Issues

Quark-diquark models and fluctuations in QCD

Eugenio Megías\textsuperscript{1*}
E. Ruiz Arriola\textsuperscript{1} and L.L. Salcedo\textsuperscript{1}

\textsuperscript{1}Department of Atomic, Molecular and Nuclear Physics, and Carlos I Institute of Theoretical and Computational Physics, University of Granada, Spain.
*Supported by the Ramón y Cajal Program of the Spanish MINEICO.

8\textsuperscript{th} International Conference on New Frontiers in Physics (ICNFP2019) “Workshop on QCD”
August 22, 2019, Kolymbari, Crete, Greece.

1 Introduction
- QCD Thermodynamics
- Hadron spectrum

2 Fluctuations of Conserved Charges in a Thermal Medium
- Fluctuations of conserved charges
- Fluctuations within the HRG approach

3 Quark-Diquark model for Baryons and Fluctuations
- Quark-diquark model for baryons
- Baryon spectrum
- Baryonic susceptibilities
- Semiclassical estimates of the susceptibilities

4 Conclusions
1. **Introduction**
   - QCD Thermodynamics
   - Hadron spectrum

2. **Fluctuations of Conserved Charges in a Thermal Medium**
   - Fluctuations of conserved charges
   - Fluctuations within the HRG approach

3. **Quark-Diquark model for Baryons and Fluctuations**
   - Quark-diquark model for baryons
   - Baryon spectrum
   - Baryonic susceptibilities
   - Semiclassical estimates of the susceptibilities

4. **Conclusions**
1 Introduction
   - QCD Thermodynamics
   - Hadron spectrum

2 Fluctuations of Conserved Charges in a Thermal Medium
   - Fluctuations of conserved charges
   - Fluctuations within the HRG approach

3 Quark-Diquark model for Baryons and Fluctuations
   - Quark-diquark model for baryons
   - Baryon spectrum
   - Baryonic susceptibilities
   - Semiclassical estimates of the susceptibilities

4 Conclusions
1 Introduction
   - QCD Thermodynamics
   - Hadron spectrum

2 Fluctuations of Conserved Charges in a Thermal Medium
   - Fluctuations of conserved charges
   - Fluctuations within the HRG approach

3 Quark-Diquark model for Baryons and Fluctuations
   - Quark-diquark model for baryons
   - Baryon spectrum
   - Baryonic susceptibilities
   - Semiclassical estimates of the susceptibilities

4 Conclusions
References

- **Fluctuations**: 1711.09837; 1805.05214; PRD99 (2019) 074020.
1. Introduction
   - QCD Thermodynamics
   - Hadron spectrum

2. Fluctuations of Conserved Charges in a Thermal Medium
   - Fluctuations of conserved charges
   - Fluctuations within the HRG approach

3. Quark-Diquark model for Baryons and Fluctuations
   - Quark-diquark model for baryons
   - Baryon spectrum
   - Baryonic susceptibilities
   - Semiclassical estimates of the susceptibilities

4. Conclusions
The partition function of QCD

\[ Z_{\text{QCD}} = \text{Tr} \, e^{-H_{\text{QCD}}/T} = \sum_n e^{-E_n/T}, \quad H_{\text{QCD}} \psi_n = E_n \psi_n. \]

Spectrum of QCD \( \rightarrow \) Thermodynamics

Hadron Resonance Gas Model

In the confined phase: Colour singlet states (hadrons + \cdots)


\[ Z = Z_0 \cdot Z_{[q\bar{q}]} \cdot Z_{[qqq]} \cdot Z_{[q\bar{q}\bar{q}]} \cdot Z_{[q\bar{g}]} \cdot Z_{[q\bar{q}q\bar{q}]} \cdot \cdots \]

In the deconfined phase: quarks and gluons \( \rightarrow \) quark-gluon plasma.

Phase transition is a crossover \( \rightarrow \) Do we see quark-gluon substructure BELOW the “phase transition”? 

Eugenio Megías
Issues

1. Introduction
   - QCD Thermodynamics
   - Hadron spectrum

2. Fluctuations of Conserved Charges in a Thermal Medium
   - Fluctuations of conserved charges
   - Fluctuations within the HRG approach

3. Quark-Diquark model for Baryons and Fluctuations
   - Quark-diquark model for baryons
   - Baryon spectrum
   - Baryonic susceptibilities
   - Semiclassical estimates of the susceptibilities

4. Conclusions
Introduction
Fluctuations of Conserved Charges in a Thermal Medium
Quark-Diquark model for Baryons and Fluctuations
Conclusions

QCD Thermodynamics
Hadron spectrum

Hadron spectrum (u,d,s)

- Particle Data Group (PDG) compilation 2016

- Relativized Quark Model (RQM), Isgur, Godfrey, Capstik 1985
Introduction
Fluctuations of Conserved Charges in a Thermal Medium
Quark-Diquark model for Baryons and Fluctuations
Conclusions

QCD Thermodynamics
Hadron spectrum

Cumulative number of states

- Cumulative number \( \equiv \) number of bound states below \( M \).

\[
N(M) = \sum_n \Theta(M - M_n),
\]

- Which states count?

\[
N(M) = N[q\bar{q}](M) + N[qqq](M) + N[\bar{q}\bar{q}\bar{q}](M) + \cdots,
\]

Assumption: \( N_{hadrons} \sim e^{M/T_H} \)

\[
\frac{Z_{HRG}}{\Tr} = \frac{A}{T_H - T} \quad \text{as} \quad T \to T_H^-; \quad T_H \sim 150 \text{ MeV} \equiv \text{Hagedorn temperature}
\]

- Non-interacting Hadron-Resonance Gas works for \( T \lesssim 0.8T_c \).
**Introduction**

Fluctuations of Conserved Charges in a Thermal Medium

Quark-Diquark model for Baryons and Fluctuations

**Conclusions**

QCD Thermodynamics

Hadron spectrum

**Cumulative number of states**

- **$n$-parton Hamiltonian:**
  \[
  H_n = \sum_{i=1}^{n} \sqrt{p_i^2 + m^2} + \sum_{i<j} v_{ij}(r_{ij}), \quad n = 2 \text{ (mesons)}, \quad n = 3 \text{ (baryons)}.
  \]

- **Within a semiclassical approximation:**
  \[
  N_n(M) \sim g_n \int \prod_{i=1}^{n} \frac{d^3x_i d^3p_i}{(2\pi)^3} \delta(\sum_{i=1}^{n} x_i) \delta(\sum_{i=1}^{n} p_i) \theta(M - H_n(p, x)) \sim \left(\frac{M^2}{\sigma}\right)^{3n-3}.
  \]

- **Large mass expansion:**
  \[
  N_{[q\bar{q}]} \sim M^6, \quad N_{[qqq]} \sim M^{12}, \quad N_{[q\bar{q}q\bar{q}]} \sim M^{18}.
  \]

- **Data indicate**
  \[
  N_{\text{baryons}}(M) \sim M^6 - M^8 \ll M^{12} \quad \text{qqq excited spectrum effectively behaves as a 2-body system.}
  \]
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
</tr>
<tr>
<td></td>
<td>- QCD Thermodynamics</td>
</tr>
<tr>
<td></td>
<td>- Hadron spectrum</td>
</tr>
<tr>
<td>2</td>
<td>Fluctuations of Conserved Charges in a Thermal Medium</td>
</tr>
<tr>
<td></td>
<td>- Fluctuations of conserved charges</td>
</tr>
<tr>
<td></td>
<td>- Fluctuations within the HRG approach</td>
</tr>
<tr>
<td>3</td>
<td>Quark-Diquark model for Baryons and Fluctuations</td>
</tr>
<tr>
<td></td>
<td>- Quark-diquark model for baryons</td>
</tr>
<tr>
<td></td>
<td>- Baryon spectrum</td>
</tr>
<tr>
<td></td>
<td>- Baryonic susceptibilities</td>
</tr>
<tr>
<td></td>
<td>- Semiclassical estimates of the susceptibilities</td>
</tr>
<tr>
<td>4</td>
<td>Conclusions</td>
</tr>
</tbody>
</table>
Fluctuations of conserved charges

- **Conserved charges** \( [Q_a, H] = 0 \).
- In the *uds* sector the only conserved charges are:
  
  \[ Q \equiv \text{Electric charge}, \quad B \equiv \text{Baryon number}, \quad S \equiv \text{Strangeness}. \]

- In the hot vacuum (no chemical potential):
  
  \[ \langle Q \rangle_T = 0, \quad \langle B \rangle_T = 0, \quad \langle S \rangle_T = 0. \]

- **Fluctuations** \( \rightarrow \) **Susceptibilities**:

\[
\chi_{ab}(T) \equiv \frac{1}{VT^3} \langle \Delta Q_a \Delta Q_b \rangle_T, \quad \Delta Q_a = Q_a - \langle Q_a \rangle_T.
\]

- At high temperature \((N_c = 3\) and flavors \(u, d, s\)):

\[
\chi_{BB}(T) \rightarrow 1/3, \quad \chi_{BQ}(T) \rightarrow 0, \quad \chi_{BS}(T) \rightarrow -1/3, \\
\chi_{SS}(T) \rightarrow 1, \quad \chi_{QS}(T) \rightarrow 1/3, \quad \chi_{QQ}(T) \rightarrow 2/3.
\]
Fluctuations of conserved charges

[M. Asakawa, M. Kitazawa, Prog. Part. Nucl. Phys. 90 (2016)].

- Fluctuations of conserved charges can be computed from the grand-canonical partition function:

\[
Z = \text{Tr} \exp \left( - \frac{H - \sum_a \mu_a Q_a}{T} \right), \quad \Omega = -T \log Z,
\]

by differentiation

\[
- \frac{\partial \Omega}{\partial \mu_a} \bigg|_{\mu_a=0} = \langle Q_a \rangle_T, \quad -T \frac{\partial^2 \Omega}{\partial \mu_a \partial \mu_b} \bigg|_{\mu_a=0=\mu_b} = \langle \Delta Q_a \Delta Q_b \rangle_T \equiv VT^3 \chi_{ab}(T).
\]

- \( Q_a \in \{ Q, B, S \} \), or, in the quark-flavor basis, \( Q_a \in \{ u, d, s \} \) where

\[
B = \frac{1}{3} (u + d + s), \quad Q = \frac{1}{3} (2u - d - s), \quad S = -s.
\]
Issues

1. Introduction
   - QCD Thermodynamics
   - Hadron spectrum

2. Fluctuations of Conserved Charges in a Thermal Medium
   - Fluctuations of conserved charges
   - Fluctuations within the HRG approach

3. Quark-Diquark model for Baryons and Fluctuations
   - Quark-diquark model for baryons
   - Baryon spectrum
   - Baryonic susceptibilities
   - Semiclassical estimates of the susceptibilities

4. Conclusions
Fluctuations within the HRG approach

\[ Q_a = \sum_{i \in \text{Hadrons}} q_i^a N_i, \quad \chi_{ab}(T) = \sum_{i,j \in \text{Hadrons}} q_i^a q_j^b \langle \Delta N_i \Delta N_j \rangle_T, \quad a, b \in \{Q, B, S\} \]

where \( q_i^a \in \{Q_i, B_i, S_i\} \equiv \text{charge with symmetry } a \text{ of } i\text{th-hadron}. \)

- Averaged number of hadrons of type \( i \) is

\[ \langle N_i \rangle_T = V \int \frac{d^3k}{(2\pi)^3} \frac{g_i}{e^{E_{k,i}/T} - \xi_i}, \]

with \( E_{k,i} = \sqrt{k^2 + M_i^2} \), and \( \xi = \pm 1 \) for bosons/fermions.

- Susceptibilities:

\[ \chi_{ab}(T) = \frac{1}{2\pi^2} \sum_{i \in \text{Hadrons}} g_i q_i^a q_i^b \sum_{n=1}^{\infty} \zeta_i^{n+1} \frac{M_i^2}{T^2} K_2 \left( \frac{nM_i}{T} \right), \]

with \( a, b = B, Q, S \), and \( K_2(z) \) is the Bessel function of 2nd kind.
Fluctuations and missing states

- **Fluctuations of Conserved Charges**: Good description of lattice data for $T \lesssim 160$ MeV.

**Lattice data of Fluctuations** [A. Bazavov et al., PRD86 (2012)]

- Fluctuations as a diagnostic tool to study missing states.
  Example: RQM seems to have too many baryonic states, but not too many charged states.
Susceptibilities within the HRG approach

Lattice data: [A. Bazavov et al. PRD86 ’12], [S. Borsanyi et al. JHEP 01 ’12].

- Low temperature behavior $\rightarrow$ dominated by lowest-lying state

\[
\begin{align*}
\chi_{BB}(T) & \sim e^{-M_p/T} , \\
\chi_{BS}(T) & \sim e^{-M_{\Lambda^0}/T} , \\
\chi_{BQ}(T) & \sim e^{-M_p/T} , \\
\chi_{QQ}(T) & \sim e^{-M_{\pi^\pm}/T} , \\
\chi_{SS}(T) & \sim e^{-M_{K^\pm}/T} , \\
\chi_{QS}(T) & \sim e^{-M_{K^\pm}/T} .
\end{align*}
\]
Issues

1. Introduction
   - QCD Thermodynamics
   - Hadron spectrum

2. Fluctuations of Conserved Charges in a Thermal Medium
   - Fluctuations of conserved charges
   - Fluctuations within the HRG approach

3. Quark-Diquark model for Baryons and Fluctuations
   - Quark-diquark model for baryons
   - Baryon spectrum
   - Baryonic susceptibilities
   - Semiclassical estimates of the susceptibilities

4. Conclusions
Quark-diquark model for baryons

Relativistic quark-diquark model for baryons $B \equiv \underbrace{(qq)_D}_{3}$

[E. Santopinto, J. Ferretti, PRC92 ’15]:

$$H_{qD} = \sqrt{\vec{p}^2 + m^2_q} + \sqrt{\vec{p}^2 + m^2_D} + V_{qD}(r)$$

It can be proved that $V_{qD}(r) \sim V_{q\bar{q}}(r)$ (up to an additive constant)

$$3 \otimes 3 \otimes 3 = (3 \otimes \bar{3}) \oplus (3 \otimes 6),$$

$$e^{-F_{qD}(x_1,x_2,T)/T} \equiv e^{-F_{q\bar{q}}(x_1,x_2,T)/T} + e^{-F_{3\otimes 6}(x_1,x_2,T)/T} \sim e^{-F_{q\bar{q}}(x_1,x_2,T)/T}.$$ 


$$V_{qD}(r) = -\frac{\tau}{r} + \sigma r + \mu,$$ 

with $\tau = \pi/12$, $\sigma = (0.42 \text{ GeV})^2$.

Parameters of the model are controlled by:

i) $m_{\text{cons}} \equiv$ constituent quark mass

ii) $\hat{m}_s \equiv$ current quark mass for the strange quark

$$m_{u,d} = m_{\text{cons}}, \quad m_s = m_{\text{cons}} + \hat{m}_s.$$
Distinguish between two kinds of diquarks:

- **scalar** $D$: $[q_1 q_2] \equiv$ quark content notation
- **axial vector** $D_{AV}$: $\{q_1 q_2\}$
- mass difference: $\Delta m_D := m_{D_{AV}} - m_D \approx 0.21 \text{ GeV}$ \cite{R.L.Jaffe, Phys. Rep. 409 ’05}.

Take into account the breaking of flavor $SU(3)$ for diquarks:

$$m_D = m_{D, ns} + n_s \hat{m}_S,$$

where $m_{D, ns}$ is the mass of a diquark with no s-quarks.

Natural choice:

$$m_{D, ns} = 2m_{cons}.$$

Free parameters of the model:

$m_{cons}, \hat{m}_S, \mu$ and $m_{D, ns}$. 
Quark-diquark model for baryons

<table>
<thead>
<tr>
<th>Baryon</th>
<th>Total deg.</th>
<th>$Q = -1$</th>
<th>$Q = 0$</th>
<th>$Q = 1$</th>
<th>$Q = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[nn]n$</td>
<td>4</td>
<td>-</td>
<td>2</td>
<td>2</td>
<td>-</td>
</tr>
<tr>
<td>${nn}n$</td>
<td>36</td>
<td>6</td>
<td>12</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>$[nn]s$</td>
<td>2</td>
<td>-</td>
<td>2</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${nn}s$</td>
<td>18</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>-</td>
</tr>
<tr>
<td>$[ns]n$</td>
<td>8</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>-</td>
</tr>
<tr>
<td>${ns}n$</td>
<td>24</td>
<td>6</td>
<td>12</td>
<td>6</td>
<td>-</td>
</tr>
<tr>
<td>$[ns]s$</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${ns}s$</td>
<td>12</td>
<td>6</td>
<td>6</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${ss}n$</td>
<td>12</td>
<td>6</td>
<td>6</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${ss}s$</td>
<td>6</td>
<td>6</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table: Spin-isospin degeneracies of the baryonic states within the quark-diquark model. The columns from $Q = -1$ to $Q = 2$ contain the degeneracies by distinguishing between the electric charges of the states. $n \equiv$ light flavors $u, d$ $s \equiv$ strange quark.

Quantum numbers of these states: $Q, B = 1, S = -ns$. 
Issues

1 Introduction
   - QCD Thermodynamics
   - Hadron spectrum

2 Fluctuations of Conserved Charges in a Thermal Medium
   - Fluctuations of conserved charges
   - Fluctuations within the HRG approach

3 Quark-Diquark model for Baryons and Fluctuations
   - Quark-diquark model for baryons
   - Baryon spectrum
   - Baryonic susceptibilities
   - Semiclassical estimates of the susceptibilities

4 Conclusions
Baryon spectrum

Variational method → Diagonalize the Hamiltonian:

\[ H_{qD} = \sqrt{\mathbf{p}^2 + m_q^2} + \sqrt{\mathbf{p}^2 + m_D^2} + V_{qD}(r). \]

For the diagonalization process, we use as basis the isotropic harmonic oscillator

\[ R_{n\ell}(r) = \frac{e^{-\frac{r^2}{2b^2}}}{\sqrt{\frac{4^n}{\sqrt{\pi}}} \binom{\ell}{n-1} \sqrt{\frac{2^{\ell+n+1}}{b^3(2\ell + 2(n-1) + 1)}}} L_{n-\frac{1}{2}}^{\ell+\frac{1}{2}}(r^2/b^2). \]

Matrix elements

\[ \langle n\ell | H_{qD} | n'\ell \rangle = \int_0^\infty dp \, \hat{u}^*_{n\ell}(p) \hat{u}_{n'\ell}(p) \left[ \sqrt{p^2 + m_q^2} + \sqrt{p^2 + m_D^2} \right] \]
\[ + \int_0^\infty dr \, u^*_{n\ell}(r) u_{n'\ell}(r) V_{qD}(r), \]

with \( u_{n\ell}(r) = r \, R_{n\ell}(r). \)

\( b \equiv \) minimize energy levels. Typically: \( 0.55 \text{ fm} \lesssim b \lesssim 0.65 \text{ fm}. \)
Typical choice of parameters:

\[ m_{D,ns} = 0.6 \text{ GeV}, \quad m_{u,d} = 0.3 \text{ GeV}, \]
\[ \hat{m}_s = 0.10 \text{ GeV}, \quad \mu = -0.459 \text{ GeV}. \]

Spectrum of baryons

Baryonic susceptibilities:

Solid lines: qD model.
Dotted lines: RQM spectrum.

Issues

1 Introduction
   - QCD Thermodynamics
   - Hadron spectrum

2 Fluctuations of Conserved Charges in a Thermal Medium
   - Fluctuations of conserved charges
   - Fluctuations within the HRG approach

3 Quark-Diquark model for Baryons and Fluctuations
   - Quark-diquark model for baryons
   - Baryon spectrum
   - Baryonic susceptibilities
   - Semiclassical estimates of the susceptibilities

4 Conclusions
Baryonic susceptibilities

- Spectrum of baryons within the quark-diquark model → baryonic susceptibilities:

\[ \chi_{BB}(T), \quad \chi_{BQ}(T), \quad \chi_{BS}(T), \]

with \[ \chi_{ab}(T) = \frac{1}{2\pi^2} \sum_{i \in \text{Hadrons}} g_i q_i^a q_i^b \sum_{n=1}^{\infty} \gamma_i^{n+1} \frac{M_i^2}{T^2} K_2 \left( \frac{nM_i}{T} \right). \]

- 4 free parameters in the model → perform the best fit to the lattice data → \( \tilde{\chi}^2 \) minimization:

\[ \tilde{\chi}^2 = \tilde{\chi}_{BB} + \tilde{\chi}_{BQ} + \tilde{\chi}_{BS}, \quad \text{where} \quad \tilde{\chi}_{ab}^2 = \sum_{j=1}^{j_{\max}} \frac{(\chi_{ab}\text{lat}(T_j) - \chi_{ab}\text{HRG}(T_j))^2}{(\Delta \chi_{ab}\text{lat}(T_j))^2}. \]

- Best fit:

\[ T \lesssim 125 \text{ MeV} - 150 \text{ MeV}. \]
**Baryonic susceptibilities**

**Figure**: $\chi^2/\nu$ as a function $\hat{m}_s$. We have fixed:

$$m_{\text{cons}} = 0.3 \text{ GeV} \quad \text{and} \quad m_{D,\text{ns}} = 2m_{\text{cons}}.$$  

Lattice data of [A.Bazavov et al. PRD86 '12].

- **Best fit**: $\hat{m}_s = 0.10 \text{ GeV}$ and $\mu = -0.459 \text{ GeV}$.  

Eugenio Megías  
Quark-diquark models and fluctuations in QCD
General fits to study the full validity of the model:

**Figure**: $\chi^2/\nu$ from a fit to the lattice data of the baryonic fluctuations from [A.Bazavov et al. PRD86 ’12] with $T \leq 150$ MeV. The dashed lines correspond to $\chi^2/\nu = 0.77$ (blue), $\chi^2/\nu = 1$ (red) and $1 + \sqrt{2/\nu}$ (green).

- **Left panel**: plane $(\hat{m}_s, m_{\text{cons}})$ as free parameters, with $m_{D,ns} = 2m_{\text{cons}}$.
- **Right panel**: plane $(m_{D,ns}, m_{\text{cons}})$ with $\hat{m}_s = 0.10$ GeV $\rightarrow m_{D,ns} = 2m_{\text{cons}}$ not assumed.
Baryonic susceptibilities of 4th order, $\chi^B_4$, $\chi^{BQ}_{22}$, $\chi^{BQ}_{31}$, and $\chi^{BQS}_{121}$, from the quark-diquark model (solid black).

Dots $\rightarrow$ lattice data of [S. Borsanyi et al. JHEP 10 (2018) 205].

Dashed green lines $\rightarrow$ result from the spectrum of the RQM.
Issues

1. Introduction
   - QCD Thermodynamics
   - Hadron spectrum

2. Fluctuations of Conserved Charges in a Thermal Medium
   - Fluctuations of conserved charges
   - Fluctuations within the HRG approach

3. Quark-Diquark model for Baryons and Fluctuations
   - Quark-diquark model for baryons
   - Baryon spectrum
   - Baryonic susceptibilities
   - Semiclassical estimates of the susceptibilities

4. Conclusions
Semiclassical estimates of the susceptibilities

- **Cumulative number**

\[ N(M) = \text{Tr}(\Theta(M - \hat{H})) \simeq \int \frac{d^3xd^3p}{(2\pi)^3} \Theta(M - H) + \cdots, \]

with

\[ H = \sqrt{\vec{p}^2 + m_q^2} + \sqrt{\vec{p}^2 + m_D^2} + \sigma r - \frac{4\alpha_S}{3r} + \mu. \]

- The integral can be performed analytically as an expansion in \( M^2/\sigma \gg 1 \). When \( m_q = 0 = m_D \):

\[ N_{\text{WKB}}(M) = \frac{M^6}{720\pi\sigma^3} + \frac{\alpha_S M^4}{36\pi\sigma^2} + \frac{\alpha_S^2 2M^2}{9\pi\sigma} - \frac{M^2}{9\pi\sigma} + \cdots. \]

- This leads to the baryonic susceptibility:

\[ \chi_{BB}^{\text{WKB}}(T) = \frac{127\pi^5}{94500} \left( \frac{T^2}{\sigma} \right)^3 + \frac{31\pi^3}{5670} \alpha_S \left( \frac{T^2}{\sigma} \right)^2 + \frac{7\pi}{405} \alpha_S^2 \frac{T^2}{\sigma} - \frac{7\pi}{810} \frac{T^2}{\sigma} + \cdots. \]
When considering finite mass effects:

\[ N_{\text{WKB}}(M) = \frac{64\sqrt{2}}{945\pi\sigma^3} \left( \frac{m_q m_D}{m_q + m_D} \right)^{3/2} (M - m_q - m_D)^{9/2} + \cdots, \]

and

\[ \chi_{BB}^{\text{WKB}}(T) = \frac{(m_q m_D)^{3/2} T^3}{\pi^2\sigma^3} e^{-(m_q + m_D)/T} + \cdots, \text{ for } T \ll m_q + m_D. \]
Conclusions

- At low temperatures hadrons can be considered as a complete basis of states in terms of a Hadron Resonance Gas (HRG) approach. The HRG works at $T \lesssim 0.8 T_c$.

- Close $T_c$ many hadrons are needed to saturate the sum rule $\Rightarrow$ previous studies suggest that there are in the QCD spectrum:
  i) conventional missing states ($q\bar{q}$ and $qqq$),
  ii) hybrid states ($q\bar{q}g$ and $qqqqg$),
  iii) tetraquarks, pentaquarks, etc ($q\bar{q}q\bar{q}$, $q\bar{q}qqq$, $\cdots$).

- Cumulative number of baryons suggests that $qqq$ excited spectrum effectively behaves as a 2-body system $\Rightarrow$ quark-diquark picture of baryons.

- Fluctuations of conserved charges in confined phase of QCD allow to study hadron spectrum and missing states in 3 sectors:
  i) electric charge, ii) baryon number, and iii) strangeness.

- We have studied the spectrum of baryons within a quark-diquark model, and used it to compute the baryonic susceptibilities $\Rightarrow$ good agreement with lattice data of susceptibilities.
Thank You!