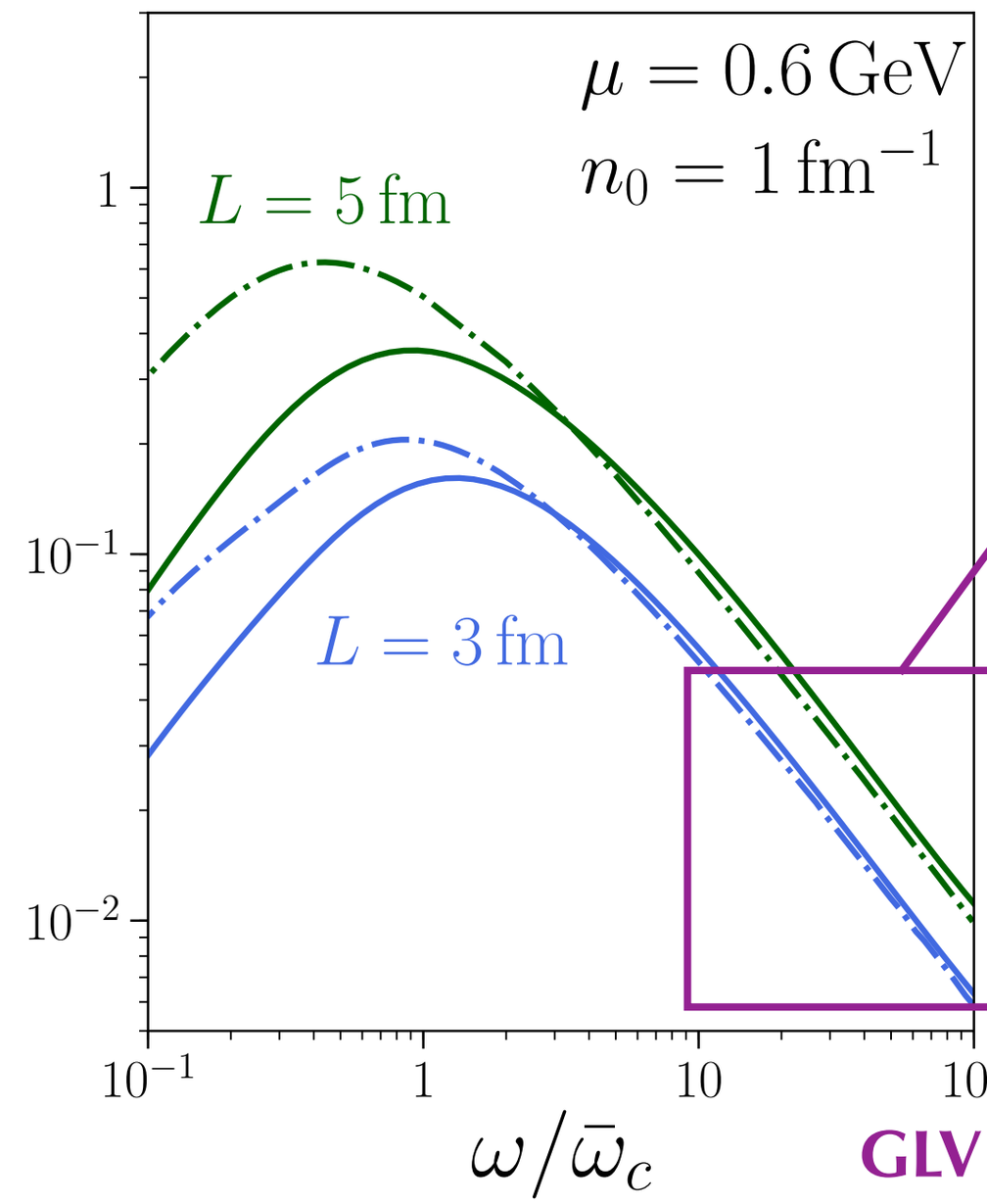
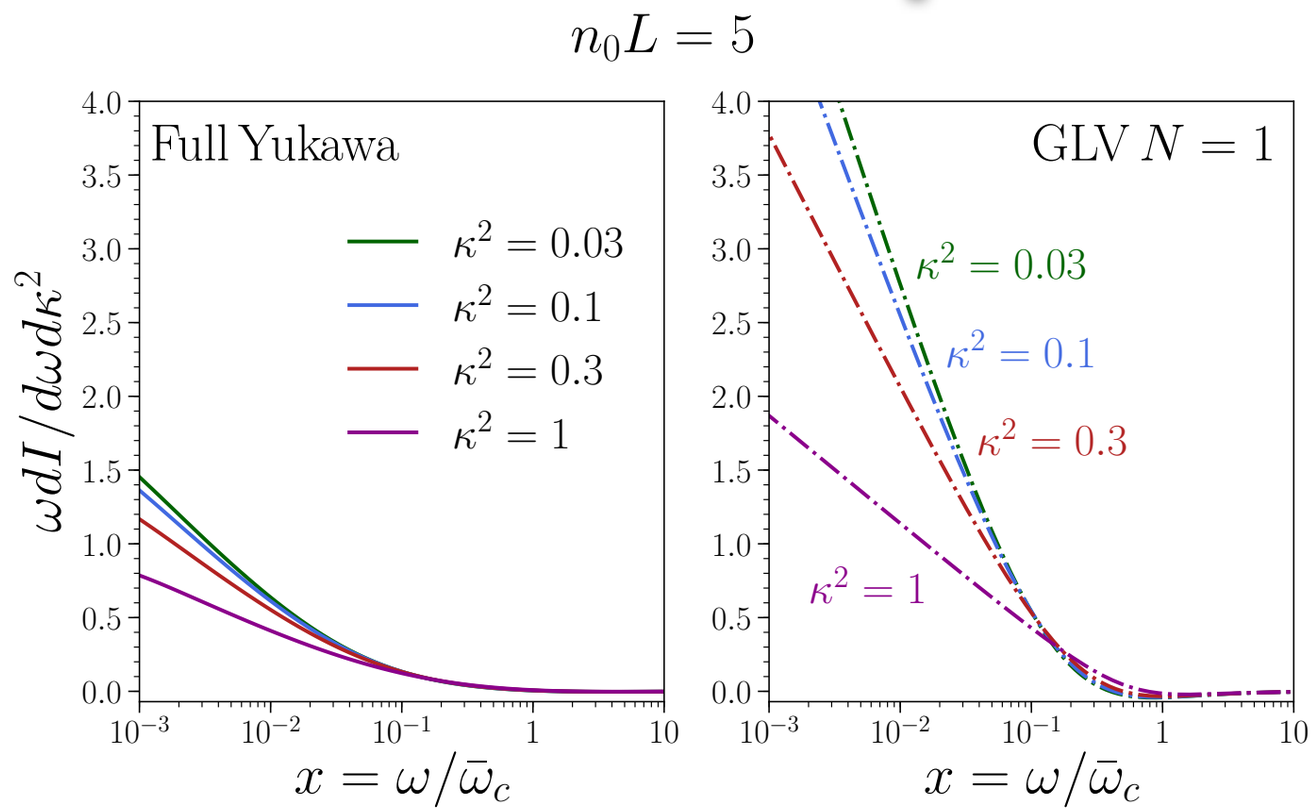




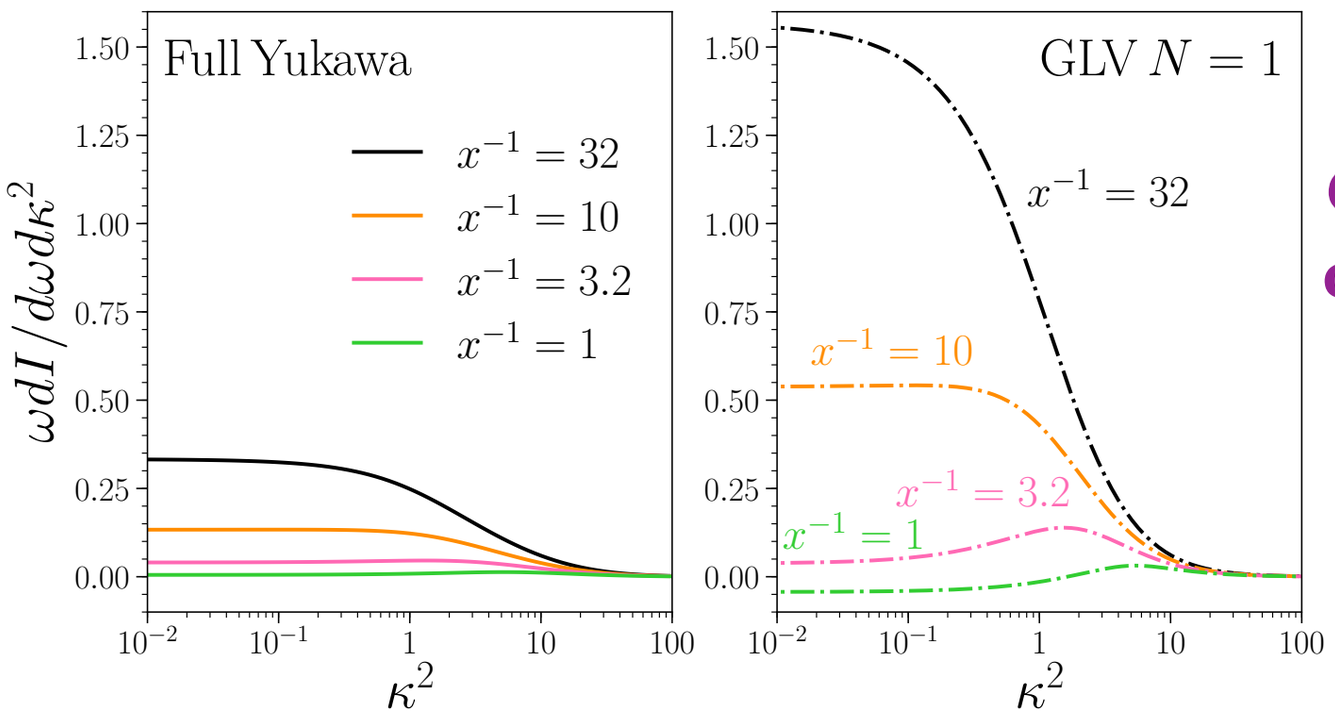
Expected $1/\omega$ behavior

GLV works well for high frequencies

k_T-differential spectrum



Discrepancies increase when decreasing ω



GLV and the full solution coincide at large traverse momentum

$$\bar{\omega}_c = \frac{1}{2}\mu^2 L$$

$$\kappa^2 = \frac{k^2}{\mu^2}$$

Comparing with the HO

- The natural parameters of the HO are: $\hat{q}\,,\ L$

$$\left.\begin{aligned} n_0\,\sigma(\mathbf{r}) &= \frac{1}{2}\hat{\mathbf{q}}\,r^2 \\ \sigma(\mathbf{r}) &= \int_{\mathbf{q}} V(\mathbf{q})\,\left(1-e^{i\mathbf{q}\mathbf{r}}\right) \end{aligned}\right\} \quad \hat{q}L \approx \int_{\mathbf{q}} \frac{1}{2}\mathbf{q}^2\cos^2\phi V(\mathbf{q}) \sim (n_0 L)\,\mu^2 \ln\sqrt{\frac{q_{max}}{\mu}}$$

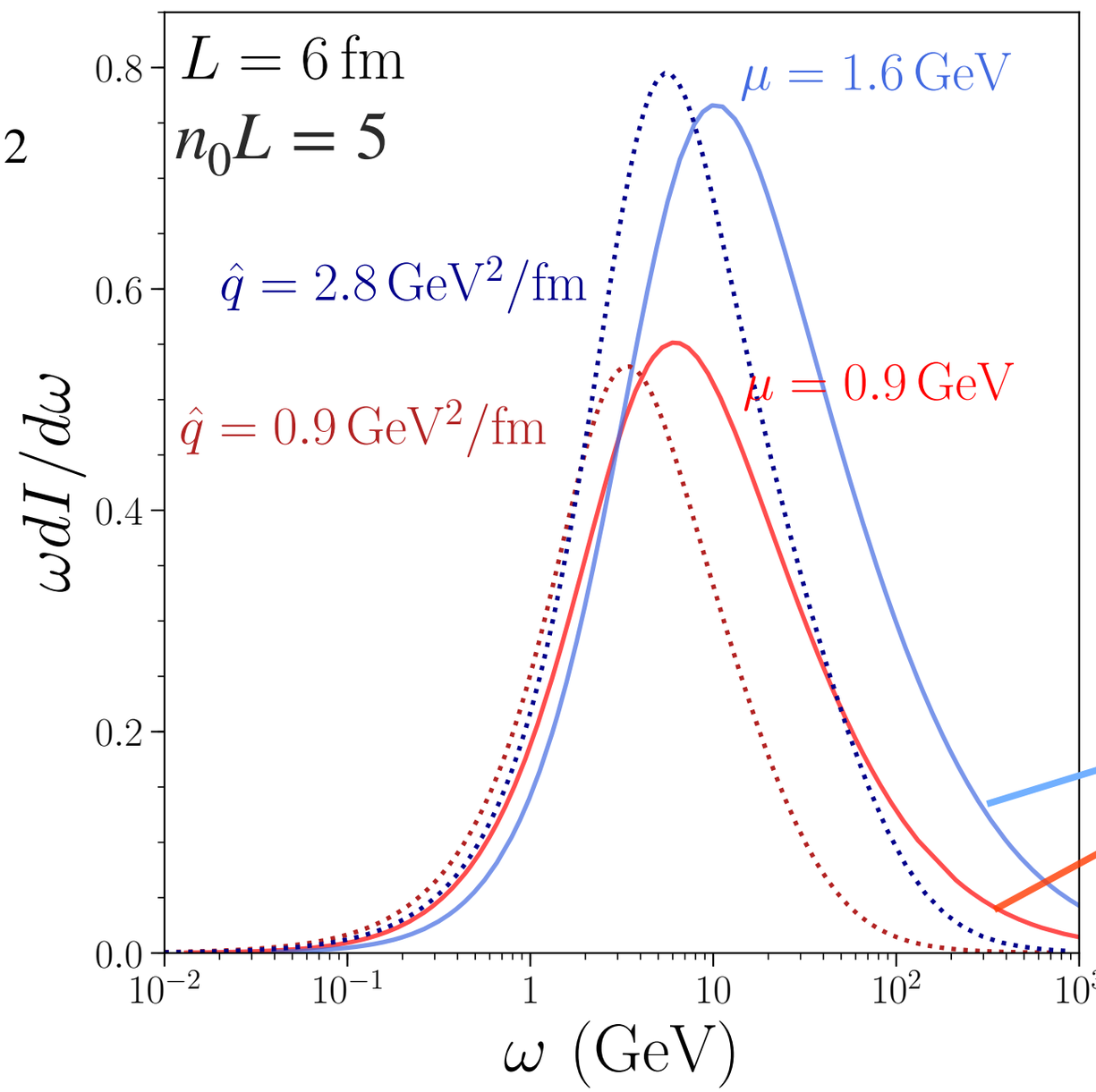
- We set $\ln\sqrt{\frac{q_{max}}{\mu}} = 1.3$

- This allows us to make **only qualitative comparisons** between the full-resumed spectrum and the HO approximation

Energy distribution

WARNING: Not a direct comparison!

$$\hat{q}L = 1.3\,(n_0 L)\mu^2$$



$$\bar{\omega}_c = \frac{1}{2}\mu^2 L$$

Full spectrum

$\bar{\omega}_c = 12\,\mathrm{GeV}$

$\bar{\omega}_c = 39\,\mathrm{GeV}$

At frequencies lower than $\bar{\omega}_c$ the HO works better than GLV

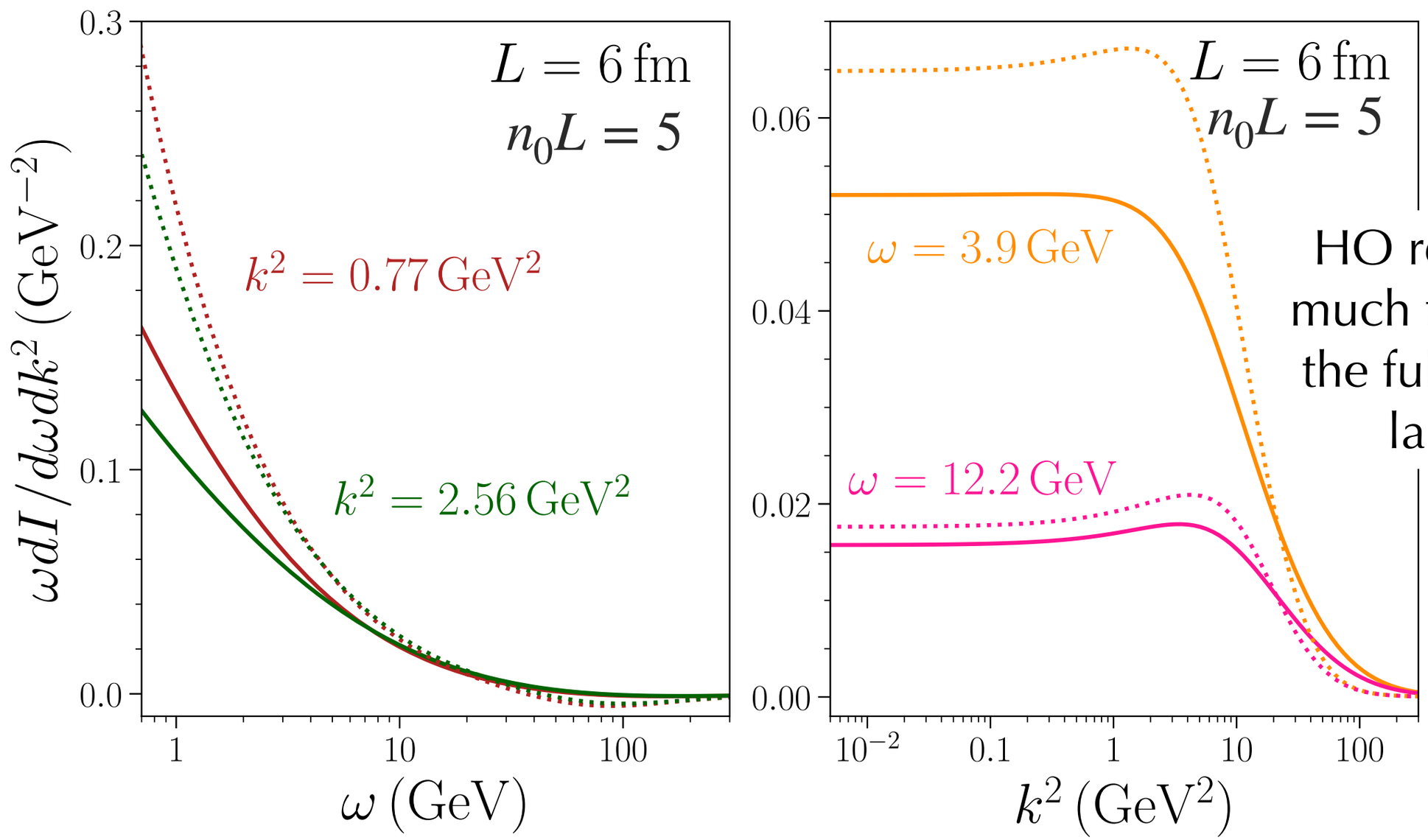
k_T-differential spectrum

WARNING: Not a direct comparison!

···· **HO**

—— **Full spectrum**

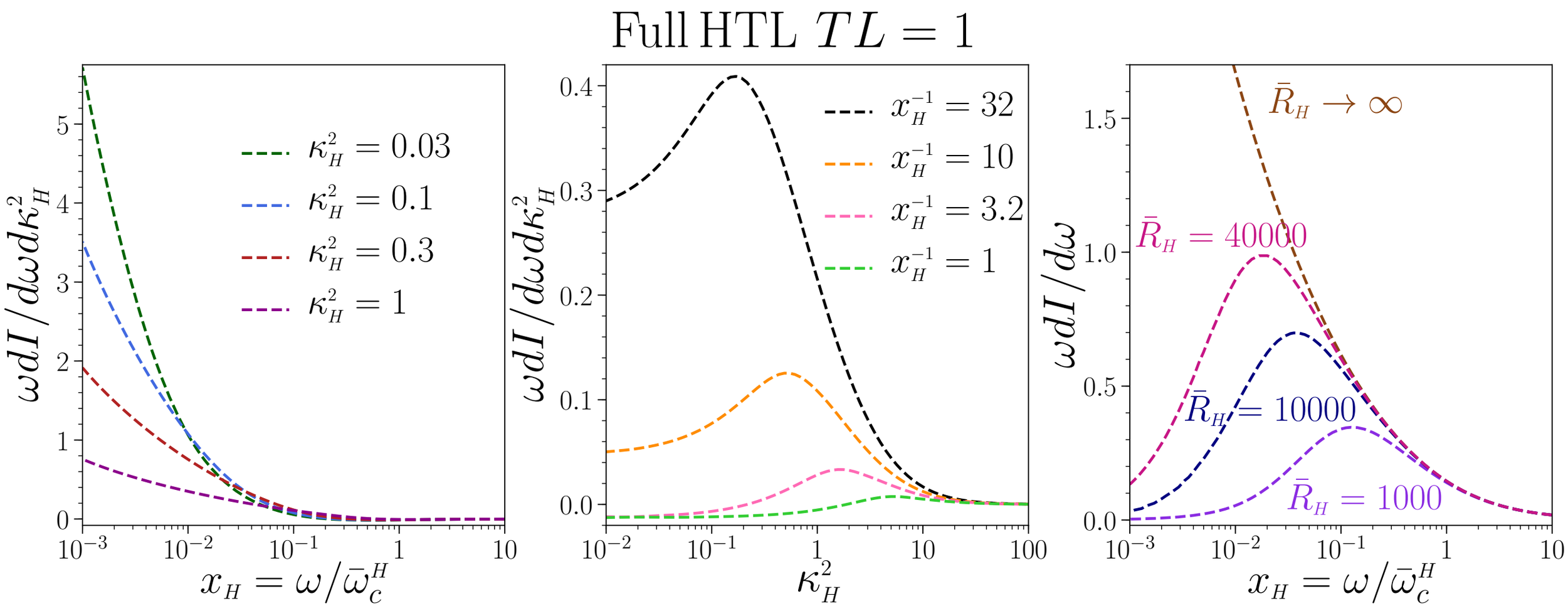
$\mu = 1.6\,\mathrm{GeV}$



HO result falls much faster than the full result at large k^2

HTL-type interaction

HTL spectra



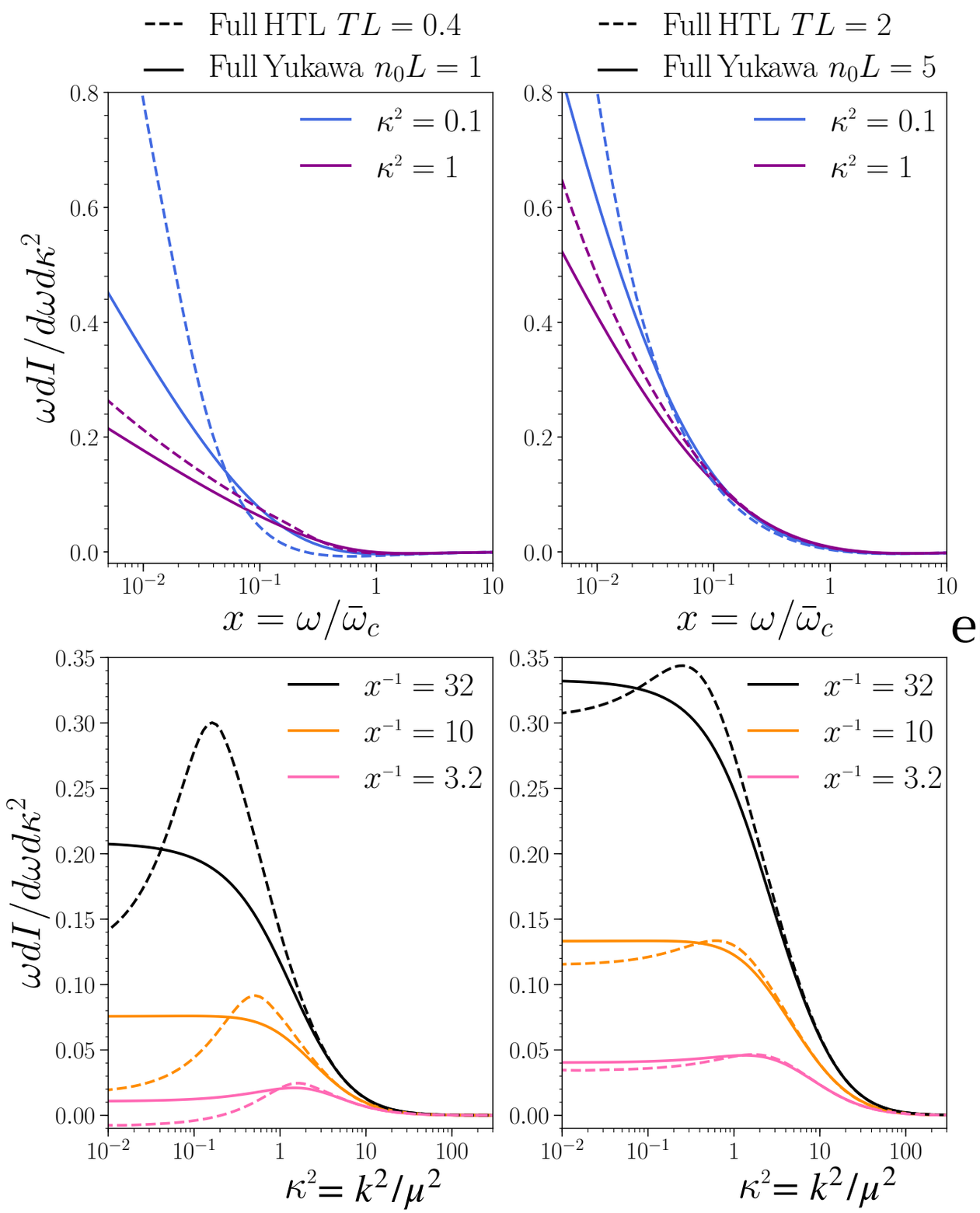
$$\bar{\omega}_c^H = \frac{1}{2} m_D^2 L \qquad \kappa_H^2 = \frac{k^2}{m_D^2} \qquad \bar{R}_H = \bar{\omega}_c L$$

HTL and Yukawa comparison

$$TL = \frac{n_0 L}{\alpha_s e N_c}$$

$$\omega_c^H = e\omega_c$$

$$\kappa_H^2 = \frac{\kappa^2}{e}$$



Agreement at large energies and transverse momenta

Conclusions

- We derive an **analytical expression** for the medium-induced gluon radiation spectrum in the soft limit with **resummation of all multiple scatterings**

- By solving differential equations, we can evaluate this spectrum for **realistic parton-medium interactions**

- Two interaction models analyzed: Yukawa and HTL

- The full-resummed spectrum is smaller than the GLV result

  GLV works well for large gluon energies and transverse momenta

- The HO goes faster to zero for large gluon energies than the full-resummed one

  HO is a better approximation than GLV for lower gluon energies

Thanks!

GLV spectrum

- With kinematic constraint

$$\omega \frac{dI^{N=1}}{d\omega} = 2\frac{\alpha_s C_R}{\pi}(n_0 L)\,\gamma \int_0^\infty dr\, \frac{r - sin(r)}{r^2} \times \left(\frac{1}{r+\gamma} - \frac{1}{\sqrt{((\bar{R}/2\gamma) + r + \gamma)^2 - 4r\bar{R}/2\gamma}} \right)$$

- No kinematic constraint

$$\lim_{\bar{R}\to\infty} \omega \frac{dI^{N=1}}{d\omega} = 2\,\frac{\alpha_s C_R}{\pi}\,(n_0\,L)\,\gamma \int_0^\infty dr\, \frac{1}{r+\gamma}\,\frac{r - sin(r)}{r^2}$$

$$\simeq 2\,\frac{\alpha_s C_R}{\pi}\,(n_0\,L) \begin{cases} \log\left[\frac{\bar{\omega}_c}{\omega}\right] & \text{for} \quad \bar{\omega}_c > \omega \\ \frac{\pi}{4}\,\frac{\bar{\omega}_c}{\omega} & \text{for} \quad \bar{\omega}_c < \omega \end{cases}$$

HO spectrum

- No kinematic constraint

$$\lim_{R\to\infty} \omega \frac{dI}{d\omega} = \frac{2\alpha_s C_R}{\pi} \ln \left| \cos\left[(1+i)\sqrt{\frac{\omega_c}{2\omega}} \right] \right| .$$

$$\omega_c = \frac{1}{2}\hat{q}L^2$$

$$R = \omega_c L$$

$$\lim_{R\to\infty} \omega \frac{dI}{d\omega} \simeq \frac{2\alpha_s C_R}{\pi} \begin{cases} \sqrt{\frac{\omega_c}{2\,\omega}} & \text{for} \quad \omega < \omega_c \\ \frac{1}{12}\left(\frac{\omega_c}{\omega}\right)^2 & \text{for} \quad \omega > \omega_c \end{cases}$$