Numerical study of the $\mathcal{N}=2$ Landau–Ginzburg model

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O. M. and Hiroshi Suzuki, PTEP 2018 (2018) 083B05 [1805.10735]; O. M., JHEP 12 (2018) 045 [1810.02519].

1. Introduction

- In the IR limit, any quantum field theory is expected to become scale invariant, which would be described by conformal field theory.
- The original theory is called the Landau–Ginzburg (LG) model or LG description; e.g., 2D $\mathcal{N}=2$ Wess–Zumino (WZ) model.
  - *String world sheet*: $\mathcal{N}=2$ superconformal field theory (SCFT)
- Non-perturbative phenomenon
  $\rightarrow$ no complete proof of *conjectured* LG/SCFT correspondence.
- To this issue, lattice field theory provides an alternative approach.
  - Preceding numerical simulations [Kawai-Kikukawa '10, Kamata-Suzuki '11]

▶ Applying a SUSY-preserving numerical method to the WZ model, we directly measure the scaling dimension and the central charge; we obtain the results that are consistent with the conjecture.

2. Landau–Ginzburg description

- CFT on the complex plane (2D Euclidean space) is invariant under $z\rightarrow$ holomorphic function $f(z)$, where $z,\bar{z}=x_0\pm ix_1$.
- Virasoro algebra for generators $L_n\sim -z^{n+1}\partial/\partial z$ ($n\in\mathbb{Z}$):

$$[L_m,L_n]=(m-n)L_{m+n}+(c/12)m(m^2-1)\delta_{m+n,0},$$

where c is the central charge (center of the central extension).

- LG model: *strongly interacting* Lagrangian description of CFT.
  - 2D $\mathcal{N}=2$ WZ model (dimensional reduction of the 4D $\mathcal{N}=1$ WZ model) is *believed* to provide the LG description of SCFT.

ADE-type theories

- WZ model ($\{\Phi_I\}_{I=1,\dots,N_\Phi}$; $\partial_{z,\bar{z}}=\frac{1}{2}(\partial_0\mp i\partial_1)$)

$$S=\int d^2x\sum_I\left[4\partial_z A_I^*\partial_{\bar{z}}A_I+\frac{\partial W(A)^*}{\partial A_I^*}\frac{\partial W(A)}{\partial A_I}\right.$$
$$\left.+(\bar{\psi}_{\dot{1}},\psi_2)_I\sum_J\begin{pmatrix}2\delta_{IJ}\partial_z & \frac{\partial^2W(A)^*}{\partial A_I^*\partial A_J^*}\\ \frac{\partial^2W(A)}{\partial A_I\partial A_J} & 2\delta_{IJ}\partial_{\bar{z}}\end{pmatrix}\begin{pmatrix}\psi_1\\ \bar{\psi}_{\dot{2}}\end{pmatrix}_J\right]$$

$\xrightarrow{\text{IR limit}}$ $\mathcal{N}=2$ minimal model (simple and *solvable* SCFT model)

Algebra	Superpotential W	Central charge c
A_n	$\frac{\lambda_1}{n+1}\Phi_1^{n+1}$, $n\geqq 1$	$3-6/(n+1)$
D_n	$\frac{\lambda_1}{n-1}\Phi_1^{n-1}+\frac{\lambda_2}{2}\Phi_1\Phi_2^2$, $n\geqq 3$	$3-6/2(n-1)$
E_7	$\frac{\lambda_1}{3}\Phi_1^3+\frac{\lambda_2}{3}\Phi_1\Phi_2^3$	$3-6/18$

($E_6\cong A_2\otimes A_3$, $E_8\cong A_2\otimes A_4$)

- Another correspondence: LG/Calabi–Yau
  e.g., $\begin{cases}\text{LG model with } W=\sum_{I=1}^5\Phi_I^5\\ \text{6D Calabi–Yau manifold defined by } \sum_{I=1}^5 z_I^5=0\end{cases}$

3. Lattice formulation [Kadoh-Suzuki '09]

- System: finite physical box L^2. We work in the momentum space,

$$p_\mu=2\pi n_\mu/L,\quad n_\mu=0,\pm1,\dots,\pm L/2a \text{ (Momentum cutoff)}.$$

Then, the action is given by ($*$ denotes the convolution)

$$S=S_B+\frac{1}{L^2}\sum_{p,I,J}(\bar{\psi}_{\dot{1}},\psi_2)_I(-p)\begin{pmatrix}2i\delta_{IJ}p_z & \frac{\partial^2W(A)^*}{\partial A_I^*\partial A_J^*}*\\ \frac{\partial^2W(A)}{\partial A_I\partial A_J}* & 2i\delta_{IJ}p_{\bar{z}}\end{pmatrix}\begin{pmatrix}\psi_1\\ \bar{\psi}_{\dot{2}}\end{pmatrix}_J(p),$$

$$S_B=\frac{1}{L^2}\sum_{p,I}N_I^*(-p)N_I(p),\qquad \underbrace{N_I(p)\equiv 2ip_zA_I(p)+\frac{\partial W(A)^*}{\partial A_I^*}(p)}_{\text{Nicolai map } (A,A^*)\rightarrow(N,N^*)\text{ ['80]}}\quad (\star)$$

- Partition function ($\{A\}_k$: solutions of the algebraic eq. ($\star$)):

$$\mathcal{Z}=\int\prod_{|p_\mu|\leq\pi/a}|dN(p)|^2\underbrace{e^{-S_B}}_{\text{Gaussian}}\sum_k \operatorname{sign}\det\left.\frac{\partial(N,N^*)}{\partial(A,A^*)}\right|_{\{A\}=\{A\}_k}$$

- This formulation manifestly preserves *translational inv. and SUSY*.
- Numerical setup
  1. Consider $A_{2,3}$-, $D_{3,4}$- and E_7-type models with the couplings $a\lambda_{1,2}=0.3$
  2. Take L/a as various even integers (typically ~ 30)
  3. Generate ~ 640 confs. of $N(p)$ using *the Gaussian random number*
  4. To solve Eq. ($\star$) numerically with respect to A, employ the Newton method

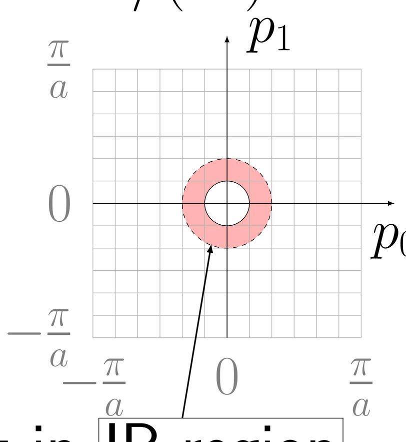
4. Scaling dimension Δ

- Two-point correlation function

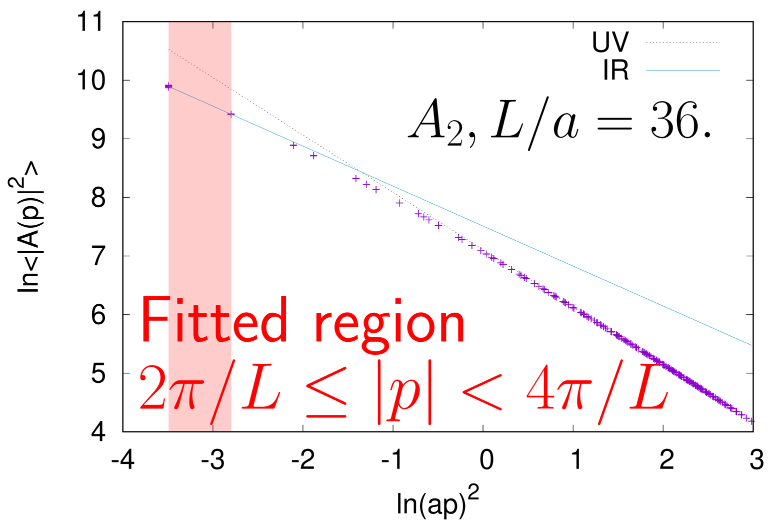
$$\underbrace{\langle\varphi_1(p)\varphi_2(-p)\rangle}_{\text{Numerical simulation}}=L^2\int_{L^2}d^2x\,e^{-ipx}\underbrace{\langle\varphi_1(x)\varphi_2(0)\rangle}_{\text{Relations in SCFT}}$$

- e.g., $\langle A(x)A^*(0)\rangle\propto 1/(x^2)^\Delta$

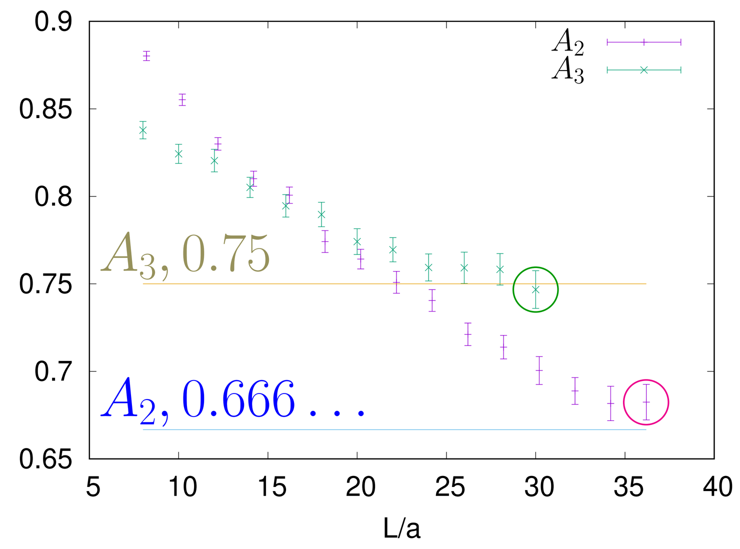
$$\Rightarrow\langle A(p)A^*(-p)\rangle\propto\frac{1}{(p^2)^{1-\Delta}}$$



- Δ obtained from the fit in IR region



	$1-\Delta$	Expected value
A_2	0.682(10)(7)	0.666. . .
A_3	0.747(11)(12)	0.75



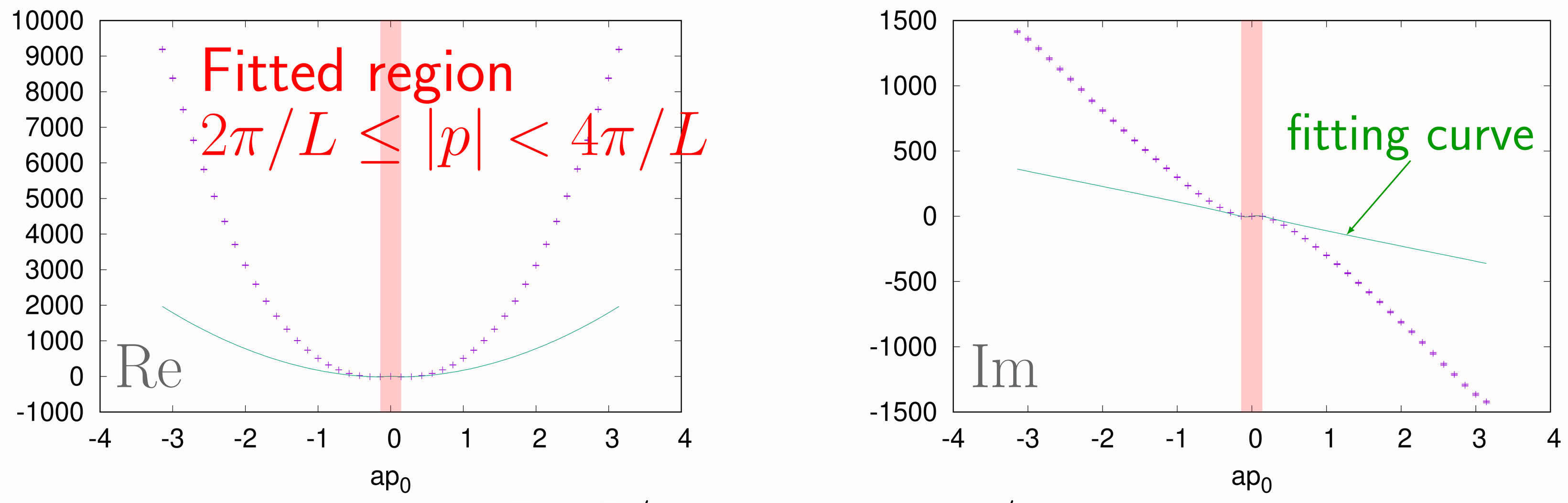
(cf. Kawai-Kikukawa (A_2): 0.660(11), Kamata-Suzuki (A_2): 0.616(25)(13))

5. Central charge c

- Energy-momentum tensor $T_{\mu\nu}$ such that $T_{\mu\mu}=0$ in the UV limit:

$$T_{zz}=\tfrac{1}{4}(T_{00}-iT_{01}-iT_{10}-T_{22})$$
$$=-4\pi\partial A^*\partial A-\pi\psi_2\partial\bar{\psi}_{\dot{2}}+\pi\partial\psi_2\bar{\psi}_{\dot{2}},\quad\dots$$

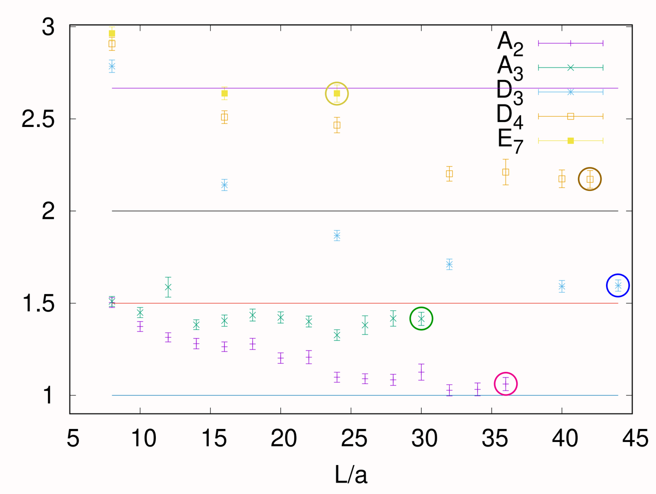
- $\langle T_{zz}(x)T_{zz}(0)\rangle=c/2z^4\rightarrow\langle T_{zz}(p)T_{zz}(-p)\rangle=L^2\pi\,c\,p_z^3/12p_{\bar{z}}$



D_3, $L/a=44$, $ap_1=\pi/22$.

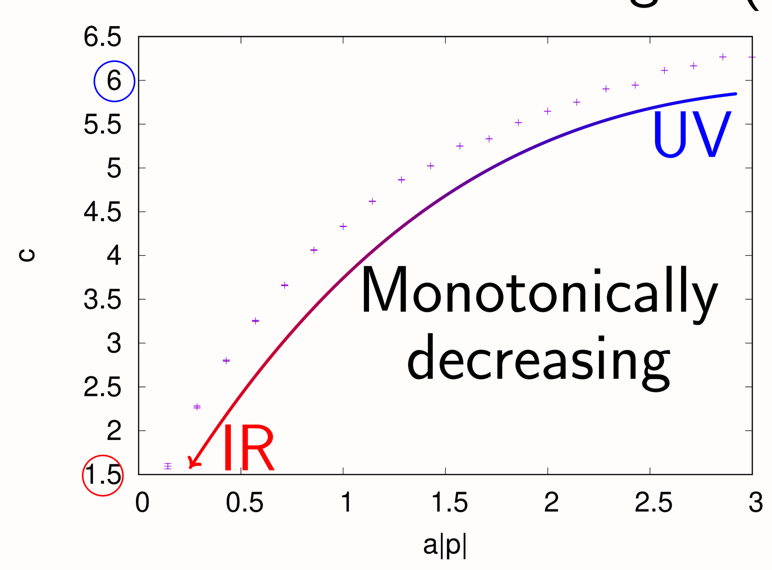
- Numerical determination of the central charge

	Central charge	Expected value
A_2	1.061(36)(34)	1
A_3	1.415(36)(36)	1.5
D_3	1.595(31)(41)	1.5
D_4	2.172(48)(39)	2
E_7	2.638(47)(59)	2.666. . .



(cf. Kamata-Suzuki (A_2): 1.09(14)(31))

- "Effective central charge" (D_3, $L/a=44$)



- Various fitted regions ($n=1,2,\dots$):
  $\frac{2\pi}{L}n\leq|p|\leq\frac{2\pi}{L}(n+1)$
- $c_{\mathrm{IR}}\approx 1.5$
  $\Leftrightarrow c_{\mathrm{UV}}\approx 6=3N_\Phi$ (free theory)
- Analogous to Zamolodchikov C-function

Zamolodchikov C-function

- General forms ($\tau=\ln z\bar{z}$):

$$\langle T_{zz}(x)T_{zz}(0)\rangle=F(\tau)/z^4\xrightarrow{\text{IR}}c/2z^4,$$
$$\langle T_{zz}(x)T_{z\bar{z}}(0)\rangle=G(\tau)/4z^3\bar{z}\rightarrow 0,\qquad\langle T_{z\bar{z}}(x)T_{z\bar{z}}(0)\rangle=H(\tau)/z^2\bar{z}^2\rightarrow 0$$

- Zamolodchikov C-function: $C(\tau)\equiv 2F(\tau)-G(\tau)-3H(\tau)/8$.
  Conservation laws and unitarity imply $dC/d\tau\leq 0$ (c-theorem).

6. Conclusion

- We numerically studied the IR behavior of 2D $\mathcal{N}=2$ WZ model, and determined Δ and c.
- This study supports the conjectured LG correspondence.
- *We hope that this numerical approach will be useful to investigate superstring theory via the LG/Calabi–Yau correspondence.*