

The layered phase of anisotropic gauge theories

A model for topological insulators

Stam Nicolis

Institut Denis Poisson
Université de Tours, Université d'Orléans, CNRS (UMR7013)
Parc Grandmont 37200 Tours, France

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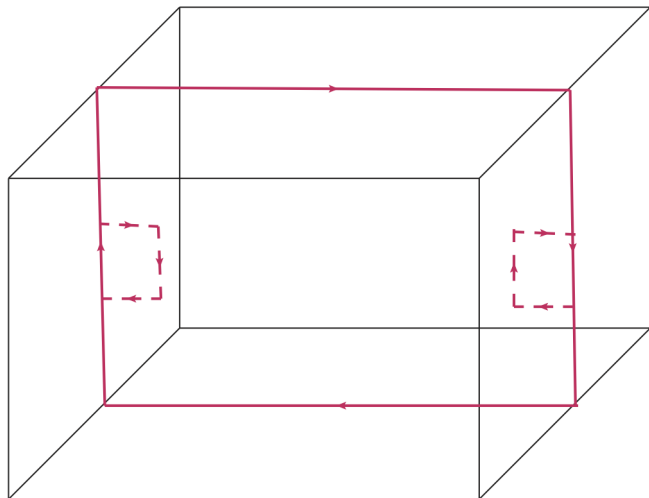
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Transport of matter in the layered phase

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- ▶ Trying to understand the phase structure of gauge theories, beyond perturbation theory—and for fields that take values in the group, rather than in the algebra. Therefore it's natural to work with a lattice regularization, that keeps gauge invariance manifest. Relevant for “exotic” phases of matter, that have attracted recent attention, in particular where transport is realized over boundaries, not through the bulk.
- ▶ Introducing anisotropic couplings in the continuum sounds strange; on the lattice it's more natural.

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- ▶ Anisotropic couplings for gauge theories haven't been studied as much. The problems seem to fall into the fault lines between subdisciplines—good to raise awareness!
- ▶ Analytic approximations can shed considerable light into what issues are the best suited for extensive numerical work—that can become non-trivial, when working in more than four dimensions. Or when insertions of non-trivial quantities (e.g. Wilson lines, surface operators) are required.

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Lattice gauge theories : Review

Let us consider the Wilson action of a lattice gauge theory

$$S = \sum_n \sum_{\mu < \nu} \beta_{\mu\nu} (1 - \text{Re } U_{\mu\nu}(n))$$
$$Z = \int \prod [dU_\mu(n)] e^S[U]$$

- ▶ If $\beta_{\mu\nu} = \beta$ for all plaquettes we have hypercubic invariance \rightarrow rotation invariance \rightarrow Lorentz invariance.
- ▶ If $\beta_{\mu\nu}$ is different, depending on the orientation of the plaquettes, can we have interesting scaling limits?
- ▶ In particular, what about scaling limits, in the presence of anisotropy, when the size of all the dimensions can become infinite? Can the infinite volume limit depend on the ratio between the couplings?

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Lattice gauge theories : Review

- ▶ In all cases gauge symmetry is manifest ; Poincaré symmetry is broken. Not a problem in more than four dimensions ; nor in less than four.
- ▶ In particular, it's interesting to probe ways, in which it can become “emergent” at some scale and become “hidden” in another.

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From anisotropic couplings to “extra” dimensions

- ▶ We shall consider the case of $d_{\parallel} + d_{\perp}$ dimensions ; $\beta_{\mu\nu}$ is the identity matrix, with the first d_{\parallel} entries equal to β and the rest equal to β' . So there are d_{\perp} “extra dimensions”. More general matrices describe “twists”, through fluxes.
These can describe topological defects in elastic media.

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Lattice gauge theories : The constraints

Fundamental reference : J.-M. Drouffe and J.-B. Zuber, *Phys. Repts.* **102** (1983) 1.

The link variables are constrained—they parametrize the group manifold :

- ▶ For $U(1)$ they satisfy $[U_\mu(n)^R]^2 + [U_\mu(n)^I]^2 = 1$
(the unit circle S^1)
- ▶ For $SU(2)$ they satisfy
 $[U_\mu(n)^0]^2 + [U_\mu(n)^1]^2 + [U_\mu(n)^2]^2 + [U_\mu(n)^3]^2 = 1$
(the unit 3-sphere S^3)

N.B. By “deforming” these constraints, it’s possible to introduce “defects” in potentially interesting ways (viz. “squashed” 3-sphere as target space; that was used, in fact to describe negative curvature defects, disclinations, in metallic glasses, using the Hopf fibration!).

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Solving the constraints

For the case of the $U(1)$ gauge group

$$\begin{aligned} Z &= \int [dU_\mu(n)] e^{S[U]} \times \\ &\quad dV_l^R dV_l^I \delta(V_l^R - U_\mu(n)^R) \delta(V_l^I - U_\mu(n)^I) = \\ &= \int [dU_\mu(n)] e^{S[V_l^R, V_l^I]} \times \\ &\quad dV_l^R dV_l^I d\alpha_l^R d\alpha_l^I e^{i[\alpha_l^R(-V_l^R + U_\mu(n)^R) + \alpha_l^I(-V_l^I + U_\mu(n)^I)]} = \\ &= \int dV_l^R dV_l^I d\alpha_l^R d\alpha_l^I e^{S[V_l^R, V_l^I] - i[\alpha_l^R V_l^R + \alpha_l^I V_l^I]} \times \\ &= \int dU_\mu(n)^R dU_\mu(n)^I e^{i[\alpha_l^R U_\mu(n)^R + \alpha_l^I U_\mu(n)^I]} \end{aligned}$$

The last line defines the “1-link integral”,
 $e^{w(\alpha_l^R, \alpha_l^I)} = e^{w([\alpha_l^R]^2 + [\alpha_l^I]^2)}$, over the gauge group. It's
useful to perform a “Wick rotation” and integrate the α_l
along the imaginary axis.

This formulation highlights the “dual” variables, α_l ; that
their action factorizes over the links means that these
are “confined”. Can be identified as the monopoles.

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Analytic calculations : The mean field approximation

- ▶ Assume factorization over the links of the Wilson action.
- ▶ Translationally invariant factorization :
 $V_j^R = V, \alpha_j^R = \alpha$ and $V_j^I = \tilde{V}, \alpha_j^I = \tilde{\alpha}$ across the links.
- ▶ Anisotropic factorization : $V_j^R = V, \alpha_j^R = \alpha,$
 $V_j^I = \tilde{V}, \alpha_j^I = \tilde{\alpha},$ for $\mu = 1, 2, \dots, d_{\parallel}$ and
 $V_j^R = V', \alpha_j^R = \alpha',$ with $V_j^I = \tilde{V}', \alpha_j^I = \tilde{\alpha}'$ for
 $\mu = d_{\parallel} + 1, \dots, d_{\parallel} + d_{\perp}.$
- ▶ We can check that, in both cases, the action depends only on the $U(1)$ invariant combinations $V^2 + \tilde{V}^2, V'^2 + \tilde{V}'^2, \alpha^2 + \tilde{\alpha}^2, \alpha'^2 + \tilde{\alpha}'^2$ (the scalar products between the V and α are invariant). We can take $\tilde{V} = 0 = \tilde{\alpha}$ and $\tilde{V}' = 0 = \tilde{\alpha}'.$

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The action

If we consider the case of only two couplings, β and β' , we have

$$S = \beta \frac{d_{\parallel}(d_{\parallel}-1)}{2} (V^4 - 1) + \beta' \frac{d_{\perp}(d_{\perp}-1)}{2} (V'^4 - 1) + \beta' d_{\parallel} d_{\perp} (V^2 V'^2 - 1) + d_{\parallel} (w(\alpha) - V\alpha) + d_{\perp} (w(\alpha') - \alpha' V')$$

The equations of motion :

$$V = dw(\alpha)/d\alpha$$

$$V' = dw(\alpha')/d\alpha'$$

$$\alpha = 2\beta(d_{\parallel} - 1)V^3 + 2\beta'd_{\perp}VV'^2$$

$$\alpha' = 2\beta'(d_{\perp} - 1)V'^3 + 2\beta'd_{\parallel}V^2V'$$

They have solutions $(V, \alpha) = (0, 0)$ and $(V', \alpha') = (0, 0)$; $(V, \alpha) \neq (0, 0)$ and $(V', \alpha') \neq (0, 0)$; and $(V, \alpha) \neq (0, 0)$ but $(V', \alpha') = (0, 0)$.

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Clearing up misunderstandings (1) : What about Elitzur's theorem ?

All these solutions are **consistent** with Elitzur's theorem. The reason is, that Elitzur's theorem assumes that it is possible to perform *local* transformations. In the mean field approximation, only *global* transformations are possible. It isn't the local symmetry that's broken-which is impossible ; it's the global symmetry. But it's "spontaneously" broken, which means it's "hidden". So the Ward identities for the moments $\langle V^m V'^n \rangle$ and so on, that describe the invariance under $SO(2) \simeq U(1)$ transformations still hold.

The solution with $(V, \alpha) \neq (0, 0)$ and $(V', \alpha') \neq (0, 0)$ simply describes the Coulomb phase, since a Wilson loop of perimeter L is equal to V^L or $V^{L_{\parallel}} V'^{L_{\perp}}$, with $L_{\parallel} + L_{\perp} = L$.

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A new phase : The layered phase

The solution with $(V, \alpha) \neq (0, 0)$ but $(V', \alpha') = (0, 0)$ describes the “layered phase”, where the layers (or branes!) are defined by the property that the Wilson loops display perimeter law within the layer and area law perpendicular to the layer.

Therefore, while the extent of the lattice along all directions is, in fact, infinite, the theory gets “dimensionally reduced” along any layer.

However the couplings β and β' are related in a very special way—that’s how the “extra dimensions” can be probed.

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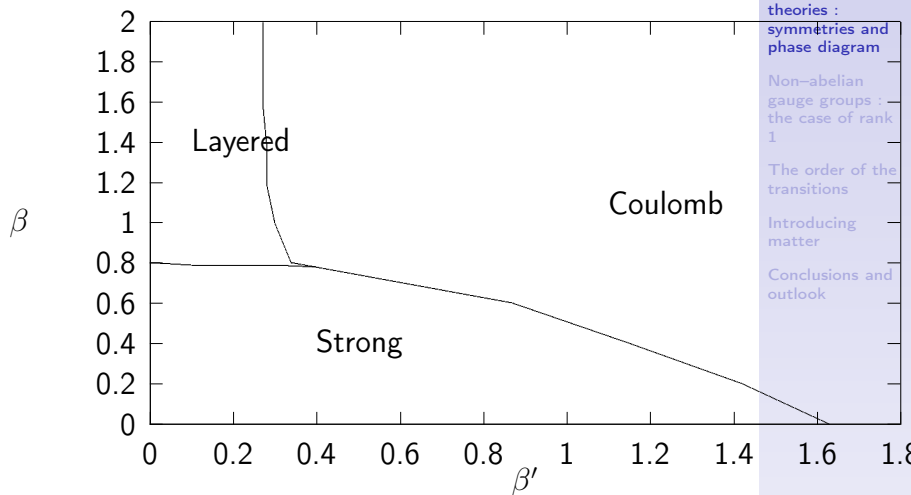
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Clearing up misunderstandings (2) : The Coulomb phase of non-abelian gauge theories

The *exact* expression for the action, in terms of the V and α variables, that are, now, matrices, reads

$$S = S_{\text{Wilson}}[V] - \text{Tr}[\alpha^T \cdot V + V^T \cdot \alpha] + w[\text{Tr}[\alpha^T \alpha]]$$

It is invariant under local $SU(2)$ transformations, that act as $V \rightarrow uVu^\dagger$ and $\alpha \rightarrow u\alpha u^\dagger$, with $u \in SU(2)$. The “mixed” term breaks the $SU(2) \times SU(2)$ group to $SU(2)$. In the mean field approximation, it is necessary and sufficient to work within the Cartan subalgebra of $SU(2)$, in order to generate all extrema of the action. The Cartan subalgebra of $SU(2)$ is $U(1)$. That’s why, in the mean field approximation, $SU(2)$ lattice gauge theory has the same qualitative properties as the $U(1)$ lattice gauge theory. In particular, it has a Coulomb phase. 

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The equivalence between $U(1)$ and $SU(2)$ in the mean field approximation

More precisely : If $V \equiv V_0 I_{2 \times 2} + V_3 \sigma_3$ and $\alpha = \alpha_0 I_{2 \times 2} + \alpha_3 \sigma_3$, then

$$S_{SU(2)}[V, \alpha] = S_{\text{Wilson}}[V_0^2 + V_3^2] - (\alpha_0 V_0 + \alpha_3 V_3) + w_{SU(2)}(\alpha_0^2 + \alpha_3^2)$$

which is invariant under $SO(2)$ transformations that act in the same way on (V_0, V_3) and (α_0, α_3) . So, every solution to the equations of motion of this action can be mapped to a solution to the equations of motion of the $U(1)$ theory and vice versa. Only the value of the couplings, β (and β') will change. So the discovery of a Coulomb phase isn't surprising—it's mandatory. It's just the Coulomb phase of the Cartan subalgebra. Only the presence of matter, charged under the gauge group, that isn't confined, can probe beyond the Cartan.

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Non-abelian gauge theories, beyond the mean-field approximation : Can they have a layered phase ?

The first answer seems to be, No, since the layered phase was defined by plaquettes having a perimeter law within the layer and area law outside it—that's how the layer is defined, in the first place ! The way out of this difficulty is by noticing that

Yes—If they have vacua with non-zero flux, that can confine chirality and can lead to current flow, only along the boundaries !

This is an example of flux compactification.

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In the mean field approximation, the transition between the confining phase and the Coulomb phase is first order. The reason is that the minimum of the action is at the origin and the absence of a quadratic term implies that this minimum can never become a maximum ; the new minimum is degenerate with the old.

In the presence of anisotropy this is, no longer, true. There does exist a quadratic contribution, that can destabilize the minimum at the origin : It's due to the term

$$\beta' d_{\parallel} d_{\perp} (V^2 V'^2 - 1)$$

that can compete with the term $d_{\perp} (w(\alpha') - \alpha' V')$.

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So let us expand the action about the point $(\alpha \neq 0, \alpha' = 0)$ in powers of α' only. We find

$$S[V, V', \alpha, \alpha'] \approx S[V, 0, \alpha, 0] + \alpha'^2 w''(0) d_{\perp} \left(\frac{1}{2} - \beta' d_{\parallel} V^2 w''(0) \right)$$

which shows how the (local) maximum becomes a (local) minimum along a line in the (β, β') plane, since $V = dw(\alpha)/d\alpha$ and $\alpha = \alpha(\beta)$.

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Fermions

There are *two* ways to couple fermions to the actions just constructed and describe *chiral* fermions—even on the lattice :

- ▶ Introduce domain walls along the “extra” dimensions, that modify the mass term to

$$\bar{\psi}M(x_{\perp})\psi$$

so that $M(x_{\perp})$ changes sign as a function of x_{\perp} . Then only one chirality can define normalizable states along the manifolds where $M(x_{\perp})$ vanishes. The Nielsen–Ninomiya theorem is evaded by the fact that the two chiralities are separated along the “extra” dimensions. In the presence of *dynamical* gauge fields, in the layered phase, the chiral zeromode(s) are eliminated from the spectrum.

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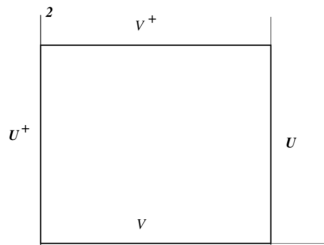
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- ▶ By introducing fluxes in the bulk : $VUV^\dagger U^\dagger = e^{i\Phi}$.



$$\phi_{p,x_5+1}^R = - \left[aM(x_5) + 5 + \sum_{\mu=1}^4 \cos(\Phi x_5 + p_\mu a) \right] \phi_{p,x_5}^R$$
$$\phi_{p,x_5+1}^L = \frac{1}{(aM(x_5 + 1) + 5 + \sum_{\mu=1}^4 \cos(\Phi x_5 + p_\mu a))} \phi_{p,x_5}^L$$

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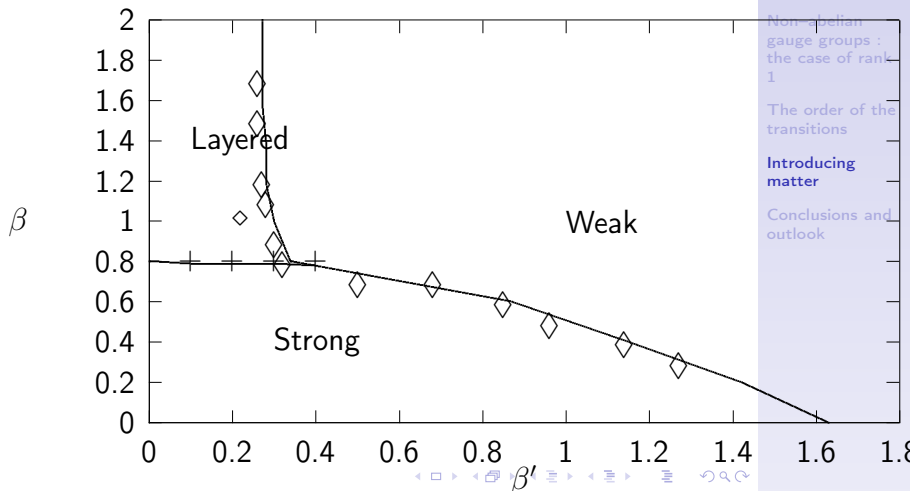
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The phase diagram in the presence of fermions



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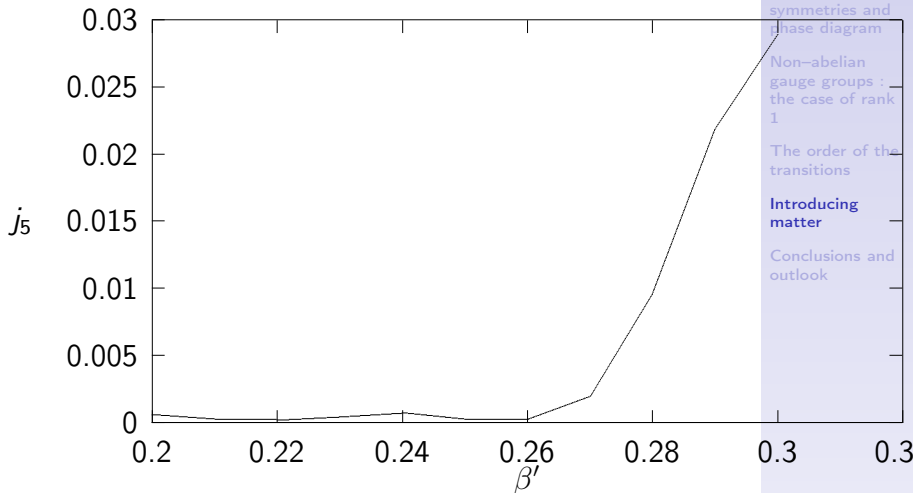
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Nothing goes through the bulk

Trying to cross the bulk at $\beta = 1.2$. In the layered phase nothing crosses; in the bulk Coulomb phase something does. An example of anomaly flow.



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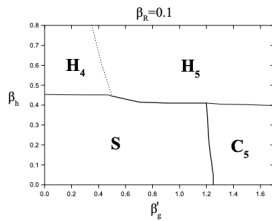
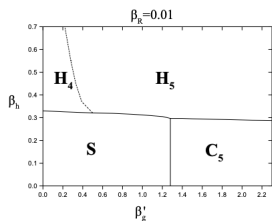
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Scalars

Typical phase diagrams in the presence of scalars (5D Abelian Higgs model) :



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- ▶ Anisotropic couplings for lattice gauge theories describe spatially non-uniform phases.
- ▶ The anisotropy implies the existence of a new phase—where strongly coupled theories can be defined.
- ▶ The layered phase is the natural setting for describing topological insulators : By construction it prohibits transport through the bulk and allows only chiral transport through the boundary. Only topological information flows through the bulk, if the anomalies aren't cancelled on the boundaries.

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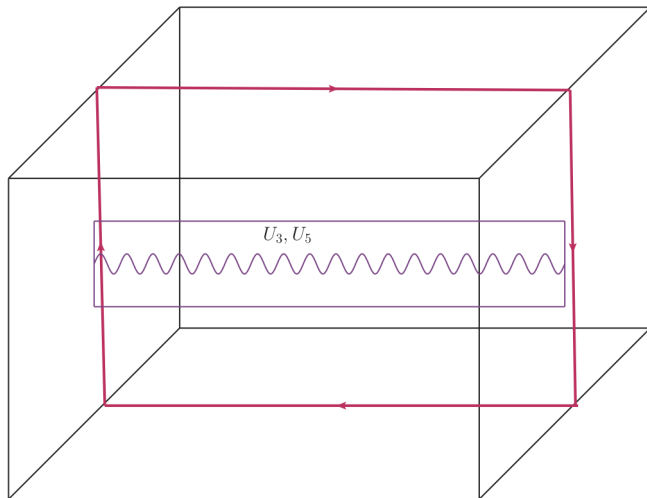
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Anomalous Transport of matter in the layered phase

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- ▶ The transitions from the layered phase to the bulk phases are continuous, so along the transition lines live conformal theories. What their detailed properties are remains to be elucidated.
- ▶ It is remarkable that the scaling properties of these theories remain quite obscure—so it isn't known, how the anisotropy of the couplings becomes visible in the scaling limit. One might think of dilaton coupling, since that, typically, couples in the “right way”, viz. $e^\Phi F^2 \leftrightarrow (1/g^2)F^2$. However this seems too naïve, and more complicated—and interesting!—couplings of the gauge field to fluctuations of the medium may be relevant for describing the anisotropy. The conceptual issues remain to be worked out.

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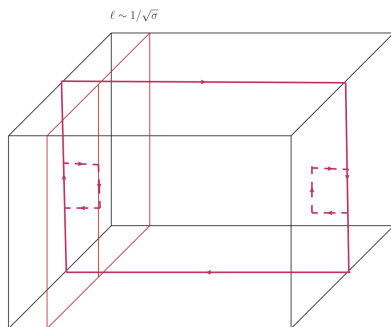
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- ▶ While the layers are sharp in the mean field approximation, corrections will make the string tension along the transition lines between the layered and the bulk phases finite. This would imply that the layers could acquire a finite thickness, leading to non-local effects on the layer.



Conclusions and outlook

- ▶ Plaquette terms that have links along extra dimensions describe mixing between gauge fields on the brane and “dark photons”.
- ▶ These considerations imply that fermions can be described by the Chern-Simons term(s); or, vice versa that the effects of the Chern-Simons terms can be described by the fermions.
- ▶ Lattice gauge theories can describe, quantitatively, transport of topological information about gauge fields and fermions. Adapting the Monte Carlo code to compute the relevant correlators is a natural next step, in order to obtain the corresponding transport coefficients (cf. work by M. A. Zubkov).

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A host of interesting problems, that are relevant for new issues in condensed matter physics as well as for high energy physics and that can be addressed with present day technology and concepts—it's just necessary to realize that they're there!

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“Now this is not the end. It is not even the beginning of the end. But it is, perhaps, the end of the beginning.”

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