Wigner - Weyl formalism and the propagator of Wilson fermions in the presence of varying external electromagnetic field

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Expectation value

$$\langle\Psi|\,\hat{A}\,|\Psi\rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \langle\Psi|x\rangle\, \langle x|\,\hat{A}\,|y\rangle\, \langle y|\Psi\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp A_W(x,p)\rho_W(x,p) \quad (1)$$

The Weyl symbol of operator $A_W(x,p)$ and Wigner distribution $W(x,p)$ are defined as follows

$$A_W(x,p) \equiv \int_{-\infty}^{\infty} dy e^{-ipy}\, \langle x + \frac{y}{2}|\,\hat{A}\,|x - \frac{y}{2}\rangle = \int_{-\infty}^{\infty} dq e^{iqx}\, \langle p + \frac{q}{2}|\,\hat{A}\,|p - \frac{q}{2}\rangle \quad (2)$$

$$W(x,p) = \int_{-\infty}^{\infty} dy e^{ipy}\, \langle x - \frac{y}{2}|\Psi\rangle\, \langle\Psi|x + \frac{y}{2}\rangle = \int_{-\infty}^{\infty} dq e^{-iqx}\, \langle p - \frac{q}{2}|\Psi\rangle\, \langle\Psi|p + \frac{q}{2}\rangle \quad (3)$$

The Moyal product (1D)

$$(\hat{A}\hat{B})_W = \int_{-\infty}^{\infty} dq e^{iqx} \langle p + \frac{q}{2} | \hat{A}\hat{B} | p - \frac{q}{2} \rangle = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dq e^{iqx} \langle p + \frac{q}{2} | \hat{A} | k \rangle \langle k | \hat{B} | p - \frac{q}{2} \rangle \tag{4}$$

changing variables

$$q = u + v \qquad k = p - u/2 + v/2 \tag{5}$$

$$(\hat{A}\hat{B})_W = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv e^{iux} \langle p + \frac{u}{2} + \frac{v}{2} | \hat{A} | p - \frac{u}{2} + \frac{v}{2} \rangle e^{ivx} \langle p - \frac{u}{2} + \frac{v}{2} | \hat{B} | p - \frac{u}{2} - \frac{v}{2} \rangle = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \left[e^{iux} \langle p + \frac{u}{2} | \hat{A} | p - \frac{u}{2} \rangle \right] e^{\frac{i}{2}\left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right)} \left[e^{ivx} \langle p + \frac{v}{2} | \hat{B} | p - \frac{v}{2} \rangle \right] \tag{6}$$

$$(AB)_W(x,p) \equiv A_W(x,p) \star B_W(x,p) = A_W(x,p) e^{\overleftrightarrow{\Delta}} B_W(x,p) \tag{7}$$

where

$$\overleftrightarrow{\Delta} \equiv \frac{i}{2}\left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right) \tag{8}$$

Partition function in Euclidean space

$$Z = \int D\bar{\Psi} D\Psi e^{-S(\bar{\Psi},\Psi)} = \int D\bar{\Psi} D\Psi \exp\left(- \sum_{\mathbf{r}_n,\mathbf{r}_m} \bar{\Psi}(\mathbf{r}_m) \left(\mathcal{D}_{\mathbf{r}_n,\mathbf{r}_m}\right) \Psi(\mathbf{r}_n) \right) \quad (9)$$

where the Dirac operator in coordinates space is defined as follows

$$\mathcal{D}_{\mathbf{x},\mathbf{y}} = -\frac{1}{2} \sum_{i=1,2,3,4} \left[(1+\gamma^i)\delta_{\mathbf{x}+\mathbf{e}_i,\mathbf{y}} + (1-\gamma^i)\delta_{\mathbf{x}-\mathbf{e}_i,\mathbf{y}} \right] U_{\mathbf{x},\mathbf{y}} + (m^{(0)}+4)\delta_{\mathbf{x},\mathbf{y}} \quad (10)$$

γ^k are the Euclidean Dirac gamma matrices, $m^{(0)}$ is the parameter that has the meaning of mass, and, $U_{\mathbf{x},\mathbf{y}} = Pe^{i\int_{\mathbf{x}}^{\mathbf{y}} d\boldsymbol{\xi}\mathbf{A}(\boldsymbol{\xi})}$ is a parallel transport operator.
In the momentum space the partition function, using Peierls substitution, takes the form

$$Z = \int D\bar{\Psi} D\Psi exp\left[-\int_{\mathcal{M}} \frac{d^4\mathbf{p}}{(2\pi)^4} \bar{\Psi}(\mathbf{p}) Q(\mathbf{p} - \mathbf{A}(i\partial_{\mathbf{p}}))\Psi(\mathbf{p}) \right] \quad (11)$$

where the Dirac operator in momentum is given

$$Q(\hat{\mathbf{p}}) = \sum_{\mu=1,2,3,4} i\gamma^{\mu} sin(\hat{p}_{\mu}) + \left[m^{(0)} + \sum_{\nu=1,2,3,4} (1-\cos(\hat{p}_{\nu})) \right] \quad (12)$$

Straightforward, though, approximate method

$$[\hat{A}]_W(x_n, p) = \int_{\mathcal{M}} dq e^{iqx_n} \langle p + \frac{q}{2} | \hat{A} | p - \frac{q}{2} \rangle \tag{13}$$

$$[\hat{\rho}]_W(x_n, p) = W(x, p) = \int_{\mathcal{M}} dq e^{-iqx_n} \langle p - \frac{q}{2} | \hat{\rho} | p + \frac{q}{2} \rangle \tag{14}$$

$$\langle \Psi | \hat{A} | \Psi \rangle = \sum_{x_n} \int_{\mathcal{M}} \frac{dp}{\mathcal{M}} A_W(x_n, p) \rho_W(x_n, p) \tag{15}$$

$$[f(\hat{x})]_W(x_n, p) = f(x_n) \qquad [h(\hat{p})]_W(x_n, p) = h(p) \tag{16}$$

for near diagonal operators, $qa << 1$

$$\partial_n e^{iqna} = \frac{e^{iqna}(e^{iqa} - 1)}{a} \approx iqe^{iqna} \tag{17}$$

Hence, the Moyal product may be used as in the continuous case, replacing the derivatives

$$(AB)_W(x_n, p) = A_W(x_n, p) \star B_W(x_n, p) = A_W(x_n, p) e^{\frac{i}{2}\left(\overleftarrow{\partial}_{x_n}\overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_{x_n}\right)} B_W(x_n, p) \tag{18}$$

The connection between two-point Green's function with Dirac operator

In the continuous case we have

$$(i\gamma_\mu \partial_x^\mu - m)G(x-y) = \delta(x-y) \tag{19}$$

which can be rewritten as

$$\hat{D}\hat{G} = 1 \tag{20}$$

We would like to apply the Weyl-Wigner tranformation

$$(\hat{Q}\hat{G})_W = 1_W \tag{21}$$

where

$$\hat{Q} = Q(\hat{\mathbf{p}} - \mathbf{A}(\hat{\mathbf{x}})) \tag{22}$$

Under limitations discussed above, the Moyal product is valid, hence

$$(\hat{Q}\hat{G})_W = Q_W \star G_W = 1 \tag{23}$$

The technique for calculation of G_W from Q_W will be discussed later

In case,

$$\partial_i A_i(\mathbf{x}) = 0 \tag{24}$$

we have

$$[\hat{p}_i, A_i(\hat{\mathbf{x}})] = 0 \tag{25}$$

Terms like $e^{i(\hat{p}_i - A_i(\hat{\mathbf{x}}))}$, that apear in the the Dirac operator for Wilson fermions, Q, may be written as

$$e^{i(\hat{p}_i - A_i(\hat{\mathbf{x}}))} = e^{i\hat{p}_i} e^{-iA_i(\hat{\mathbf{x}})} \tag{26}$$

Hence,

$$[e^{i(\hat{p}_i - A_i(\hat{\mathbf{x}}))}]_W = [e^{i\hat{p}_i} e^{-iA_i(\hat{\mathbf{x}})}]_W = [e^{i\hat{p}_i}] \star [e^{-iA_i(\hat{\mathbf{x}})}]_W = e^{ip_i} e^{-iA_i(\mathbf{x}_n)} \tag{27}$$

Since the, $p_i e^{\frac{i}{2}\left(\overleftarrow{\partial}_{\mathbf{r}_n} \cdot \overrightarrow{\partial}_{\mathbf{p}} - \overleftarrow{\partial}_{\mathbf{p}} \cdot \overrightarrow{\partial}_{\mathbf{r}_n}\right)} A_i(\mathbf{x}_n) = p_i A_i(\mathbf{x}_n)$, because A_i doesn't depend on x_i.

$$[Q(\hat{\mathbf{p}} - \mathbf{A}(\hat{\mathbf{x}}))]_W(x_n, p) = Q(\mathbf{p} - \mathbf{A}(\mathbf{x}_n)) \tag{28}$$

And, of course, $A_i(x_n)$ is a slowly varying function.

Approximation physical limitations in case of uniform constant magnetic field

In case of uniform magnetic field $\mathbf{B} = B\mathbf{e}_3$ (bringing back the physical units)

$$\mathbf{p} \to \mathbf{p} - e\mathbf{A} = \mathbf{p} - eB\mathbf{e}_2 x_1 \tag{29}$$

The Weyl symbol of the Dirac operator $Q(\mathbf{p} - eB\mathbf{e}_2 x_{1n})$ includes terms like

$$\exp\left(i\frac{e}{\hbar}Bx_{1n}a\right) = \exp\left(i\frac{e}{\hbar}Ba^2 n_1\right) \tag{30}$$

So, our naive method of using a continuous theory definitions in the lattice model is valid as far as the following condition remains

$$\frac{e}{\hbar}Ba^2 << 1 \tag{31}$$

$$B << \frac{\hbar}{ea^2} = \frac{6.626 \times 10^{-34}[m^2 kgs^{-1}]}{2\pi 1.6 \times 10^{-19}[C] \times 10^{-20}[m^2]} \sim 10^5[T] \tag{32}$$

For example, the highest magnetic fields in recent QHE experiments were about 20[T]

Baker–Campbell–Hausdorff formula

$$\hat{Q}(\hat{\mathbf{p}}) = \sum_{\mu=1,2,3,4} i\gamma^{\mu} sin(\hat{p}_{\mu}) + \left[m^{(0)} + \sum_{\nu=1,2,3,4} (1 - \cos(\hat{p_{\nu}}))\right] \quad (33)$$

In oreder to find the Weyl symbol of a typical expression in $\hat{Q}(\hat{\mathbf{p}} - \mathbf{A}(\hat{\mathbf{x}}))$, $\exp\left[i(\hat{p}_{\mu} - A_{\mu}(\hat{\mathbf{x}}))a\right]$, we start with $\exp\left(i(\hat{p}_{\mu} - A_{\mu} e^{i\mathbf{k}\hat{\mathbf{x}}})a\right)$.

Using a special case of Baker–Campbell–Hausdorff formula

$$e^{X+Y} = e^{X} e^{\beta Y} \quad (34)$$

where $[X, Y] = \alpha Y$, and, $\beta \equiv \frac{1-e^{-\alpha}}{\alpha}$, one gets

$$\exp\left(i(\hat{p}_{\mu} - A_{\mu} e^{i\mathbf{k}\hat{\mathbf{x}}})a\right) = e^{i\hat{p}_{\mu} a} \exp(-i\beta a A_{\mu} e^{i\mathbf{k}\hat{\mathbf{x}}}) \quad (35)$$

and the commutation relations

$$[ia\hat{p}_{\mu}, iaA_{\mu} e^{i\mathbf{k}\hat{\mathbf{x}}}] = (ik_{\mu} a) iaA_{\mu} e^{i\mathbf{k}\hat{\mathbf{x}}} \quad (36)$$

gives $\alpha = ik_{\mu} a$

Weyl symbol of $\exp\left[i(\hat{p}_k - A_k(\hat{\mathbf{x}}))\right]$

$$\left[\exp\left(i(\hat{p}_\mu - A_\mu e^{i\mathbf{k}\hat{\mathbf{x}}})\right)\right]_W = \left[e^{i\hat{p}_\mu}\exp\left(-i\beta_\mu A_\mu e^{i\mathbf{k}\hat{\mathbf{x}}}\right)\right]_W = \exp\left[i\left(p_\mu - \frac{sinh(k_\mu/2)}{k_\mu/2}A_\mu e^{i\mathbf{k}\mathbf{x}_n}\right)\right] \quad (37)$$

$$\left[\exp\left(i(\hat{p}_\mu - \sum_{J=1}^{N} A_{J\mu} e^{\mathbf{k}_J\hat{\mathbf{x}}})\right)\right]_W = \exp\left[i\left(p_\mu - \sum_{J=1}^{N}\alpha_{J\mu}A_{J\mu}e^{\mathbf{k}_J\mathbf{x}_n}\right)\right] \quad (38)$$

where $a(k_{J\mu}) = \frac{\sinh(k_{J\mu}/2)}{k_{J\mu}/2}$

$$\left[\exp\left(i\hat{p}_\mu - i\int\left[\tilde{A}_\mu(\mathbf{k})e^{\mathbf{k}i\partial_{\mathbf{p}}} + c.c.\right]dk\right)\right]_W = \exp\left(ip_\mu - i\mathcal{A}_\mu(\mathbf{x}_n)\right) \quad (39)$$

where $\mathcal{A}_\mu(\mathbf{x}) = \int\left[a_\mu(\mathbf{k})\tilde{A}_\mu(\mathbf{k})e^{\mathbf{k}\mathbf{x}} + c.c.\right]dk$ and $a_\mu(\mathbf{k}) = \frac{\sinh(k_\mu/2)}{k_\mu/2}$

For slowly varying fields, $ka << 1$, we get the expected result $\mathcal{A}_\mu(\mathbf{x}_n) = A_\mu(\mathbf{x}_n)$

The two point Green function $\mathcal{G}(\mathbf{p}_1, \mathbf{p}_2)$ obeys equation

$$\hat{Q}(\hat{\mathbf{p}} - \mathbf{A}(i\partial_{\mathbf{p}}))\mathcal{G}(\mathbf{p}, \mathbf{q}) = \delta(\mathbf{p} - \mathbf{q}) \tag{40}$$

The Weyl symbol of $\mathcal{G}$ is defined as

$$\mathcal{G}_W(\mathbf{x}_n, \mathbf{p}) \equiv \int d\mathbf{q} e^{i\mathbf{x}_n\mathbf{q}} \mathcal{G}(\mathbf{p} + \mathbf{q}/2, \mathbf{p} - \mathbf{q}/2) \tag{41}$$

It obeys the Groenewold equation

$$\mathcal{G}_W(\mathbf{x}_n, \mathbf{p}) \star Q_W(\mathbf{x}_n, \mathbf{p}) = 1 \tag{42}$$

So, we have to find the Weyl symbol of $\hat{Q}(\mathbf{p} - \mathbf{A}(i\partial_{\mathbf{p}}))$, and then, to solve the Groenewold equation in order to find the Weyl symbol of the propagator $\mathcal{G}(\mathbf{p}, \mathbf{q})$ in the presence of $\mathbf{A}$

Let us represent $\mathcal{G}_W(x_n, p)$ as the series

$$\mathcal{G}_W(x_n,p) = \mathcal{G}_W^{(0)}(x_n,p) + \mathcal{G}_W^{(1)}(x_n,p) + \mathcal{G}_W^{(2)}(x_n,p) + \dots \tag{43}$$

Substituting it into the Groenewold equation $1 = \mathcal{G}_W(x_n,p) e^{\overleftrightarrow{\Delta}} Q_W(x_n,p)$

$$\begin{aligned} 1 =& \mathcal{G}_W^{(0)}(x_n,p) Q_W(x_n,p) \\ 0 =& \mathcal{G}_W^{(0)}(x_n,p) \overleftrightarrow{\Delta} Q_W(x_n,p) + \mathcal{G}_W^{(1)}(x_n,p) Q_W(x_n,p) \\ 0 =& \frac{1}{2}\mathcal{G}_W^{(0)}(x_n,p) \overleftrightarrow{\Delta}^2 Q_W(x_n,p) + \mathcal{G}_W^{(1)}(x_n,p) \overleftrightarrow{\Delta} Q_W(x_n,p) + \mathcal{G}_W^{(2)}(x_n,p) Q_W(x_n,p) \end{aligned} \tag{44}$$

From this sequence we obtain

$$\begin{aligned} \mathcal{G}_W^{(0)}(x_n,p) =& Q_W^{-1}(x_n,p) \\ \mathcal{G}_W^{(1)}(x_n,p) =& -\left[Q_W^{-1}(x_n,p)) \overleftrightarrow{\Delta} Q_W(x_n,p)\right] Q_W^{-1}(x_n,p) \\ \mathcal{G}_W^{(2)}(x_n,p) =& -\left[\frac{1}{2}\mathcal{G}_W^{(0)}(x_n,p) \overleftrightarrow{\Delta}^2 Q_W(x_n,p) + \mathcal{G}_W^{(1)}(x_n,p) \overleftrightarrow{\Delta} Q_W(x_n,p)\right] Q_W^{-1}(x_n,p) \\ \mathcal{G}_W^{(m)}(x_n,p) =& -\sum_{k=0\dots m-1} \frac{1}{(m-k)!}\left[\mathcal{G}_W^{(k)}(x_n,p) \overleftrightarrow{\Delta}^k Q_W(x_n,p)\right] Q_W^{-1}(x_n,p) \end{aligned} \tag{45}$$

$$\mathcal{G}_W^{(m)}(x_n, p) = \sum_{\sum_{i=1\dots M} k_i = m} \mathcal{C}^m_{k_1 k_2 \dots k_M} \left[\dots \left[Q_W^{-1} \overleftrightarrow{\Delta}^{k_1} Q_W \right] Q_W^{-1} \overleftrightarrow{\Delta}^{k_2} Q_W \right] Q_W^{-1} \dots \overleftrightarrow{\Delta}^{k_M} Q_W \Big] Q_W^{-1} \tag{46}$$

where

$$\mathcal{C}^{k_1 + \dots + k_M}_{k_1 k_2 \dots k_M} = \frac{(-1)^M}{k_1! k_2! \dots k_M!} \tag{47}$$

We obtain the final form of the solution:

$$\mathcal{G}_W(x_n, p) = Q_W^{-1} + \sum_{m=1}^{\infty} \sum_{\substack{M = 1 \dots m \\ \sum_1^M k_i = m \\ k_i \neq 0}} \frac{(-1)^M}{k_1! k_2! \dots k_M!} \left[\dots \left[Q_W^{-1} \overleftrightarrow{\Delta}^{k_1} Q_W \right] Q_W^{-1} \overleftrightarrow{\Delta}^{k_2} Q_W \right] Q_W^{-1} \dots \overleftrightarrow{\Delta}^{k_M} Q_W \Big] Q_W^{-1} \tag{48}$$

The obtained solution allows to reconstruct the Green function

$$\mathcal{G}(p + q/2, p - q/2) = \frac{1}{(2\pi)^4} \sum_n e^{-i x_n q} \mathcal{G}_W(x_n, p) \tag{49}$$

$$\tilde{\mathcal{G}}(z_n, y_n) = \int \frac{d^4 p}{(2\pi)^4} e^{i(z_n - y_n)p} \mathcal{G}_W\left(\frac{z_n + y_n}{2}, p \right) \tag{50}$$

We have obtained an approximate expression for the Weyl symbol of lattice Wilson Dirac operator and the explicit expression for the fermion propagator in the presence of *arbitrary* external electromagnetic field.

Projects currently in progress

1. Formal Weyl-Wigner formalism on the lattice
2. Exact solution of the Groenewold equation

Several possible generalizations:

1. Non-Abelian external gauge field
2. Taking interactions between the fermions into account
3. More complicated lattice models - condensed matter