Zilch currents in CKT

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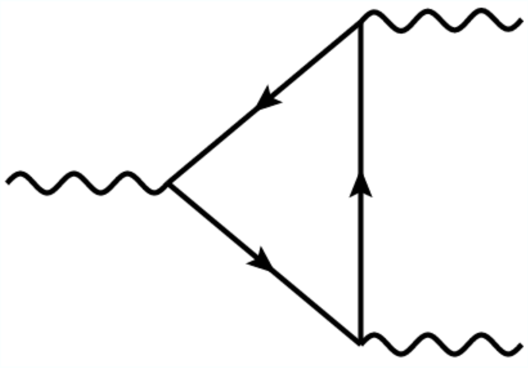
Chiral Effects

Chiral Anomaly

$$\mathcal{L} = \bar{\psi}\, i\gamma_\mu D^\mu \psi - \frac{1}{4} F^2$$

$$\downarrow$$

$$\partial_\mu \bar{\psi} \gamma^\mu \gamma_5 \psi = 0$$



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$$\partial_\mu \bar{\psi}\gamma^\mu\gamma_5\psi = \frac{e^2}{2\pi^2} E \cdot B$$

In chiral media anomaly results in transport phenomena

$$J^\mu = \sigma_B B^\mu + \sigma_\omega \omega^\mu \quad , \quad J_5^\mu = \sigma_{5,B} B^\mu + \sigma_{5,\omega} \omega^\mu$$

$$\sigma_B = \frac{\mu_5}{2\pi^2} \quad , \quad \sigma_\omega = \frac{\mu \mu_5}{\pi^2}$$

$$\sigma_{5,B} = \frac{\mu}{2\pi^2} \quad , \quad \sigma_{5,\omega} = \left(\frac{\mu^2 + \mu_5^2}{2\pi^2} + \frac{T^2}{6} \right)$$

where $B^\mu = \tilde{F}^{\mu\nu} u_\nu$ and $\omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta$.

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Chiral effects were studied in various approaches:

- Free Dirac gas, linear response and strong field limit;
- Holographic plasma;
- Collisionless kinetic theory;
- Hydrodynamics;

appearing to be pretty robust and always proportional to the anomalous coefficient

$$\partial_\mu J_5^\mu = C\, E \cdot B$$

$$\downarrow$$

$$\sigma_B \sim \sigma_\omega \sim \sigma_{5,\,B} \sim \sigma_{5,\,\omega} - \frac{T^2}{6} \sim C$$

- Chiral effects are a macroscopic manifestation of quantum anomaly
- Time parity of $\boldsymbol{B}$ and $\boldsymbol{\Omega} \rightarrow$ chiral effects are dissipationless
- The origin of vortical effect is less clear
- tCVE $\rightarrow$ connection with gravitational anomalies?

Anomaly from Berry curvature in CKT

The semiclassical action of a single particle:

$$S = \int dt \, (\boldsymbol{p} \cdot \dot{\boldsymbol{x}} + \boldsymbol{A}(\boldsymbol{x}) \cdot \dot{\boldsymbol{x}} - \boldsymbol{a}_p \cdot \dot{\boldsymbol{p}} - H(p,x))$$

A single left-/right-handed fermion satisfies the Weyl equation

$$(\boldsymbol{\sigma} \cdot \boldsymbol{p}) u_p = \pm |\boldsymbol{p}| u_p$$

The intersection of energy levels produces Berry connection

$$i \boldsymbol{a}_p \equiv u_p^\dagger \boldsymbol{\nabla}_p u_p$$

with a monopole-like curvature in momentum space

$$\boldsymbol{b} = \boldsymbol{\nabla} \times \boldsymbol{a}_p = \pm \frac{\hat{p}}{2|\boldsymbol{p}|^2}$$

Poisson brackets for this action are

$$\{p_i,p_j\} = -\frac{\epsilon_{ijk}B_k}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}} \quad \{x_i,x_j\} = \frac{\epsilon_{ijk}\Omega_k}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}} \quad \{p_i,x_j\} = \frac{\delta_{ij}+\Omega_i B_j}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}}$$

where $B_i = -\epsilon_{ijk}\frac{\partial A_j}{\partial x_k}, \Omega_i = -\epsilon_{ijk}\frac{\partial a_{\boldsymbol{p}j}}{\partial x_k}$. Using these brackets one can proceed to develop a kinetic theory[1] for Fermi-liquid and obtain kinetic equation which implies non-conservation of the particles current:

$$\partial_t n + \boldsymbol{\nabla j} = \frac{k}{4\pi^2}\boldsymbol{E}\cdot\boldsymbol{B}$$

where k is the number of quanta of Berry curvature through the Fermi surface.

[1]Son, Yamamoto, (2012)

Equations of motion can be written as

$$\sqrt{G}\dot{\boldsymbol{x}} = \frac{\partial\varepsilon}{\partial\boldsymbol{p}} + \boldsymbol{E}\times\boldsymbol{b} + \boldsymbol{B}(\hat{\boldsymbol{p}}\cdot\boldsymbol{b})$$

$$\sqrt{G}\dot{\boldsymbol{p}} = \boldsymbol{E} + \frac{\partial\varepsilon}{\partial\boldsymbol{p}}\times\boldsymbol{B} + \boldsymbol{b}(\boldsymbol{E}\cdot\boldsymbol{B})$$

where $G = (1+\boldsymbol{B}\cdot\boldsymbol{b})^2$. The factor of $\sqrt{G}$ plays role of a Jacobian in the phase space

$$d^3x d^3p/(2\pi)^3 \to \sqrt{G}d^3x d^3p/(2\pi)^3$$

and needed to have a measure satisfying Liouville equation[2]:

$$\partial_t\sqrt{G} + \boldsymbol{\nabla}_x(\sqrt{G}\dot{\boldsymbol{x}}) + \boldsymbol{\nabla}_p(\sqrt{G}\dot{\boldsymbol{p}}) = 2\pi\boldsymbol{E}\cdot\boldsymbol{B}\,\delta^{(3)}(\boldsymbol{p})$$

[2] M. Stephanov et al, PRL, 2012

While the modified Liouville equation already indicates the axial anomaly, we can evaluate the current

$$\boldsymbol{j} = \int_p \sqrt{G}\,\dot{\boldsymbol{x}} f(\boldsymbol{p}, \boldsymbol{x})$$

The explicit expression involves the dispersion which should also include the magnetization term

$$\varepsilon = |\boldsymbol{p}| \left(1 - \boldsymbol{b} \cdot B\right)$$

Taking the equilibrium limit and setting $E = 0$ one finds the same CME current

$$\boldsymbol{j}_{\pm} = \pm \frac{\mu_{\pm}}{4\pi^2} \boldsymbol{B} \quad \Rightarrow \quad \boldsymbol{j}_{el} = \frac{\mu_5}{2\pi^2} \boldsymbol{B}$$

as in other approaches.

The simplest intuitive approach to describe vorticity via CKT[3] relies on the substitution

$$\boldsymbol{B} \to 2|\boldsymbol{p}|\boldsymbol{\Omega}$$

transforming the Lorentz force into the Coriolis force

$$\dot{\boldsymbol{p}} = \boldsymbol{E}_{eff} + 2|\boldsymbol{p}|\dot{\boldsymbol{x}} \times \boldsymbol{\Omega}$$

Concentrating on the polarization currents we finally find

$$\boldsymbol{j}_{\pm} = \pm\left(\frac{\mu_{\pm}^2}{4\pi^2}\boldsymbol{\Omega} + \frac{T^2}{12}\right) \quad \Rightarrow \quad \boldsymbol{j}_5 = \left(\frac{\mu^2 + \mu_5^2}{2\pi^2} + \frac{T^2}{6}\right)\boldsymbol{\Omega}$$

which agrees with other derivations of chiral effects.

[3] M. Stephanov et al, PRL, 2012

- One may be interested in the response of the helicity current of massless particles of arbitrary spin - say, photons

- Vortical effect for photons can indeed be found via Kubo formula[4] for the helicity current

$$K_\mu = \epsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta \qquad K^\mu = \frac{T^2}{6}\omega^\mu$$

- The approach in CKT is also applicable for theory with constituents of an arbitrary spin[5]

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[4] A. Avkhadiev, A. Sadofyev, PRD, 2017

[5] X. G. Huang and A. V. Sadofyev, JHEP (2019)

Zilch currents

In 1964 Lipkin pointed out[6] that there is additional conserved current in the free electrodynamics

$$\zeta = \boldsymbol{H} \cdot \boldsymbol{B} + \boldsymbol{G} \cdot \boldsymbol{E}$$

$$J_\zeta = -\boldsymbol{H} \times \boldsymbol{E} + \boldsymbol{G} \times \boldsymbol{B},$$

with $\boldsymbol{H} = \nabla \times \boldsymbol{B}$, $\boldsymbol{G} = \nabla \times E$

[6] H. Lipkin, Journal of Mathematical Physics 5, (1964)

Later it was found[7] that there is an infinite number of related currents. In the covariant form they can be written as

$$Z^{\mu} = F^{\mu\nu}\partial_0^{2n+1}\tilde{F}_{0\nu} - \tilde{F}^{\mu\nu}\partial_0^{2n+1}F_{0\nu}$$

In a fixed guage after quantization one can see that corresponding charge can serve as a specific measure of helicity

$$Q_h = \int d^3x : h := \sum_{J,\lambda}(-1)^{\lambda}\hat{a}^{\dagger}_{\lambda}(J)\hat{a}_{\lambda}(J)$$

$$Q_{\zeta} = \int d^3x : \zeta^{(n)} := 2(-1)^n\sum_{J,\lambda}(-1)^{\lambda}\omega^{(2n+2)}\hat{a}^{\dagger}_{\lambda}(J)\hat{a}_{\lambda}(J)$$

[7] T.W.B. Kibble, Journal of Mathematical Physics 6, (1965).

Vortical effect for such a current of spin 3 was recently calculated explicitly[8]

$$\boldsymbol{J}_{\zeta}(0) = \frac{8\pi^2 T^4}{45}\boldsymbol{\Omega}$$

[8] M. N. Chernodub, A. Cortijo and K. Landsteiner, Phys. Rev. D (2018)

Chiral kinetic theory

We would like to obtain this result in CKT in order to connect ZVE with Berry phase. We need to construct a current of a charge

$$Q_\zeta = \int d^3x : \zeta^{(n)} := 2(-1)^n \sum_{J,\lambda} (-1)^\lambda \omega^{(2n+2)} \hat{a}^\dagger_\lambda(J) \hat{a}_\lambda(J)$$

However, abundance of symmetries in a free electrodynamics implies abundance of conserved currents of the same charge. Therefore we redefine zilch of spin $2n+3$ as

$$\mathbb{Z}_i = \tilde{F}^\mu_{(i} \partial_0^{2n+1} F_{0)\mu} - F^\mu_{(i} \partial_0^{2n+1} \tilde{F}_{0)\mu}$$

The net value on the axis calculated in field theory is

$$\boldsymbol{J}_\zeta(0) = \frac{(2n+5)}{(2n+3)} \frac{2(-1)^n}{3\pi^2} \Omega T^{2n+4} (2n+4)! \zeta(2n+4)$$

We expect that the current in kinetic theory for a particle of certain helicity is

$$\mathbb{Z}_i = 2(-1)^n \int_p p_{(0}^{2n+2} j_{i)}$$

In order for j_i to be genuine vector it has to include a magnetization current[9]

$$j_\mu = p_\mu f + S_{\mu\nu} \partial^\nu f$$

[9] J. Y. Chen, D. T. Son, M. A. Stephanov, H. U. Yee, Y. Yin, "Lorentz Invariance in Chiral Kinetic Theory," PRL (2014)

Here $f = n\left(p_\mu u^\mu - \frac{1}{2} S^{\mu\nu}\omega_{\mu\nu}\right)$ is a distribution function, $S_{ij} = \epsilon^{ijk}\frac{p^k}{p}$ and $\omega_{ij} = \epsilon^{ijk}\omega^k$ is a vorticity tensor.

The measure of integration is

$$\int\limits_p = \int \frac{d^4 p}{(2\pi)^3}\, 2\delta(p^2)\, \theta(u \cdot p)$$

$$\mathbb{Z}^i(0) = \frac{(2n+5)}{(2n+3)}\frac{2(-1)^n}{3\pi^2}\Omega^i T^{2n+4}(2n+4)!\zeta(2n+4)$$

The final expression for the current on the axis coincides with results obtained in the field theory

Conclusions

- CKT allows to calculate CVE for particles of an arbitrary spin
- ZVE - as a vortical effect for gauge invariant measure of helicity of photons - can be reproduced as well
- To do so one has to introduce a notion of the zilch current in CKT
- Strinkingly, the vortical effect in the zilch current is related to the Berry phase and the topological properties of the system in analogy with other chiral effects