VACUUM STRUCTURE IN 3D SUPERSYMMETRIC GAUGE THEORIES

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See the review in *Uspekhi* [arXiv: 1312.1804]

MOTIVATION

- Widely known : Maldacena duality
  $AdS_5 \times S^5 \leftrightarrow \mathcal{N} = 4$ 4d SYM

- New duality (Bagger + Lambert, 07; Aharony + Bergman + Jafferis + Maldacena, 08)

$AdS_4 \times S^7$ or $AdS_4 \times CP^3 \leftrightarrow$
$\mathcal{N} = 8$ or $\mathcal{N} = 6$ 3d SYMCS

DYNAMICS ?

THE SIMPLEST VARIANT

N = 1 3d SYM + CS

$$S = \frac{1}{g^2}\mathrm{Tr}\int d^3x \left\{-\frac{1}{2}F_{\mu\nu}^2 + i\bar{\lambda}\not{D}\lambda\right\} +$$
$$\kappa\mathrm{Tr}\int d^3x \left\{\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) - \bar{\lambda}\lambda\right\}$$

$\lambda_{1,2}$ - Majorana fermion.

- A chiral gauge theory.
- Mass scale $m = \kappa g^2$.
- Gauge invariance dictates for $k = 4\pi\kappa$ (the level) to be integer or sometimes half-integer.

Vacuum dynamics ?

Witten index

$$I_W = \mathrm{Tr}\left\{(-1)^F e^{-\beta H}\right\} = \sum_{n,\mathrm{bos}} e^{-\beta E_n} - \sum_{n,\mathrm{ferm}} e^{-\beta E_n}$$
$$= \; N_{\mathrm{bos}}(E=0) - N_{\mathrm{ferm}}(E=0)$$

- $I_W \neq 0 \;\Rightarrow$ supersymmetry is not broken.
- $I_W = 0 \;\Rightarrow$ supersymmetry may be broken.

The result (Witten, 99)

$$I(k,N) \;=\; (\mathrm{sgn}(k))^{N-1} \begin{pmatrix} |k| + N/2 - 1 \\ N-1 \end{pmatrix}$$

- for $SU(N)$ group

if $|k| \geq N/2$.

- $I(k,N) = 0$ if $|k| < N/2$.

- Spontaneous SUSY breaking at $|k| < N/2$.

Witten's derivation

• Consider the theory in a LARGE spatial box, $g^2 L \gg 1$

• Integrate mentally over fermions. k is renormalized. For positive k,

$$k \;\to\; k - \frac{N}{2}\,.$$

• The coupling g^2 is also renormalized. New higher derivative terms appear. Irrelevant at large volume !

• We obtain a pure CS theory with renormalized coupling.

• Topological theory, a finite number of states.

$$\#\text{of vacuum states in SYMCS}(k) \;=\; \#\text{of states in CS}(k{-}N/2)\,.$$

Clever people can calculate the # of states in pure CS using the correspondence

pure CS $\leftrightarrow$ 2d WZNW $\leftrightarrow$ 2d conformal theories

• The number of states on the left equals the number of conformal blocks on the right.

• This gives $(k_{CS} > 0)$

$$\#_{CS} \ = \left(\begin{array}{c} N + k_{CS} - 1 \\ N - 1 \end{array} \right) .$$

canonical quantization

(Elitzur + Moore + Schwimmer + Seiberg, 89; Labastida + Ramallo, 89)

pure CS Lagrangian

$$\mathcal{L}_{CS} \;=\; -\kappa\epsilon_{jk}\left[\mathrm{Tr}\{A_j\dot{A}_k\} + \mathrm{Tr}\{A_0 F_{jk}\}\right]$$

Canonical momenta

$$\Pi_j^a \;=\; \frac{\kappa}{2}\epsilon_{jk}A_k^a \;.$$

$\Pi_j^a \to -i\delta/(\delta A_j^a)$ as usual.

$G_j^a = \Pi_j^a - (\kappa/2)\epsilon_{jk}A_k^a = 0$ are second class constraints.

Only a half of them can be implemented

$$(\hat{G}_1^a + i\hat{G}_2^a)\Psi[A] \;=\; 0 \quad \text{or} \quad (\hat{G}_1^a - i\hat{G}_2^a)\Psi[A] \;=\; 0$$

• WE calculate the index via the effective Hamiltonian in small finite volume, $g^2 L \ll 1$. Similar to the index calculation for 4d theories (Witten, 82).

• Consider the $4d$ pure SYM theory. Gluons $A_j(\vec{x})$ and gluinos $\lambda_\alpha(\vec{x})$, $\alpha = 1, 2$.

• Small volume $g^2(L) \ll 1$

• Periodic b.c. $A_j(x + L, y, z) = A_j(x, y, z)$, etc.

• Expand in modes.

Born–Oppenheimer approach

slow variables: $C_j^{a=1,\ldots,r} \equiv A_j^{\mathrm{Cartan}(\vec{0})}$ and $\lambda_\alpha^{a=1,\ldots,r} \equiv \lambda_\alpha^{\mathrm{Cartan}(\vec{0})}$

fast variables: all the rest

• for SU(2), just 3 bosonic (C_j) and 2 holomorphic fermion (λ_α) variables.

• Topologically nontrivial gauge transformations (N = 2)

$$t^3 C_1 \;\to\; t^3 C_1 - i e^{-4\pi i t^3 x/L} \partial_j e^{4\pi i t^3 x/L} = t^3 \left(C_1 + \frac{4\pi}{L} \right)$$

and similar shifts for $C_{2,3}$.

• Wave functions are invariant:

$$\Psi \left(C_1 + \frac{4\pi}{L}, C_{2,3} \right) \;=\; \Psi \left(C_1 + \frac{4\pi}{L}, C_{2,3} \right) \;=\Psi \left(C_j \right)$$

and similarly for the shifts of C_2 and C_3

Effective Hamiltonian on the dual torus to the leading BO order:

$$H^{\rm eff} \;=\; \frac{1}{2} (\hat{P}_j^a)^2, \quad a = 1, \ldots, r \;\; \text{(rank of the group)},$$

$$\hat{P}_j^a = -i \partial / \partial C_j^a \, .$$

• For $SU(2)$, one sees four vacuum states: $1, \lambda_\alpha, \lambda_\alpha \lambda^\alpha$.

• Weyl transformations—another kind of top. nontrivial gauge transformations

• For $SU(2)$ — gauge rotation by π around the 2-nd color axis.

• Effective wave functions are Weyl invariant

$$\Psi(-C_j, -\lambda_\alpha) = \Psi(C_j, \lambda_\alpha)$$

• Two bosonic Weyl invariant functions: 1 and $\lambda_\alpha \lambda^\alpha$

• $I_W = 2$.

• $I_W = N$ for SU(N).

SYMCS

Two complications:

- The effective Hamiltonian is more complicated.
- The tree (leading-order BO) approximation is not enough. One-loop effects are necessary to take into account.

- Assume $g^2 L \ll 1$

Slow variables: $C_{j=1,2}^{a=1,\ldots,r} \equiv A_j^{\mathrm{Cartan}(\mathbf{0})}$ and $\lambda^{a=1,\ldots,r} \equiv \lambda_{1-i2}^{\mathrm{Cartan}(\mathbf{0})}$.

Leading order $(N=2)$

$$H^{\rm eff} \;=\; \frac{g^2}{2L^2}\left(\hat{P}_j - \frac{\kappa L^2}{2}\epsilon_{jk}C_k\right)^2 + \frac{\kappa g^2}{2}(\lambda\bar{\lambda} - \bar{\lambda}\lambda)\ .$$

- Landau problem on a finite (dual !) torus, $C_{j=1,2} \in (0, 4\pi/L)$

- Index = magnetic flux = $2k$. (Dubrovin, Krichever, Novikov, 76)

$$SU(N): \quad H = \frac{g^2}{L^2}\left[\frac{(\hat{P}^a_j + \mathcal{C}^a_j)^2}{2} + \frac{1}{2}\mathcal{B}^{ab}(\lambda^a\bar{\lambda}^b - \bar{\lambda}^b\lambda^a)\right]$$

where $\mathcal{B}^{ab} = \epsilon_{jk}\partial^a_j\mathcal{C}^b_k$.

- At the tree level,

$$\mathcal{C}^a_j = -\frac{\kappa L^2}{2}\epsilon_{jk}C^a_k \ , \qquad \mathcal{B}^{ab} = \kappa L^2\delta^{ab} \ .$$

- The index

$$I = \frac{1}{(2\pi)^r}\int_{T\times T}\prod_{ja} d\mathcal{C}^a_j \ \det\|\mathcal{B}^{ab}\| = Nk^{N-1} \ .$$

(Cecotti + Girardello, 82).

Vacuum wave functions, SU(2)

Quasiperiodic boundary conditions

$$\begin{aligned} \Psi(X+1,Y) &= e^{-2\pi ikY}\Psi(X,Y) \\ \Psi(X,Y+1) &= e^{2\pi ikX}\Psi(X,Y) \end{aligned}$$

$(X = C_1 L/(4\pi),\ Y = C_2 L/(4\pi))$.

$$\Psi_m \sim \sum_n \exp\left\{-2\pi k\left(n+Y+\frac{m}{2k}\right)\right. \\ \left. -2\pi ikXY - 4\pi ikX\left(n+\frac{m}{2k}\right)\right\}$$

(θ–functions)

• θ - functions are analytic functions living on the torus.

• They are distinguished by an integer q called *level*.

Quasiperiodic boundary conditions

$$\theta^q(z+1) = \theta^q(z)\,, \tag{1}$$

$$\theta^q(z+i) = e^{q\pi(1-2iz)}\theta^q(z)\,. \tag{2}$$

• $\theta^q(z)$ form a linear space of dimension q.

Its basis:

$$Q_m^q(z) = \sum_{n=-\infty}^{\infty} \exp\left\{-\pi q\left(n+\frac{m}{q}\right)^2 + 2\pi iqz\left(n+\frac{m}{q}\right)\right\}$$

$m = 0, \ldots, q-1.$

• A generic $\theta^q(z)$ has q simple zeroes.

• $\theta^q(z)$ are "*elliptic polynomials*" of degree q.

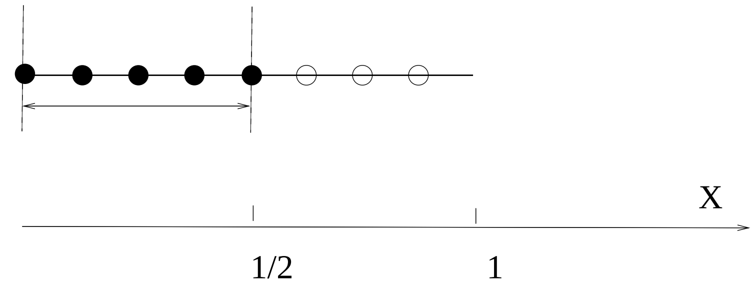

Figure 1: Maximal torus, Weyl alcove and vacuum states for $SU(2)$ ($k = 4$).

Properties of the wave functions:

- $\Psi_{m+2k}(X, Y) \;=\; \Psi_m(X, Y)$.
- $\Psi_m(X, Y) \;=\; \Psi_{2k-m}(-X, -Y)$.
- $\Psi_{m+1}(X, Y) \;=\; e^{i\pi Y}\Psi_m\left(X + \frac{1}{2k}, Y\right)$.

Weyl-invariant combinations

$$\Psi_0,\ \Psi_k,\ \text{and}\ \ \Psi_m + \Psi_{2k-m}\quad (m = 1, \ldots, k-1)\ .$$

This gives $I^{\text{tree}}[SU(2)] \;=\; k + 1\,.$

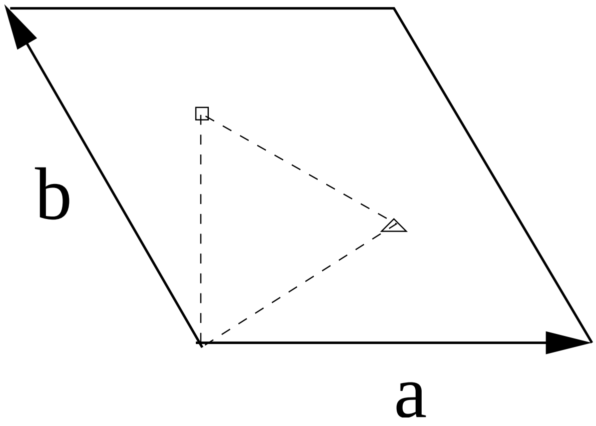

Figure 2: Maximal torus and Weyl alcove for $SU(3)$. **a** and **b** - simple coroots. The points $\Box$ and $\triangle$ - fundamental coweights.

Vacuum wave functions, SU(N=3)

• motion on $T \times T$, T being the maximal torus formed by simple coroots $\mathbf{a} = (1, 0)$ and $\mathbf{b} = (-1/2, \sqrt{3}/2)$. They satisfy $\exp\{4\pi i a^a t^a\} \equiv \exp\{iLC^a t^a\} = 1$). Shifts along **a** or **b** are gauge transformations.

Boundary conditions

$$\begin{aligned}\Psi(\mathbf{X}+\mathbf{a},\mathbf{Y}) &= e^{-2\pi i k \mathbf{a}\mathbf{Y}}\Psi(\mathbf{X},\mathbf{Y}),\\ \Psi(\mathbf{X}+\mathbf{b},\mathbf{Y}) &= e^{-2\pi i k \mathbf{b}\mathbf{Y}}\Psi(\mathbf{X},\mathbf{Y}),\\ \Psi(\mathbf{X},\mathbf{Y}+\mathbf{a}) &= e^{2\pi i k \mathbf{a}\mathbf{X}}\Psi(\mathbf{X},\mathbf{Y}),\\ \Psi(\mathbf{X},\mathbf{Y}+\mathbf{b}) &= e^{2\pi i k \mathbf{b}\mathbf{X}}\Psi(\mathbf{X},\mathbf{Y}).\end{aligned}$$

Zero energy eigenfunctions

$$\begin{aligned}\Psi_{\vec{W}}(\mathbf{X},\mathbf{Y}) = \sum_{\mathbf{n}} \exp\Big\{&-2\pi k(\mathbf{n}+\mathbf{Y}+\mathbf{W})^2\\ &-2\pi i k\mathbf{X}\mathbf{Y} - 4\pi i k\mathbf{X}(\mathbf{n}+\mathbf{W})\Big\},\end{aligned}$$

with $\mathbf{W}\cdot\mathbf{a}$ and $\mathbf{W}\cdot\mathbf{b}$ being integer multiples of $1/(2k)$.

- The # of Weyl invariant states is counted as the # of such points $\mathbf{W}$ within the Weyl alcove.

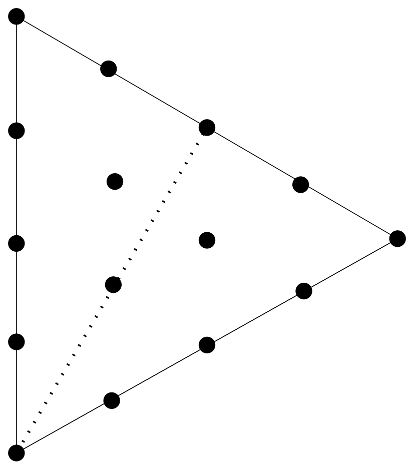

Figure 3: $SU(3)$: 15 vacuum states for $k=4$. The dotted line marks the boundary of the Weyl alcove for G_2 (9 states).

This gives

$$I^{\rm tree}_{SU(3)}(k>0) \;=\; \sum_{m=1}^{k+1} m \;=\; \frac{(k+1)(k+2)}{2} \;=\; \left(\begin{array}{c} k+2 \\ 2 \end{array} \right)$$

For any N, k,

$$I^{\rm tree}_{SU(N)}(\hbox{any } k) \;=\; [{\rm sgn}(k)]^{(N-1)} \left(\begin{array}{c} N+|k|-1 \\ N-1 \end{array} \right) .$$

Symplectic groups:

$$I^{\rm tree}_{{\rm Sp}(2r)} \;=\; (-1)^r \left(\begin{array}{c} |k|+r \\ r \end{array} \right) \; .$$

G_2:

$$I^{\rm tree}_{G_2}(k) \;=\; \left\{ \begin{array}{cc} \frac{(|k|+2)^2}{4} & \text{for even } k \\ \frac{(|k|+1)(|k|+3)}{4} & \text{for odd } k \end{array} \right\} \; .$$

LOOP CORRECTIONS

(specific for $3d$!)

1-loop renormalization
$k(>0) \to k - N/2$ (fermions) $+ N$ (gluons)

• Known in infinite volume (Pisarski + Rao, 85; Kao + Lee + Lee, 96).

• True also in finite volume.

• No renormalization beyond one loop.

• Substituting $k \to k - N/2$ in the tree level index (taking into account only fermion corrections) gives Witten's result.

??? What about gluon loop corrections ???

Heuristic rebuttal

(Witten, private communication)

The shift $k \to k + N$ due to gluon loops is an immanent feature of pure CS (shows up in Wilson loop expectation values, etc.) and should not be counted twice.

• Fine, but assumes the large volume approach, $g^2 L \gg 1$.

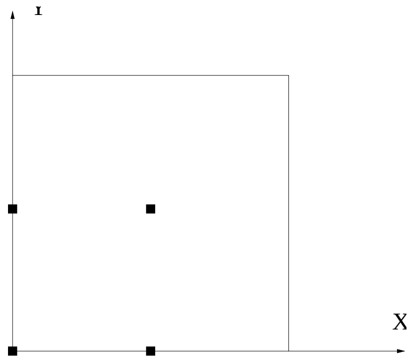

Figure 4: Dual torus $T_{SU(2)} \times T_{SU(2)}$ and its "corners".

Resolution of the paradox

- Take $N = 2$.
- Extra fluxes brought about by fermion and gluon loops are concentrated in the corners of the dual torus, where the BO approximation breaks down.

$$\mathbf{X} = (0,0); \quad \mathbf{X} = (1/2,0); \quad \mathbf{X} = (0,1/2); \quad \mathbf{X} = (1/2,1/2)\ .$$

• In the massless limit $\rightarrow$ flux lines .
• Flux +1 in each corner due to gluon loops
• Flux -1/2 in each corner due to fermion loops

• *Extra zero modes due to the presence of integer flux lines are singular at the "corners"* $\Psi \sim 1/\sqrt{X^2+Y^2}$ *and are not admissible.*

Heuristically: Integer flux lines are not observable (like Dirac strings). A singular fractional flux line is not admissible. Four of them are admissible, but their presence make it "more difficult" for the globally defined solutions to the Schrödinger equation to exist.

$$2k \;\rightarrow\; 2k-2$$

as a result.

- Amusing physics and mathematics for $N > 2$ and for nonunitary groups. Multidimensional Dirac strings.

- for $SU(2)$:

$$\mathcal{A}_3 = -\frac{i}{2z^3}$$

- for $SU(3)$:

$$\mathcal{A}_3 = -\frac{i}{2}\left(\frac{1}{z^3} + \frac{1}{z^3 + z^8\sqrt{3}} + \frac{1}{z^3 - z^8\sqrt{3}}\right)$$

$$\mathcal{A}_8 = \frac{i\sqrt{3}}{2}\left(\frac{1}{z^3 - z^8\sqrt{3}} - \frac{1}{z^3 + z^8\sqrt{3}}\right)$$

- Singular at some complex lines.

Theories with matter

- Two new effects:

- An extra matter-induced renormalization of k.

- The appearance of extra Higgs vacua due to nontrivial Yukawa interactions.

- $\mathcal{N} = 2$

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K.Intriligator + N.Seiberg, 2013

- generic $\mathcal{N} = 1$

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A.S., 2013