Zilch currents in CKT

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Chiral effects

CKT for fermions

Vortical effect in the CKT

Vortical effects for photons

Definition of zilch

Zilch in the CKT

Conclusions
Chiral Effects
Chiral Anomaly

\[ \mathcal{L} = \bar{\psi} i \gamma_\mu D^\mu \psi - \frac{1}{4} F^2 \]

\[ \downarrow \]

\[ \partial_\mu \bar{\psi} \gamma^\mu \gamma_5 \psi = 0 \]
\[ L = \bar{\psi} i \gamma_{\mu} D^{\mu} \psi - \frac{1}{4} F^2 \]

\[ \partial_{\mu} \bar{\psi} \gamma^{\mu} \gamma^5 \psi = 0 \]

\[ \partial_{\mu} \bar{\psi} \gamma^{\mu} \gamma^5 \psi = \frac{e^2}{2\pi^2} E \cdot B \]
In chiral media anomaly results in transport phenomena

\[ J^\mu = \sigma_B B^\mu + \sigma_\omega \omega^\mu, \quad J_5^\mu = \sigma_{5,B} B^\mu + \sigma_{5,\omega} \omega^\mu \]

\[ \sigma_B = \frac{\mu_5}{2\pi^2}, \quad \sigma_\omega = \frac{\mu \mu_5}{\pi^2} \]

\[ \sigma_{5,B} = \frac{\mu}{2\pi^2}, \quad \sigma_{5,\omega} = \left( \frac{\mu^2 + \mu_5^2}{2\pi^2} + \frac{T^2}{6} \right) \]

where \( B^\mu = \tilde{F}^{\mu\nu} u_\nu \) and \( \omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta \).
In chiral media anomaly results in transport phenomena

\[ J^\mu = \sigma_B B^\mu + \sigma_\omega \omega^\mu , \quad J^\mu_5 = \sigma_5, B B^\mu + \sigma_5, \omega \omega^\mu \]

\[ \sigma_B = \frac{\mu_5}{2\pi^2} , \quad \sigma_\omega = \frac{\mu \mu_5}{\pi^2} \]

\[ \sigma_5, B = \frac{\mu}{2\pi^2} , \quad \sigma_5, \omega = \left( \frac{\mu^2 + \mu_5^2}{2\pi^2} + \frac{T^2}{6} \right) \]

where \( B^\mu = \tilde{F}^{\mu\nu} u_\nu \) and \( \omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta \).
Chiral effects were studied in various approaches:

- Free Dirac gas, linear response and strong field limit;
- Holographic plasma;
- Collisionless kinetic theory;
- Hydrodynamics;

appearing to be pretty robust and always proportional to the anomalous coefficient

\[
\partial_\mu J^\mu_5 = C E \cdot B
\]

\[
\downarrow
\]

\[
\sigma_B \sim \sigma_\omega \sim \sigma_5, B \sim \sigma_5, \omega - \frac{T^2}{6} \sim C
\]
- Chiral effects are a macroscopic manifestation of quantum anomaly

- Time parity of $B$ and $\Omega \rightarrow$ chiral effects are dissipationless

- The origin of vortical effect is less clear

- tCVE $\rightarrow$ connection with gravitational anomalies?
Anomaly from Berry curvature in CKT

The semiclassical action of a single particle:

\[ S = \int dt \left( p \cdot \dot{x} + A(x) \cdot \dot{x} - a_p \cdot \dot{p} - H(p,x) \right) \]

A single left-/right-handed fermion satisfies the Weyl equation

\[(\sigma \cdot p) u_p = \pm |p| u_p\]

The intersection of energy levels produces Berry connection

\[ i a_p \equiv u_p^{\dagger} \nabla_p u_p \]

with a monopole-like curvature in momentum space

\[ b = \nabla \times a_p = \pm \frac{\hat{p}}{2|p|^2} \]
Poisson brackets for this action are

\[
\{ p_i, p_j \} = -\frac{\epsilon_{ijk} B_k}{1 + B \cdot \Omega}, \quad \{ x_i, x_j \} = \frac{\epsilon_{ijk} \Omega_k}{1 + B \cdot \Omega}, \quad \{ p_i, x_j \} = \frac{\delta_{ij} + \Omega_i B_j}{1 + B \cdot \Omega}
\]

where \( B_i = -\epsilon_{ijk} \frac{\partial A_j}{\partial x_k} \), \( \Omega_i = -\epsilon_{ijk} \frac{\partial a_{pj}}{\partial x_k} \). Using these brackets one can proceed to develop a kinetic theory\(^1\) for Fermi-liquid and obtain kinetic equation which implies non-conservation of the particles current:

\[
\partial_t n + \nabla j = \frac{k}{4\pi^2} E \cdot B
\]

where \( k \) is the number of quanta of Berry curvature through the Fermi surface.

\(^1\)Son, Yamamoto, (2012)
Equations of motion can be written as

\[
\sqrt{G} \dot{x} = \frac{\partial \epsilon}{\partial p} + E \times b + B (\hat{p} \cdot b)
\]

\[
\sqrt{G} \dot{p} = E + \frac{\partial \epsilon}{\partial p} \times B + b (E \cdot B)
\]

where \( G = (1 + B \cdot b)^2 \). The factor of \( \sqrt{G} \) plays role of a Jacobian in the phase space

\[
d^3x d^3p / (2\pi)^3 \rightarrow \sqrt{G} d^3x d^3p / (2\pi)^3
\]

and needed to have a measure satisfying Liouville equation\(^2\): 

\[
\partial_t \sqrt{G} + \nabla_x (\sqrt{G} \dot{x}) + \nabla_p (\sqrt{G} \dot{p}) = 2\pi E \cdot B \delta^{(3)}(p)
\]

\(^2\)M. Stephanov et al, PRL, 2012
While the modified Liouville equation already indicates the axial anomaly, we can evaluate the current

\[
j = \int_p \sqrt{G} \dot{x} f(p, x)
\]

The explicit expression involves the dispersion which should also include the magnetization term

\[\varepsilon = |p| (1 - b \cdot B)\]

Taking the equilibrium limit and setting \(E = 0\) one finds the same CME current

\[
j_{\pm} = \pm \frac{\mu_{\pm}}{4\pi^2} B \quad \Rightarrow \quad j_{el} = \frac{\mu_5}{2\pi^2} B
\]

as in other approaches.
The simplest intuitive approach to describe vorticity via CKT\(^3\)
relies on the substitution

\[ B \rightarrow 2|p|\Omega \]

transforming the Lorentz force into the Coriolis force

\[ \dot{p} = E_{\text{eff}} + 2|p|\dot{x} \times \Omega \]

Concentrating on the polarization currents we finally find

\[ j_\pm = \pm \left( \frac{\mu_\pm^2}{4\pi^2} \Omega + \frac{T^2}{12} \right) \Rightarrow j_5 = \left( \frac{\mu^2 + \mu_5^2}{2\pi^2} + \frac{T^2}{6} \right) \Omega \]

which agrees with other derivations of chiral effects.

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\(^3\)M. Stephanov et al, PRL, 2012
One may be interested in the response of the helicity current of massless particles of arbitrary spin – say, photons.

Vortical effect for photons can indeed be found via Kubo formula\(^4\) for the helicity current

\[
K_\mu = \epsilon^{\mu \nu \alpha \beta} A_\nu \partial_\alpha A_\beta \quad K^\mu = \frac{T^2}{6} \omega^\mu
\]

The approach in CKT is also applicable for theory with constituents of an arbitrary spin\(^5\)

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\(^4\) A. Avkhadiev, A. Sadofyev, PRD, 2017

\(^5\) X. G. Huang and A. V. Sadofyev, JHEP (2019)
Zilch currents
In 1964 Lipkin pointed out\textsuperscript{6} that there is additional conserved current in the free electrodynamics

\[ \zeta = H \cdot B + G \cdot E \]

\[ J_\zeta = -H \times E + G \times B, \]

with \( H = \nabla \times B, \ G = \nabla \times E \)

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\textsuperscript{6}H. Lipkin, Journal of Mathematical Physics 5, (1964)
Later it was found\textsuperscript{7} that there is an infinite number of related currents. In the covariant form they can be written as

\[ Z^\mu = F^{\mu\nu} \partial_0^{2n+1} \tilde{F}_{0\nu} - \tilde{F}^{\mu\nu} \partial_0^{2n+1} F_{0\nu} \]

In a fixed gauge after quantization one can see that corresponding charge can serve as a specific measure of helicity

\[ Q_h = \int d^3x : h := \sum_{J,\lambda} (-1)^\lambda \hat{a}_\lambda^\dagger (J) \hat{a}_\lambda (J) \]

\[ Q_\zeta = \int d^3x : \zeta^{(n)} := 2(-1)^n \sum_{J,\lambda} (-1)^\lambda \omega^{(2n+2)} \hat{a}_\lambda^\dagger (J) \hat{a}_\lambda (J) \]

\textsuperscript{7}T.W.B. Kibble, Journal of Mathematical Physics 6, (1965).
Vortical effect for such a current of spin 3 was recently calculated explicitly\textsuperscript{8}

\[ J_\zeta(0) = \frac{8\pi^2 T^4}{45} \Omega \]

Chiral kinetic theory

We would like to obtain this result in CKT in order to connect ZVE with Berry phase. We need to construct a current of a charge

\[ Q_\zeta = \int d^3x : \zeta^{(n)} := 2(-1)^n \sum_{J, \lambda} (-1)^\lambda \omega^{(2n+2)} \hat{a}_\lambda^\dagger(J) \hat{a}_\lambda(J) \]

However, abundance of symmetries in a free electrodynamics implies abundance of conserved currents of the same charge. Therefore we redefine zilch of spin \(2n + 3\) as

\[ Z_i = \tilde{F}^\mu_{(i} \partial_0^{2n+1} F_{0)}\mu - F^\mu_{(i} \partial_0^{2n+1} \tilde{F}_{0)}\mu \]

The net value on the axis calculated in field theory is

\[ J_\zeta(0) = \frac{(2n + 5)}{(2n + 3)} \frac{2(-1)^n}{3\pi^2} \Omega T^{2n+4}(2n + 4)! \zeta(2n + 4) \]
We expect that the current in kinetic theory for a particle of certain helicity is

\[ Z_i = 2(-1)^n \int_p p^{2n+2} j_i \]

In order for \( j_i \) to be genuine vector it has to include a magnetization current\(^9\)

\[ j_\mu = p_\mu f + S_{\mu\nu} \partial^\nu f \]

Here \( f = n \left( p_\mu u^\mu - \frac{1}{2} S^{\mu\nu} \omega_{\mu\nu} \right) \) is a distribution function, 
\( S_{ij} = \epsilon^{ijk} \frac{p^k}{p} \) and \( \omega_{ij} = \epsilon^{ijk} \omega^k \) is a vorticity tensor.

The measure of integration is

\[
\int_p = \int \frac{d^4p}{(2\pi)^3} 2\delta(p^2) \theta(u \cdot p)
\]
\[ Z^i(0) = \frac{(2n + 5)}{(2n + 3)} \frac{2(-1)^n}{3\pi^2} \Omega^i T^{2n+4} (2n + 4)! \zeta(2n + 4) \]

The final expression for the current on the axis coincides with results obtained in the field theory.
Conclusions

- CKT allows to calculate CVE for particles of an arbitrary spin

- ZVE - as a vortical effect for gauge invariant measure of helicity of photons - can be reproduced as well

- To do so one has to introduce a notion of the zilch current in CKT

- Strikingly, the vortical effect in the zilch current is related to the Berry phase and the topological properties of the system in analogy with other chiral effects