Wigner - Weyl formalism and the propagator of Wilson fermions in the presence of varying external electromagnetic field

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Overview

1. Introduction
   - Weyl-Wigner phase space formalism and Moyal product in the continuous case
   - Wilson fermions and the connection between two-point Green’s function with Dirac operator

2. Using Weyl-Wigner formalism for fermions on the lattice
3. Calculation of the Weyl symbol of Dirac operator for Wilson fermions
4. Propagator expression using an iterative solution of the Groenewold equation

The Weyl-Wigner formalism in the continuous case

Expectation value

\[
\langle \Psi | \hat{A} | \Psi \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \langle \Psi | x \rangle \langle x | \hat{A} | y \rangle \langle y | \Psi \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp A_W(x, p) \rho_W(x, p)
\]

The Weyl symbol of operator \( A_W(x, p) \) and Wigner distribution \( W(x, p) \) are defined as follows

\[
A_W(x, p) \equiv \int_{-\infty}^{\infty} dy e^{-ipy} \langle x + \frac{y}{2} | \hat{A} | x - \frac{y}{2} \rangle = \int_{-\infty}^{\infty} dq e^{iqx} \langle p + \frac{q}{2} | \hat{A} | p - \frac{q}{2} \rangle
\]

\[
W(x, p) = \int_{-\infty}^{\infty} dy e^{ipy} \langle x - \frac{y}{2} | \Psi \rangle \langle \Psi | x + \frac{y}{2} \rangle = \int_{-\infty}^{\infty} dq e^{-iqx} \langle p - \frac{q}{2} | \Psi \rangle \langle \Psi | p + \frac{q}{2} \rangle
\]
Moyal product in the continuous case

The Moyal product (1D)

\[
(\hat{A}\hat{B})^W = \int dq e^{iqx} \langle p + \frac{q}{2} | \hat{A}\hat{B} | p - \frac{q}{2} \rangle = \int dk \int dq e^{iqx} \langle p + \frac{q}{2} | \hat{A} | k \rangle \langle k | \hat{B} | p - \frac{q}{2} \rangle
\]

changing variables

\[q = u + v \quad k = p - u/2 + v/2\]

\[
(\hat{A}\hat{B})^W = \int du \int dv e^{iux} \langle p + \frac{u}{2} + \frac{v}{2} | \hat{A} | p - \frac{u}{2} + \frac{v}{2} \rangle e^{ivx} \langle p - \frac{u}{2} + \frac{v}{2} | \hat{B} | p - \frac{u}{2} - \frac{v}{2} \rangle =
\]

\[
\int du \int dv \left[ e^{iux} \langle p + \frac{u}{2} | \hat{A} | p - \frac{u}{2} \rangle \right] e^{i} \left( \frac{\delta_x}{\delta_p} - \frac{\delta_p}{\delta_x} \right) \left[ e^{ivx} \langle p + \frac{v}{2} | \hat{B} | p - \frac{v}{2} \rangle \right]
\]

\[(AB)^W(x, p) \equiv A^W(x, p) \ast B^W(x, p) = A^W(x, p) e^{\nabla} B^W(x, p)\]

where

\[
\nabla \equiv \frac{i}{2} \left( \frac{\delta_x}{\delta_p} - \frac{\delta_p}{\delta_x} \right)
\]
Wilson fermions

Partition function in Euclidean space

\[
Z = \int D\bar{\Psi} D\Psi e^{-S(\bar{\Psi},\Psi)} = \int D\bar{\Psi} D\Psi \exp \left( - \sum_{r_n, r_m} \bar{\Psi}(r_m) (D_{r_n, r_m}) \Psi(r_n) \right)
\]  

(9)

where the Dirac operator in coordinates space is defined as follows

\[
D_{x,y} = -\frac{1}{2} \sum_{i=1,2,3,4} \left[ (1 + \gamma^i)\delta_{x+e_i,y} + (1 - \gamma^i)\delta_{x-e_i,y} \right] U_{x,y} + (m^{(0)} + 4)\delta_{x,y}
\]  

(10)

\(\gamma^k\) are the Euclidean Dirac gamma matrices, \(m^{(0)}\) is the parameter that has the meaning of mass, and, \(U_{x,y} = Pe^{i \int_x^y d\xi A(\xi)}\) is a parallel transport operator.

In the momentum space the partition function, using Peierls substitution, takes the form

\[
Z = \int D\bar{\Psi} D\Psi \exp \left[ - \int_{\mathcal{M}} \frac{d^4 \mathbf{p}}{(2\pi)^4} \bar{\Psi}(\mathbf{p}) Q(\mathbf{p} - A(i\partial_\mu))\Psi(\mathbf{p}) \right]
\]  

(11)

where the Dirac operator in momentum is given

\[
Q(\mathbf{p}) = \sum_{\mu=1,2,3,4} i\gamma^\mu \sin(\hat{p}_\mu) + \left[ m^{(0)} + \sum_{\nu=1,2,3,4} (1 - \cos(\hat{p}_\nu)) \right]
\]  

(12)
Applying the Weyl-Wigner formalism to Wilson fermions on the lattice

Straightforward, though, approximate method

\[
[\hat{A}]_W(x_n, p) = \int_{\mathcal{M}} dq e^{iqx_n} \langle p + \frac{q}{2} | \hat{A} | p - \frac{q}{2} \rangle
\]  

(13)

\[
[\hat{\rho}]_W(x_n, p) = W(x, p) = \int_{\mathcal{M}} dq e^{-iqx_n} \langle p - \frac{q}{2} | \hat{\rho} | p + \frac{q}{2} \rangle
\]  

(14)

\[
\langle \Psi | \hat{A} | \Psi \rangle = \sum_{x_n} \int_{\mathcal{M}} dp \frac{dp}{\mathcal{M}} A_W(x_n, p) \rho_W(x_n, p)
\]  

(15)

\[
[f(\hat{x})]_W(x_n, p) = f(x_n) \quad [h(\hat{p})]_W(x_n, p) = h(p)
\]  

(16)

for near diagonal operators, \( qa << 1 \)

\[
\partial_n e^{iqa} = \frac{e^{iqa} (e^{iqa} - 1)}{a} \approx iqe^{iqa}
\]  

(17)

Hence, the Moyal product may be used as in the continuous case, replacing the derivatives

\[
(AB)_W(x_n, p) = A_W(x_n, p) \star B_W(x_n, p) = A_W(x_n, p)e^{i\frac{1}{2} \left(\frac{\partial}{\partial x_n} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial x_n}\right)} B_W(x_n, p)
\]  

(18)
In the continuous case we have

\[ (i\gamma_\mu \partial_\mu - m)G(x - y) = \delta(x - y) \] (19)

which can be rewritten as

\[ \hat{D}\hat{G} = 1 \] (20)

We would like to apply the Weyl-Wigner tranformation

\[ (\hat{Q}\hat{G})_W = 1_W \] (21)

where

\[ \hat{Q} = Q(\hat{p} - A(\hat{x})) \] (22)

Under limitations discussed above, the Moyal product is valid, hence

\[ (\hat{Q}\hat{G})_W = Q_W \star G_W = 1 \] (23)

The technique for calculation of \( G_W \) from \( Q_W \) will be discussed later.
Calculation of the Weyl symbol of Dirac operator for Wilson fermions

In case,
\[ \partial_i A_i(x) = 0 \]  \hspace{1cm} (24)
we have
\[ [\hat{\rho}_i, A_i(\hat{x})] = 0 \]  \hspace{1cm} (25)

Terms like \( e^{i(\hat{\rho}_i - A_i(\hat{x}))} \), that appear in the the Dirac operator for Wilson fermions, \( Q \), may be written as
\[ e^{i(\hat{\rho}_i - A_i(\hat{x}))} = e^{i\hat{\rho}_i} e^{-iA_i(\hat{x})} \]  \hspace{1cm} (26)

Hence,
\[ [e^{i(\hat{\rho}_i - A_i(\hat{x}))}]_W = [e^{i\hat{\rho}_i} e^{-iA_i(\hat{x})}]_W = [e^{i\hat{\rho}_i}]_* [e^{-iA_i(\hat{x})}]_W = e^{i\rho_i} e^{-iA_i(x_n)} \]  \hspace{1cm} (27)

Since the,
\[ p_i e^{i\frac{1}{2} \left( \partial_{r_n} \cdot \partial_p - \partial_p \cdot \partial_{r_n} \right)} A_i(x_n) = p_i A_i(x_n) \], because \( A_i \) doesn't depend on \( x_i \).
\[ [Q(\hat{p} - A(\hat{x}))]_W(x_n, p) = Q(p - A(x_n)) \]  \hspace{1cm} (28)

And, of course, \( A_i(x_n) \) is a slowly varying function.
Approximation physical limitations in case of uniform constant magnetic field

In case of uniform magnetic field $B = B e_3$ (bringing back the physical units)

$$p \rightarrow p - eA = p - eBe_{2x_1}$$

(29)

The Weyl symbol of the Dirac operator $Q(p - eBe_{2x_1n})$ includes terms like

$$\exp \left( i \frac{e}{\hbar} B x_{1n} a \right) = \exp \left( i \frac{e}{\hbar} B a^2 n_1 \right)$$

(30)

So, our naive method of using a continuous theory definitions in the lattice model is valid as far as the following condition remains

$$\frac{e}{\hbar} B a^2 << 1$$

(31)

$$B << \frac{\hbar}{ea^2} = \frac{6.626 \times 10^{-34} [m^2 kgs^{-1}]}{2\pi 1.6 \times 10^{-19}[C] \times 10^{-20}[m^2]} \sim 10^5 [T]$$

(32)

For example, the highest magnetic fields in recent QHE experiments were about 20[T]
\[
\hat{Q}(\hat{p}) = \sum_{\mu=1,2,3,4} i \gamma^\mu \sin(\hat{p}_\mu) + \left[ m^{(0)} + \sum_{\nu=1,2,3,4} (1 - \cos(\hat{p}_\nu)) \right]
\] (33)

In order to find the Weyl symbol of a typical expression in \( \hat{Q}(\hat{p} - \mathbf{A}(\hat{x})) \), \( \exp\left[ i(\hat{p}_\mu - \mathbf{A}_\mu(\hat{x}))a \right] \), we start with \( \exp \left( i(\hat{p}_\mu - \mathbf{A}_\mu e^{ik\hat{x}})a \right) \).

Using a special case of Baker–Campbell–Hausdorff formula

\[ e^{X+Y} = e^X e^Y \] (34)

where \([X, Y] = \alpha Y\), and, \( \beta \equiv \frac{1-e^{-\alpha}}{\alpha} \), one gets

\[ \exp \left( i(\hat{p}_\mu - \mathbf{A}_\mu e^{ik\hat{x}})a \right) = e^{i\hat{p}_\mu a} \exp( -i\beta a\mathbf{A}_\mu e^{ik\hat{x}} ) \] (35)

and the commutation relations

\[ [ia\hat{p}_\mu, i\mathbf{A}_\mu e^{ik\hat{x}}] = (ik_\mu a)i\mathbf{A}_\mu e^{ik\hat{x}} \] (36)

gives \( \alpha = ik_\mu a \)
Weyl symbol of $\exp\left[i(\hat{p}_k - A_k(\hat{x}))\right]$

\[
\left[ \exp \left( i(\hat{p}_\mu - A_\mu e^{ik\hat{x}}) \right) \right]_W = \left[ e^{i\hat{p}_\mu} \exp \left( -i\beta_\mu A_\mu e^{ik\hat{x}} \right) \right]_W = \exp \left[ i\left( p_\mu - \frac{\sinh(k_\mu/2)}{k_\mu/2} A_\mu e^{ikx_n} \right) \right] \quad (37)
\]

\[
\left[ \exp \left( i(\hat{p}_\mu - \sum_{J=1}^N A_{J\mu} e^{k_J\hat{x}}) \right) \right]_W = \exp \left[ i\left( p_\mu - \sum_{J=1}^N \alpha_{J\mu} A_{J\mu} e^{k_Jx_n} \right) \right] \quad (38)
\]

where $a(k_{J\mu}) = \frac{\sinh(k_{J\mu}/2)}{k_{J\mu}/2}$

\[
\left[ \exp \left( i\hat{p}_\mu - i \int [\tilde{A}_\mu(k)e^{ki\partial_p} + c.c.] \, dk \right) \right]_W = \exp \left( ip_\mu - iA_\mu(x_n) \right) \quad (39)
\]

where $A_\mu(x) = \int [a_\mu(k)\tilde{A}_\mu(k)e^{kx} + c.c.] \, dk$ and $a_\mu(k) = \frac{\sinh(k_\mu/2)}{k_\mu/2}$

For slowly varying fields, $ka << 1$, we get the expected result $A_\mu(x_n) = A_\mu(x_n)$
The two point Green function $G(p_1, p_2)$ obeys equation

$$\hat{Q}(\hat{p} - A(i\partial_p))G(p, q) = \delta(p - q)$$  \hspace{1cm} (40)

The Weyl symbol of $G$ is defined as

$$G_W(x_n, p) \equiv \int dq e^{ix_nq} G(p + q/2, p - q/2)$$  \hspace{1cm} (41)

It obeys the Groenewold equation

$$G_W(x_n, p) \star Q_W(x_n, p) = 1$$  \hspace{1cm} (42)

So, we have to find the Weyl symbol of $\hat{Q}(\hat{p} - A(i\partial_p))$, and then, to solve the Groenewold equation in order to find the Weyl symbol of the propagator $G(p, q)$ in the presence of $A$. 
Iterative solution of the Groenewold equation

Let us represent $G_W(x_n, p)$ as the series

$$G_W(x_n, p) = G_W^{(0)}(x_n, p) + G_W^{(1)}(x_n, p) + G_W^{(2)}(x_n, p) + \ldots$$ (43)

Substituting it into the Groenewold equation $1 = G_W(x_n, p) e \leftarrow \Delta Q_W(x_n, p)$

$$1 = G_W^{(0)}(x_n, p) Q_W(x_n, p)$$

$$0 = G_W^{(0)}(x_n, p) \leftarrow \Delta Q_W(x_n, p) + G_W^{(1)}(x_n, p) Q_W(x_n, p)$$ (44)

$$0 = \frac{1}{2} G_W^{(0)}(x_n, p) \leftarrow \Delta^2 Q_W(x_n, p) + G_W^{(1)}(x_n, p) \leftarrow \Delta Q_W(x_n, p) + G_W^{(2)}(x_n, p) Q_W(x_n, p)$$

From this sequence we obtain

$$G_W^{(0)}(x_n, p) = Q_W^{-1}(x_n, p)$$

$$G_W^{(1)}(x_n, p) = -\left[ Q_W^{-1}(x_n, p) \leftarrow \Delta Q_W(x_n, p) \right] Q_W^{-1}(x_n, p)$$

$$G_W^{(2)}(x_n, p) = -\left[ \frac{1}{2} G_W^{(0)}(x_n, p) \leftarrow \Delta^2 Q_W(x_n, p) + G_W^{(1)}(x_n, p) \leftarrow \Delta Q_W(x_n, p) \right] Q_W^{-1}(x_n, p)$$ (45)

$$G_W^{(m)}(x_n, p) = -\sum_{k=0}^{m-1} \frac{1}{(m - k)!} \left[ G_W^{(k)}(x_n, p) \leftarrow \Delta^k Q_W(x_n, p) \right] Q_W^{-1}(x_n, p)$$
Iterative solution of the Groenewold equation

\[ G_W^{(m)}(x_n, p) = \sum_{\sum_{i=1...M} k_i = m} C_{k_1 k_2 ... k_M}^m \left[ \ldots \left[ Q_W^{-1} \Delta k_1 Q_W \right] Q_W^{-1} \Delta k_2 Q_W \right] Q_W^{-1} \ldots \Delta k_M Q_W \] \tag{46}

where

\[ C_{k_1 k_2 ... k_M}^{k_1 + ... + k_M} = \frac{(-1)^M}{k_1! k_2! \ldots k_M!} \tag{47} \]

We obtain the final form of the solution:

\[ G_W(x_n, p) = Q_W^{-1} + \sum_{m=1}^{\infty} \sum_{M=1...m} \frac{(-1)^M}{k_1! k_2! \ldots k_M!} \left[ \ldots \left[ Q_W^{-1} \Delta k_1 Q_W \right] Q_W^{-1} \Delta k_2 Q_W \right] Q_W^{-1} \ldots \Delta k_M Q_W \] \tag{48}

The obtained solution allows to reconstruct the Green function

\[ \mathcal{G}(p + q/2, p - q/2) = \frac{1}{(2\pi)^4} \sum_n e^{-i x_n q} G_W(x_n, p) \] \tag{49}

\[ \tilde{\mathcal{G}}(z_n, y_n) = \int \frac{d^4 p}{(2\pi)^4} e^{i(z_n - y_n)p} G_W \left( \frac{z_n + y_n}{2}, p \right) \] \tag{50}
We have obtained an approximate expression for the Weyl symbol of lattice Wilson Dirac operator and the explicit expression for the fermion propagator in the presence of arbitrary external electromagnetic field.

Projects currently in progress

1. Formal Weyl-Wigner formalism on the lattice
2. Exact solution of the Groenewold equation

Several possible generalizations:

1. Non-Abelian external gauge field
2. Taking interactions between the fermions into account
3. More complicated lattice models - condensed matter