Space-time structure may be topological and not geometrical

Gabriele Carcassi and Christine Aidala
August 26, 2019
International Conference on New Frontiers in Physics
Assumptions of Physics

• This talk is part of a broader project called Assumptions of Physics (see http://assumptionsofphysics.org/)

• The aim of the project is to find a handful of physical principles and assumptions from which the basic laws of physics can be derived

• To do that we want to develop a general mathematical theory of experimental science: the theory that studies scientific theories
  • A formal framework that forces us to clarify our assumptions
  • From those assumptions the mathematical objects are derived
  • Each mathematical object has a clear physical meaning and no object is unphysical
  • Gives us concepts and tools that span across different disciplines
  • Allows us to explore what happens when the assumptions fail, possibly leading to new physics ideas

Gabriele Carcassi - University of Michigan
Experimental verifiability leads to topological spaces, sigma-algebras, ...

Infinitesimal reducibility leads to classical phase space

Irreducibility leads to quantum state space

Deterministic and reversible evolution leads to isomorphism on state space

Non-reversible evolution

Hamilton's equations
\[ \frac{d}{dt} (q, p) = (\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}) \]

Schroedinger equation
\[ i\hbar \frac{\partial}{\partial t} \psi = H\psi \]

Euler-Lagrange equations
\[ \delta \int L(q, \dot{q}, t) = 0 \]

Thermodynamics

Kinematic equivalence leads to massive particles

General mathematical theory of experimental science

State-level assumptions

Process-level assumptions
Mathematical structure for space-time

• Riemannian manifold
• Differentiable manifold + inner product
• Topological manifold + differentiable structure
• Ordered topological space + locally $\mathbb{R}^n$
• Topological space + order topology

• If we want to understand why (i.e. under what conditions) space-time has the structure it has, we first need to understand why (i.e. under what conditions) it is a topological space, it has an order topology, ...
Mathematical structure for space-time

- Riemannian manifold
- Differentiable manifold + inner product
- Topological manifold + differentiable structure
- Ordered topological space + locally $\mathbb{R}^n$
- Topological space + order topology

If we want to understand why (i.e. under what conditions) space-time has the structure it has, we first need to understand why (i.e. under what conditions) it is a topological space, it has an order topology, ...
Simple things first

• A similar hierarchy is present for other mathematical structures used in physics
  • Hilbert space – Inner product space + closure under Cauchy sequences – Vector space + inner product – ...

• If we want true understanding, then we need to understand the simpler structure first
  • This is what our project, Assumptions of Physics, is about
Outline

• In this talk we will focus on topology and order. We will:
  • Show that topologies naturally emerge from requiring experimental verifiability
  • Show that an order topology corresponds to experimental verifiability of quantities: outcomes than can be smaller, greater or equal to others
  • Then we need to understand how quantities are constructed from experimental verifiability
    • That is, find a set of necessary and sufficient conditions under which experimental verifiability gives us an order topology
  • Argue that, in the end, those conditions are untenable at Planck scale, and that ordering cannot be experimentally defined
  • Conclude that all that is built on top of an order topology (manifolds, differentiable structures, inner product) fails to be well defined at Planck scale
Verifiable statements

• The most fundamental math structures are from logic and set theory
  • All other structures are based on that
• For science, we want to extend these with experimental verifiability
• Our fundamental object will be a verifiable statement: an assertion for which we have (in principle) an experimental test that, if the statement is true, terminates successfully in a finite amount of time
• Verifiable statements do not follow standard Boolean logic:
  • We may verify “there is extra-terrestrial life” but not its negation “there is no extra-terrestrial life”
  • No negation in general, finite conjunction, countable (infinite) disjunction
What is a topology?

• Given a set $X$, a topology $T \subseteq 2^X$ is a collection of subsets of $X$ that:
  • It contains $X$ and $\emptyset$
  • In general, not closed under complement
  • It is closed under finite intersection and arbitrary (infinite) union

• How do we get to this in physics?
Start with a countable set of verifiable statements (the most we can test experimentally). We call this a basis.

<table>
<thead>
<tr>
<th>Basis $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
</tr>
<tr>
<td>$e_2$</td>
</tr>
<tr>
<td>$e_3$</td>
</tr>
<tr>
<td>...</td>
</tr>
</tbody>
</table>
Start with a countable set of verifiable statements (the most we can test experimentally). We call this a basis.

Construct all verifiable statements that can be verified from the basis (close under finite conjunction and countable disjunction). We call this an experimental domain.

<table>
<thead>
<tr>
<th>Basis $B$</th>
<th>Verifiable statements $D_X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>$s_1 = e_1 \lor e_2$</td>
</tr>
<tr>
<td>$e_2$</td>
<td></td>
</tr>
<tr>
<td>$e_3$</td>
<td>$s_2 = e_1 \land e_3$</td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>
Start with a countable set of verifiable statements (the most we can test experimentally). We call this a basis.

Construct all verifiable statements that can be verified from the basis (close under finite conjunction and countable disjunction). We call this an experimental domain.

Consider all truth assignments: it is sufficient to assign the basis
Start with a countable set of verifiable statements (the most we can test experimentally). We call this a basis.

Construct all verifiable statements that can be verified from the basis (close under finite conjunction and countable disjunction). We call this an experimental domain.

Consider all truth assignments: it is sufficient to assign the basis.

Remove truth assignments that are impossible (e.g. the distance is more than 5 and less than 3).
Start with a countable set of verifiable statements (the most we can test experimentally). We call this a basis.

Construct all verifiable statements that can be verified from the basis (close under finite conjunction and countable disjunction). We call this an experimental domain.

Consider all truth assignments: it is sufficient to assign the basis.

Remove truth assignments that are impossible (e.g. the distance is more than 5 and less than 3).

We call the remaining lines the set of possibilities for the experimental domain (what can be found experimentally).

<table>
<thead>
<tr>
<th>Basis $B$</th>
<th>Verifiable statements $D_X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>$e_2$</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Start with a countable set of verifiable statements (the most we can test experimentally). We call this a basis.

Construct all verifiable statements that can be verified from the basis (close under finite conjunction and countable disjunction). We call this an experimental domain.

Consider all truth assignments: it is sufficient to assign the basis.

Remove truth assignments that are impossible (e.g., the distance is more than 5 and less than 3).

We call the remaining lines the set of possibilities for the experimental domain (what can be found experimentally).

Each verifiable statement corresponds to a set of possibilities in which the statement is true.
Start with a countable set of verifiable statements (the most we can test experimentally). We call this a basis.

Construct all verifiable statements that can be verified from the basis (close under finite conjunction and countable disjunction). We call this an experimental domain.

Consider all truth assignments: it is sufficient to assign the basis.

Remove truth assignments that are impossible (e.g. the distance is more than 5 and less than 3).

The experimental domain $\mathcal{D}_X$ induces a natural topology on the set of possibilities $\mathcal{X}$.

The role of logic (and math) in science is to capture what is consistent (i.e. the possibilities) and what is verifiable (i.e. the verifiable statements).

We call the remaining lines the set of possibilities for the experimental domain (what can be found experimentally).

Each verifiable statement corresponds to a set of possibilities in which the statement is true.
Examples

- “the mass of the photon is less than $10^{-13}$ eV” is verifiable and corresponds to an open set (a set in the topology).
- “the mass of the photon is exactly 0 eV” is not verifiable and is not an open set (not a set in the topology).
  - However, it is falsifiable and corresponds to a closed set (the complement is in the topology).
- Topological concepts (second countability, Hausdorff spaces, interior/exterior/boundary, …) can be better understood in terms of experimental verification.
  - They are not some abstract mathematical thing: they are physically meaningful.

Gabriele Carcassi - University of Michigan
Quantities

• We can define a quantity as a measurable property of a system that has a magnitude: can be compared to another of the same kind and found to be greater or smaller

• Mathematically a quantity is formed by:
  • a set $Q$
  • a linear (total) ordering $\leq: Q \times Q \to \mathbb{B}$
  • the order topology generated by the linear ordering, whose basis elements are of the form $(-\infty, q)$ and $(q, +\infty)$; that is, we can always tell experimentally whether something is more or less than something else
    • equality, in general, is not experimentally testable: for continuous quantities corresponds to infinite precision measurements
Constructing quantities and references

• The question is: how do we operationally construct quantities? How can we model that appropriately?

• We start with the idea of a reference: something physical that partitions our range into a before, on, and after
  • E.g. a line on a ruler, the tick of a clock, a standard weight for a balance scale, a threshold on an A/D converter

• Mathematically, a reference is a tuple of three statements b/o/a; only before and after are required to be experimentally verifiable
Constructing quantities and references

• Problem 1 - In general, before/on/after are not mutually exclusive

<table>
<thead>
<tr>
<th>Before</th>
<th>On</th>
<th>After</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

In this case, the possibilities of the domain cannot correspond to distinct values
Strict references

• We say a reference is strict if before/on/after are mutually exclusive

<table>
<thead>
<tr>
<th>Before</th>
<th>On</th>
<th>After</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

• If the extent of what we measure is smaller than the extent of our reference, then we can always assume our references are strict
Multiple references

- Problem 2 - To construct a reference scale we need multiple references, but in general these would not construct a linear order.

- We need to define what it means for references to be aligned purely on the logical relationship between statements.
Ordered references

• We can say that reference 1 is before reference 2 if whenever we find something before or on the other, it must be before the second

• More precisely, if $b_1 \lor o_1 \not\equiv o_2 \lor a_2$
  • $\not\equiv$ Means the statements are incompatible, they can’t be true at the same time

• Note how $b_1 \not\leq \neg a_1 \leq b_2 \not\leq \neg a_2$
  • Where $a \not\leq b$ ($a$ is narrower than $b$) means that if $a$ then $b$ must be true as well
Aligned references

• More in general, we can say that two references are aligned if the before and not-after statement can be ordered by narrowness.

• For example, \( b_1 \preceq b_2 \preceq \neg a_1 \preceq a_2 \)
  • \( \preceq \) Means that if the first statement is true then the second statement will be true as well.
  • That is, the first statement is narrower, more specific.

• Here we see how the ordering of references is related to the logical ordering defined by the specificity (narrowness) of the statements.

• We need our references to be aligned if we want to construct a linear ordering.
Resolving the overlaps

• Problem 3a - If two different references overlap, we can’t say one is before the other: we can’t fully resolve the linear order
• Problem 3b - Conversely, if two reference don’t overlap and there can be something in between, we must be able to put a reference there
• We always need a way, then, to find (possibly finer) references to explore the full space
Refinable references

• Conceptually, a set of references is refinable if we can solve the previous problems:
  • if two references overlap we can always refine them to two that do not overlap
  • if two ordered references are not consecutive (there can be something in between) we can always construct a reference in the middle
• Mathematically is not complicated, but is tedious and not so interesting
• With these definitions and some work...
Reference ordering theorem

• An experimental domain is fully characterized by a quantity if and only if it can be generated by a set of refinable aligned strict references

<table>
<thead>
<tr>
<th>Property of references</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strict</td>
<td>The quantity is always only before/on/after the reference. This can be assumed if the extent of what we measure is smaller than the extent of the reference.</td>
</tr>
<tr>
<td>Aligned</td>
<td>The before/after statement have an ordering in term of narrowness (specificity). Necessary to have a coherent before and after over the whole range.</td>
</tr>
<tr>
<td>Refinable</td>
<td>If we have overlaps, we can always construct finer references. Necessary to create smallest mutually exclusive cases that correspond to the values.</td>
</tr>
</tbody>
</table>
Integers and reals

• If we assume that between two non-overlapping references we can only put finitely many references, then the ordering is the one of the integers
  • Equality can be tested as well

• If we assume that between two non-overlapping references we can always put another, then the ordering is the one of the reals
  • Equality cannot be tested in this case

• These are the only two orderings that are homogeneous, where all references have the same properties
  • And that is why they are the most fundamental in physics
Are these requirements tenable at Planck scale?

<table>
<thead>
<tr>
<th>Property of references</th>
<th>Meaning</th>
<th>Problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strict</td>
<td>The quantity is always only before/on/after the reference. This can be assumed if the extent of what we measure is smaller than the reference.</td>
<td>Objects measured and references are ultimately of the same kind; their extent should be comparable</td>
</tr>
<tr>
<td>Aligned</td>
<td>The before/after statements have an ordering in term of narrowness (specificity). Necessary to have a coherent before and after over the whole range.</td>
<td>If indistinguishable particles are the smallest references and are placed very close to each other, it is not clear how can be sure they haven’t switched</td>
</tr>
<tr>
<td>Refinable</td>
<td>If we have overlaps, we can always construct finer references. Necessary to create smallest mutually exclusive cases that correspond to the values.</td>
<td>The whole point of reaching Planck length is that we cannot further refine our references</td>
</tr>
</tbody>
</table>
Are these requirements tenable at Planck scale?

• If we take the quantum nature of the references into consideration, all the requirements seem untenable
  • Note that all three are necessary: if even only one fails we have a problem
• What fails is ordering itself
  • Is not that the real numbers need to be changed to rationals or integers: we don’t have numbers to begin with
Failure of ordering

• Riemannian manifold
• Differentiable manifold + inner product
• Topological manifold + differentiable structure
• Ordered topological space + locally $\mathbb{R}^n$
• Topological space + order topology

• If ordering fails, all the structures that are based on ordering fail as well. No manifold, no differentiability, no calculus, no inner product, no geometry. We need to develop a new chain of mathematical tools.
Conclusion

• Topology, the simplest mathematical structure needed for geometry, has a clear well-defined meaning in terms of experimental verifiability
  • This is appropriate as experimental verifiability is the foundation of science

• Order topology, the next required structure, formally captures the ability to experimentally compare quantities
  • The ordering is generated by logical relationships: if “x<8” then also “x<10”

• For real numbers, the requirements can only be satisfied ideally, most likely leading to a breakdown at Planck scale
  • The idea that our “measurement device” is “classical” is baked into the very nature of the order topology, which can’t then be undone up the stack
Conclusion

• The standard mathematical toolchain (i.e. manifolds, differentiability/integration, differential geometry, Riemannian geometry, ...) needs to be rethought
  • The idea that we can take something and divide it into infinitesimal contributions is intrinsically classical

• In the same way that the geometry of space-time (i.e. the metric tensor) depends on the energy/mass distribution, the topology may depend on it as well

• The foundations of physics lie in understanding the most basic mathematical structures, their physical significance and how they can be generalized
Experimental verifiability leads to topological spaces, sigma-algebras, ...

Infinitesimal reducibility leads to classical phase space

Irreducibility leads to quantum state space

Deterministic and reversible evolution leads to isomorphism on state space

Non-reversible evolution

Hamilton's equations
\[ \frac{d}{dt} (q,p) = \left( \frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q} \right) \]

Euler-Lagrange equations
\[ \delta \int L(q,\dot{q},t) = 0 \]

Schroedinger equation
\[ i\hbar \frac{\partial}{\partial t} \psi = H\psi \]

Thermodynamics

Kinematic equivalence leads to massive particles

General mathematical theory of experimental science

State-level assumptions

Process-level assumptions
For more information

- Assumptions of Physics project website: [http://assumptionsofphysics.org/](http://assumptionsofphysics.org/)

- Topology and Experimental Distinguishability