Stability and superluminality of cosmological models in generalized Galileon theories

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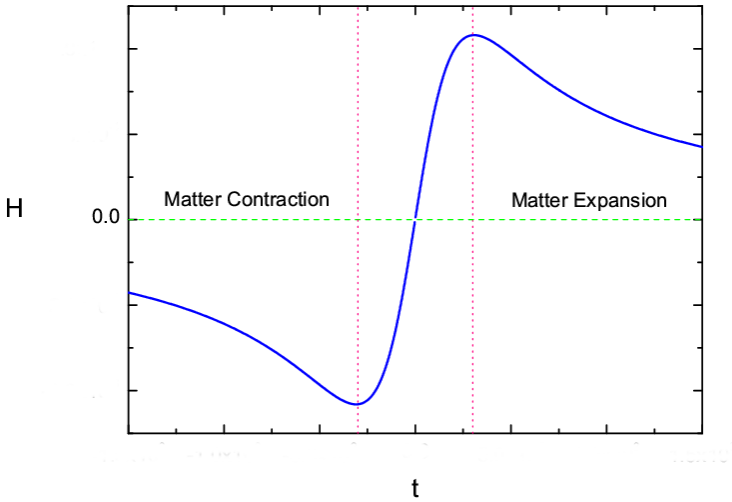
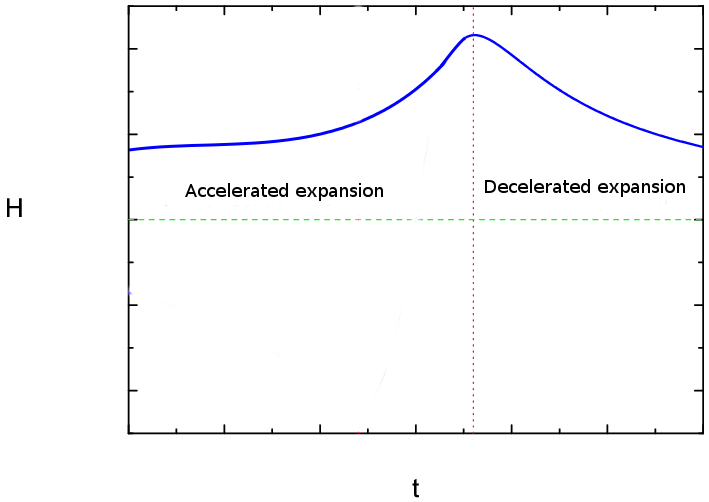
Cosmological scenarios without initial singularity

- **Motivation**: the unresolved problem of the initial singularity in the Big Bang theory.
- **Examples non-singular solutions**: cosmological bounce and Genesis.





- **Specific properties**: both solutions imply violation of Null Energy condition (NEC)

$$T_{\mu\nu}k^{\mu}k^{\nu} > 0 \quad (g_{\mu\nu}k^{\mu}k^{\nu} = 0).$$

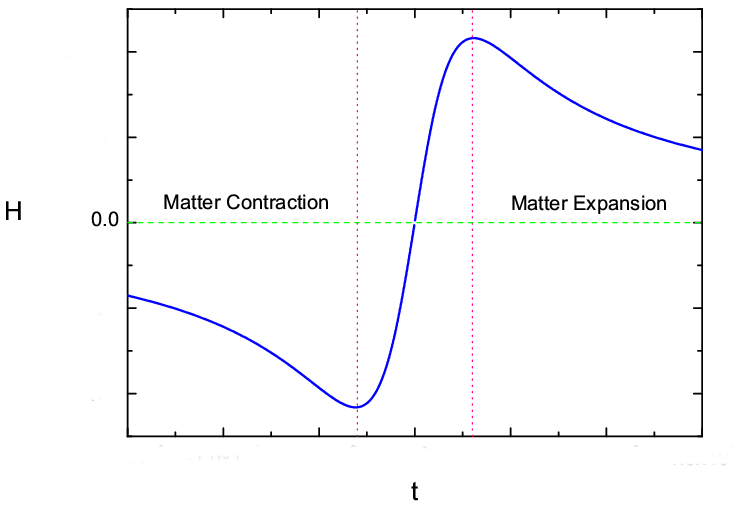
Null Energy Condition

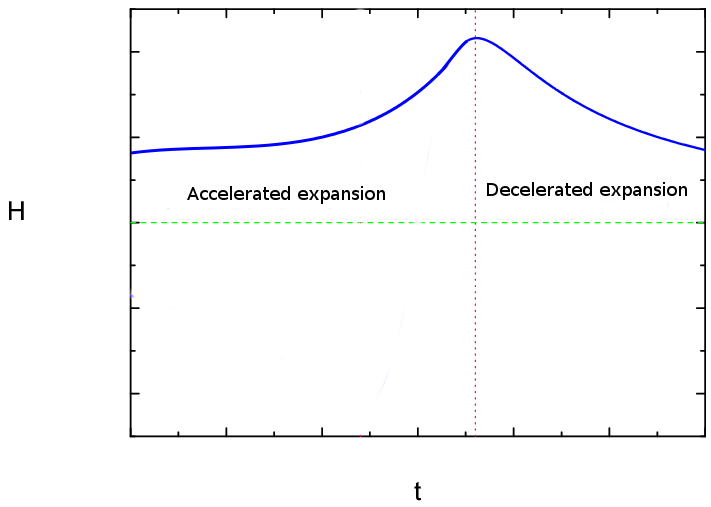
For a homogeneous, isotropic Universe:

$$T_{\mu\nu}k^{\mu}k^{\nu} > 0 \qquad \longleftrightarrow \qquad p + \rho > 0$$

If NEC holds, then cosmological bounce is impossible:

$$\dot{H} = -4\pi G(p+\rho) + \frac{\kappa}{a^2} \quad \longrightarrow \quad \dot{H} = -4\pi G(p+\rho) < 0.$$





Null Energy Condition

For a homogeneous, isotropic Universe:

$$T_{\mu\nu}k^{\mu}k^{\nu} > 0 \qquad \longleftrightarrow \qquad p + \rho > 0$$

Another cosmological consequence of NEC: energy density is always decreases in the expanding Universe, which forbids the Genesis scenario

$$\frac{d\rho}{dt} = -3H(\rho + p) < 0.$$

NEC violation either requires an exotic matter component or results in pathological behaviour of a theory.

Generalized Galileon theories / Horndeski theories

G.Horndeski, Int.J.Theor.Phys. 10, 363 (1974)

Lagrangian of the theory

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\right),$$

$$\mathcal{L}_2 = F(\pi, X),$$

$$\mathcal{L}_3 = K(\pi, X)\Box\pi,$$

$$\mathcal{L}_4 = -G_4(\pi, X)R + 2G_{4X}(\pi, X)\left[(\Box\pi)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\right],$$

$$\mathcal{L}_5 = G_5(\pi, X)G^{\mu\nu}\pi_{;\mu\nu} + \frac{1}{3}G_{5X}\left[(\Box\pi)^3 - 3\Box\pi\pi_{;\mu\nu}\pi^{;\mu\nu} + 2\pi_{;\mu\nu}\pi^{;\mu\rho}\pi_{;\rho}^{\ \nu}\right],$$

where π is a scalar field, $X = g^{\mu\nu}\pi_{,\mu}\pi_{,\nu}$, $G_{iX} = \partial G_i/\partial X$.

- EOMs are second order (hence, no Ostrogradsky ghosts).
- NEC can be violated with no pathologies arising at the linearized level.

Stability of the cosmological solution

A solution is considered *stable* if there are no ghost and gradient instabilities among the linearized perturbations about the solution.

Background set-up: a homogeneous Galileon field $\pi = \pi_b(t)$ in a spatially-flat FLRW geometry

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2.$$

Perturbations

$$h_{00} = 2\alpha, \quad h_{0i} = \partial_i\beta, \quad h_{ij} = -a^2(2\zeta \cdot \delta_{ij} + h_{ij}^T), \quad \delta\pi = \chi.$$

- The explicit form of the only scalar DOF depends on the gauge fixing: in unitary gauge ($\chi = 0$) the curvature perturbation ζ is dynamical.

Stability analysis

Quadratic action for dynamical DOFs ($\chi = 0$)

$$S^{(2)} = \int \mathrm{d}t\mathrm{d}^3x\, a^3 \Big[\frac{\mathcal{G}_\mathcal{T}}{8} \left(\dot{h}^T_{ij} \right)^2 - \frac{\mathcal{F}_\mathcal{T}}{8a^2} \left(\partial_k h^T_{ij} \right)^2 + \mathcal{G}_\mathcal{S} \dot{\zeta}^2 - \mathcal{F}_\mathcal{S} \frac{(\nabla\zeta)^2}{a^2} \Big]$$

$$\mathcal{G}_\mathcal{S} = \frac{\Sigma {\mathcal{G}_\mathcal{T}}^2}{\Theta^2} + 3\mathcal{G}_\mathcal{T},$$

$$\mathcal{F}_\mathcal{S} = \frac{1}{a} \frac{\mathrm{d}\xi}{\mathrm{d}t} - \mathcal{F}_\mathcal{T},$$

$$\xi = \frac{a {\mathcal{G}_\mathcal{T}}^2}{\Theta}.$$

Σ, Θ, $\mathcal{G}_\mathcal{T}$ и $\mathcal{F}_\mathcal{T}$ are some expressions in terms of Lagrangian functions.
Sound speeds squared for tensor and scalar modes:

$$c^2_\mathcal{T} = \frac{\mathcal{F}_\mathcal{T}}{\mathcal{G}_\mathcal{T}}, \quad c^2_\mathcal{S} = \frac{\mathcal{F}_\mathcal{S}}{\mathcal{G}_\mathcal{S}}$$

$$S^{(2)} = \int \mathrm{d}t\mathrm{d}^3x\, a^3 \Big[\frac{\mathcal{G}_\mathcal{T}}{8} \left(\dot{h}^T_{ij} \right)^2 - \frac{\mathcal{F}_\mathcal{T}}{8a^2} \left(\partial_k h^T_{ij} \right)^2 + \mathcal{G}_\mathcal{S} \dot{\zeta}^2 - \mathcal{F}_\mathcal{S} \frac{(\nabla\zeta)^2}{a^2} \Big]$$

$$c_\mathcal{T}^2 = \frac{\mathcal{F}_\mathcal{T}}{\mathcal{G}_\mathcal{T}}, \qquad c_\mathcal{S}^2 = \frac{\mathcal{F}_\mathcal{S}}{\mathcal{G}_\mathcal{S}}$$

Stability conditions

$$\mathcal{G}_\mathcal{T} \geq \mathcal{F}_\mathcal{T} > 0, \quad \mathcal{G}_\mathcal{S} \geq \mathcal{F}_\mathcal{S} > 0$$

No-go theorem

Gradient instabilities inevitably arise about any non-singular solution in Horndeski theory.

M.Libanov, S.Mironov, V.Rubakov, JCAP 1608 (2016)

T.Kobayashi, Phys.Rev. D94 (2016)

No-go theorem in Horndeski theory

$$S^{(2)} = \int \mathrm{d}t\mathrm{d}^3x\, a^3 \Big[\frac{\mathcal{G}_\mathcal{T}}{8} \left(\dot{h}_{ij}^T\right)^2 - \frac{\mathcal{F}_\mathcal{T}}{8a^2} \left(\partial_k h_{ij}^T\right)^2 + \mathcal{G}_\mathcal{S} \dot{\zeta}^2 - \mathcal{F}_\mathcal{S} \frac{(\nabla\zeta)^2}{a^2} \Big]$$

$$\mathcal{F}_\mathcal{S} = \frac{1}{a}\frac{\mathrm{d}\xi}{\mathrm{d}t} - \mathcal{F}_\mathcal{T} \qquad \longrightarrow \qquad \boxed{\frac{\mathrm{d}\xi}{\mathrm{d}t} = a\left(\mathcal{F}_\mathcal{S} + \mathcal{F}_\mathcal{T}\right) > 0}$$

$$\mathcal{F}_\mathcal{S} > \epsilon > 0, \quad \mathcal{F}_\mathcal{T} > \epsilon > 0 \quad \longrightarrow \quad \exists t_0,\ \xi(t_0) = 0$$

By definition:

$$\xi = \frac{a{\mathcal{G}_\mathcal{T}}^2}{\Theta}$$

In Horndeski theories one cannot make $\xi(t)$ behave in a way suggested by stability conditions, hence, there are always gradient instabilities arising.

Beyond Horndeski

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_{\mathcal{BH}}\right),$$

$$\mathcal{L}_2 = F(\pi, X),$$

$$\mathcal{L}_3 = K(\pi, X)\Box\pi,$$

$$\mathcal{L}_4 = -G_4(\pi, X)R + 2G_{4X}(\pi, X)\left[(\Box\pi)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\right],$$

$$\mathcal{L}_5 = G_5(\pi, X)G^{\mu\nu}\pi_{;\mu\nu} + \frac{1}{3}G_{5X}\left[(\Box\pi)^3 - 3\Box\pi\pi_{;\mu\nu}\pi^{;\mu\nu} + 2\pi_{;\mu\nu}\pi^{;\mu\rho}\pi_{;\rho}{}^{\nu}\right],$$

$$\mathcal{L}_{\mathcal{BH}} = F_4(\pi, X)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'}$$
$$+F_5(\pi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'}\pi_{;\sigma\sigma'}.$$

M.Zumalacárregui, J.García-Bellido, Phys.Rev. D 89 (2014)

J.Gleyzes, D.Langlois, F.Piazza, F.Vernizzi, Phys.Rev.Lett. 114 (2015)

Stability analysis

$$S^{(2)} = \int \mathrm{d}t\mathrm{d}^3x a^3 \Big[\frac{\mathcal{G}_\mathcal{T}}{8} \left(\dot{h}_{ij}^T \right)^2 - \frac{\mathcal{F}_\mathcal{T}}{8a^2} \left(\partial_k h_{ij}^T \right)^2 + \mathcal{G}_\mathcal{S} \dot{\zeta}^2 - \mathcal{F}_\mathcal{S} \frac{(\nabla\zeta)^2}{a^2} \Big]$$

$$\mathcal{G}_\mathcal{S} = \frac{\Sigma \mathcal{G}_\mathcal{T}{}^2}{\Theta^2} + 3\mathcal{G}_\mathcal{T},$$
$$\mathcal{F}_\mathcal{S} = \frac{1}{a}\frac{\mathrm{d}\xi}{\mathrm{d}t} - \mathcal{F}_\mathcal{T},$$
$$\xi = \frac{a\mathcal{G}_\mathcal{T}(\mathcal{G}_\mathcal{T} - \mathcal{D}\dot{\pi})}{\Theta},$$

where

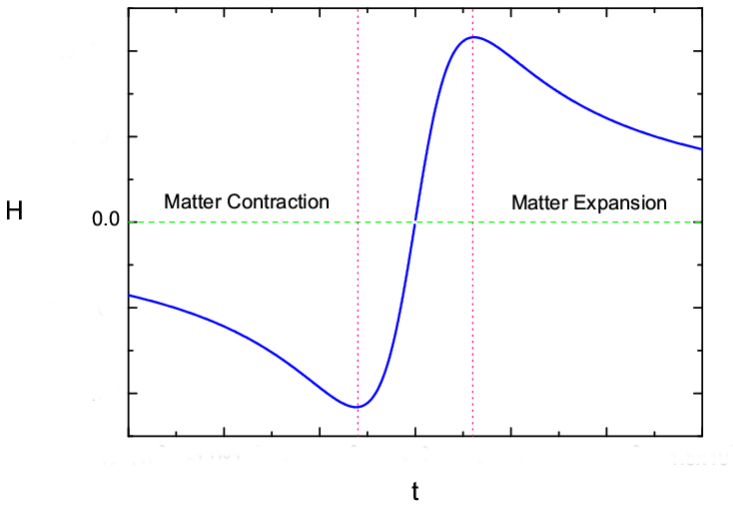
$$\mathcal{D} = 2F_4 X\dot{\pi} + 6HF_5 X^2.$$

Non-trivial $\mathcal{D}$ allows the "correct" behaviour of $\xi(t)$, which is consistent with stability conditions.

Non-singular cosmological solutions in beyond Horndeski theory

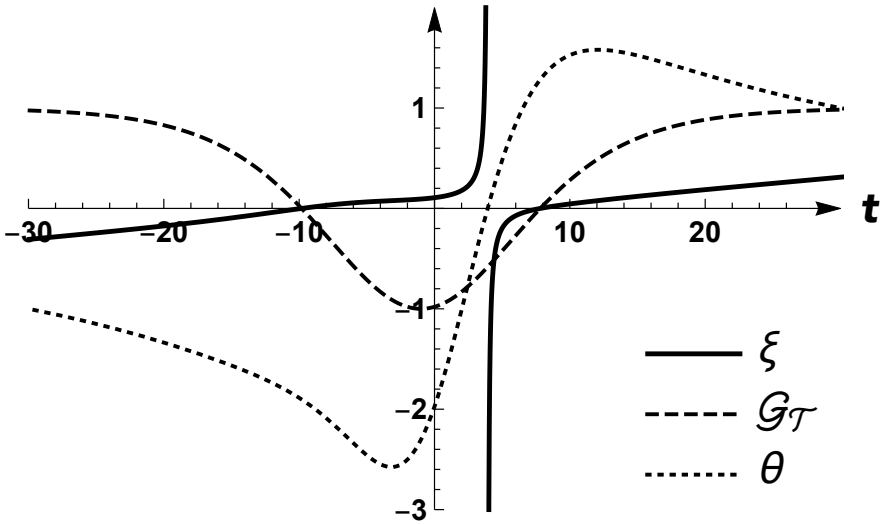
A construction technique for stable non-singular cosmologies:

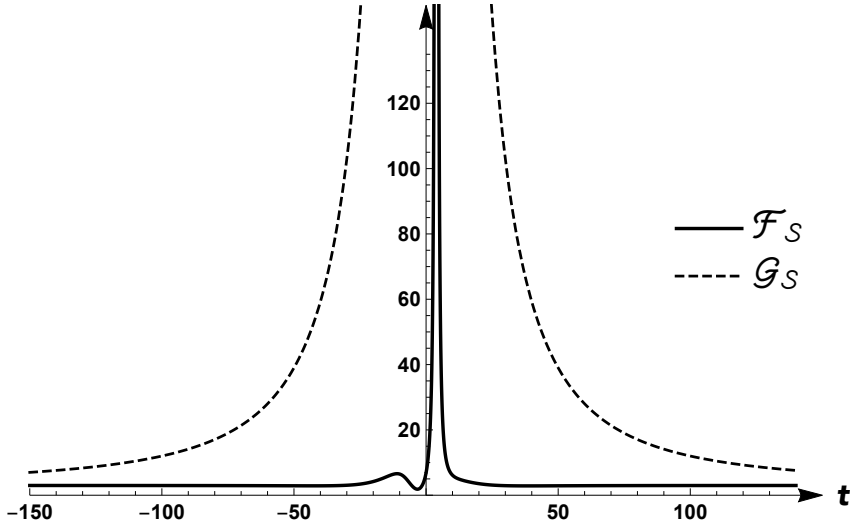
- Explicit choice of the Hubble parameter $H(t)$:

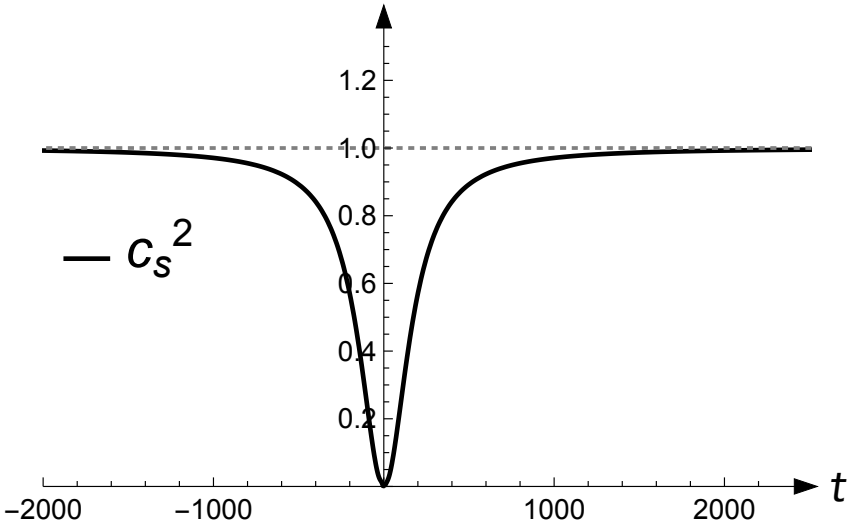


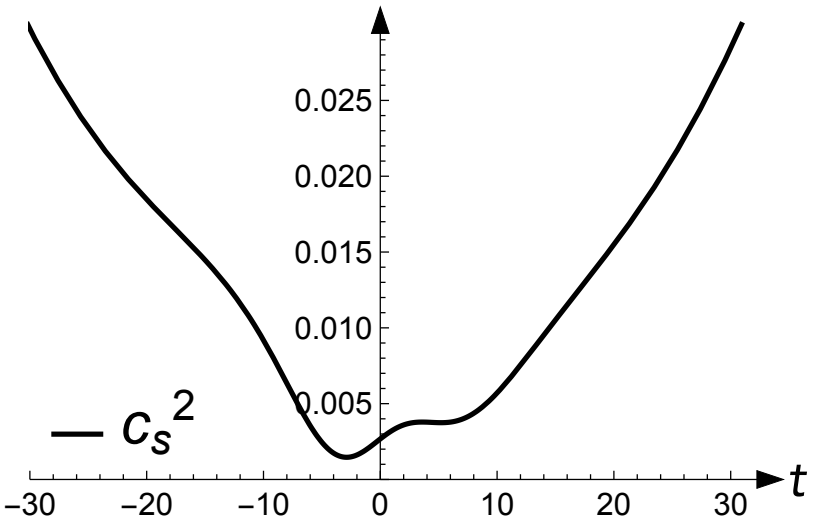
- Reconstruction of the Lagrangian functions F, G_4, G_5, F_4 and F_5, which comply with the following conditions:
  (a) background EOM
  (b) stability conditions $(\mathcal{G}_\mathcal{T} \geq \mathcal{F}_\mathcal{T} > 0,\ \mathcal{G}_\mathcal{S} \geq \mathcal{F}_\mathcal{S} > 0)$
  (c) * specific form of the asymptotics of the theory

Bouncing solution in beyond Horndeski theory: an example









Superluminality issue

$$S^{(2)} = \int \mathrm{d}t\mathrm{d}^3x a^3 \Big[\frac{\mathcal{G}_\mathcal{T}}{8} \left(\dot{h}^T_{ij} \right)^2 - \frac{\mathcal{F}_\mathcal{T}}{8a^2} \left(\partial_k h^T_{ij} \right)^2 + \mathcal{G}_\mathcal{S} \dot{\zeta}^2 - \mathcal{F}_\mathcal{S} \frac{(\nabla\zeta)^2}{a^2} \Big]$$

Stability conditions

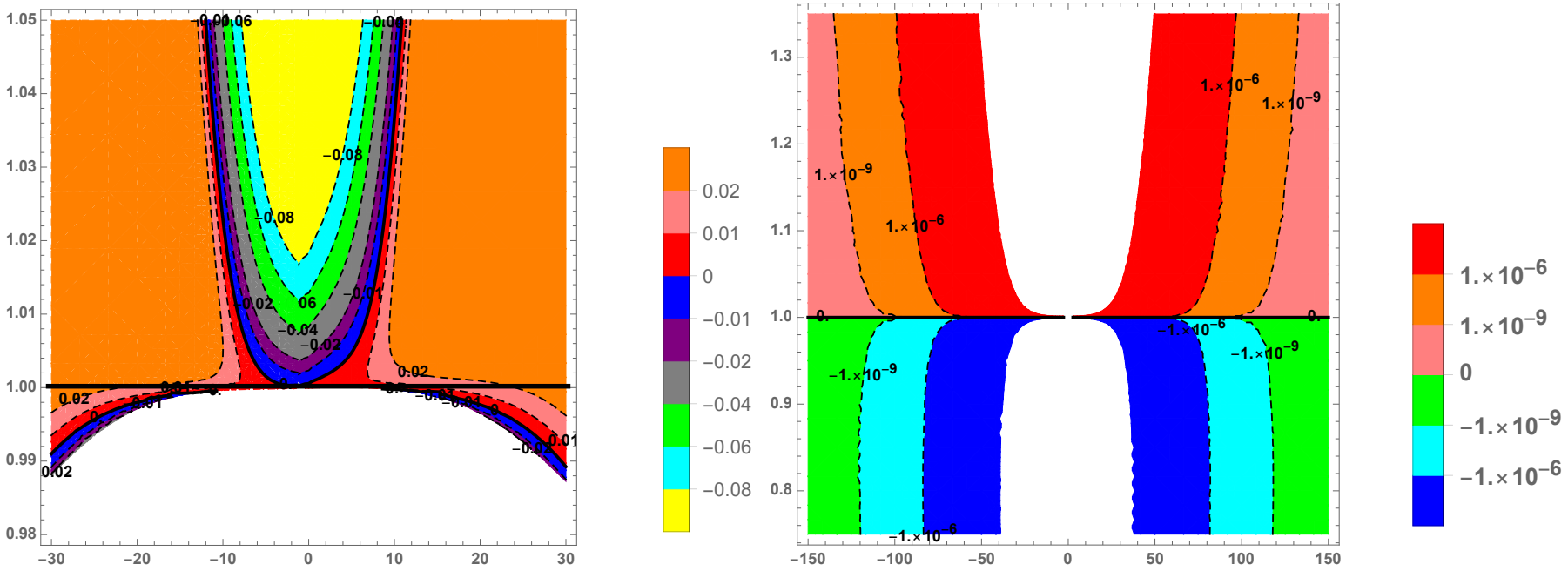
$$\mathcal{G}_\mathcal{T} \geq \mathcal{F}_\mathcal{T} > 0, \quad \mathcal{G}_\mathcal{S} \geq \mathcal{F}_\mathcal{S} > 0$$

$$c^2_\mathcal{T} = \frac{\mathcal{F}_\mathcal{T}}{\mathcal{G}_\mathcal{T}} \leq 1, \qquad c^2_\mathcal{S} = \frac{\mathcal{F}_\mathcal{S}}{\mathcal{G}_\mathcal{S}} \leq 1$$

In the solution above $\mathcal{G}_\mathcal{T} = 1$ and $\mathcal{F}_\mathcal{T} = 1$, hence, $c^2_\mathcal{T} = 1$ at all times.

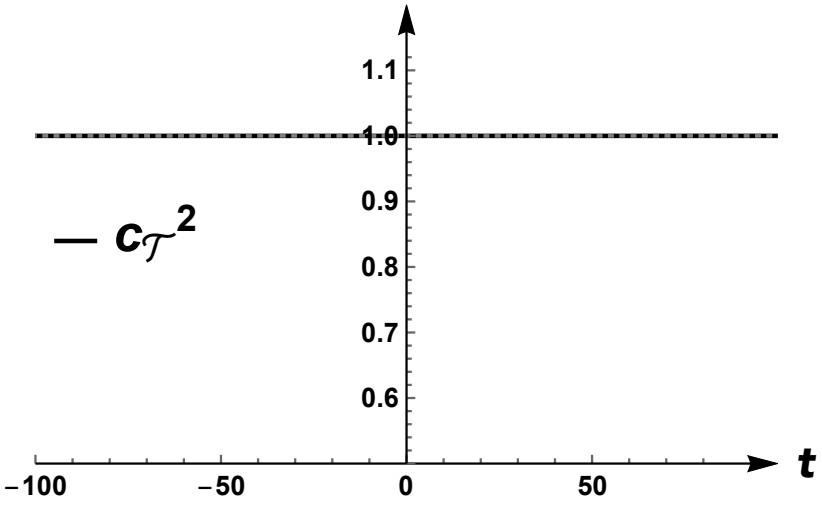
Does the bouncing solution remain safely (sub)luminal even upon minor fluctuations of the solution?

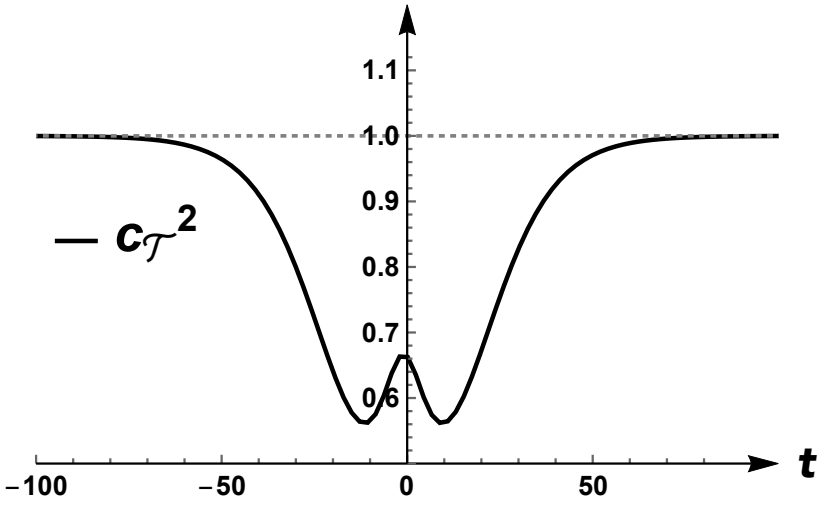
Phase space $(\pi, \dot{\pi})$ around the solution



Phase space for $c_{\mathcal{S}}^2(\pi, \dot{\pi})$ (left panel) and $(1 - c_{\mathcal{T}}^2(\pi, \dot{\pi}))$ (right panel). The bouncing solution lies right on the boundary of the superluminal domain.

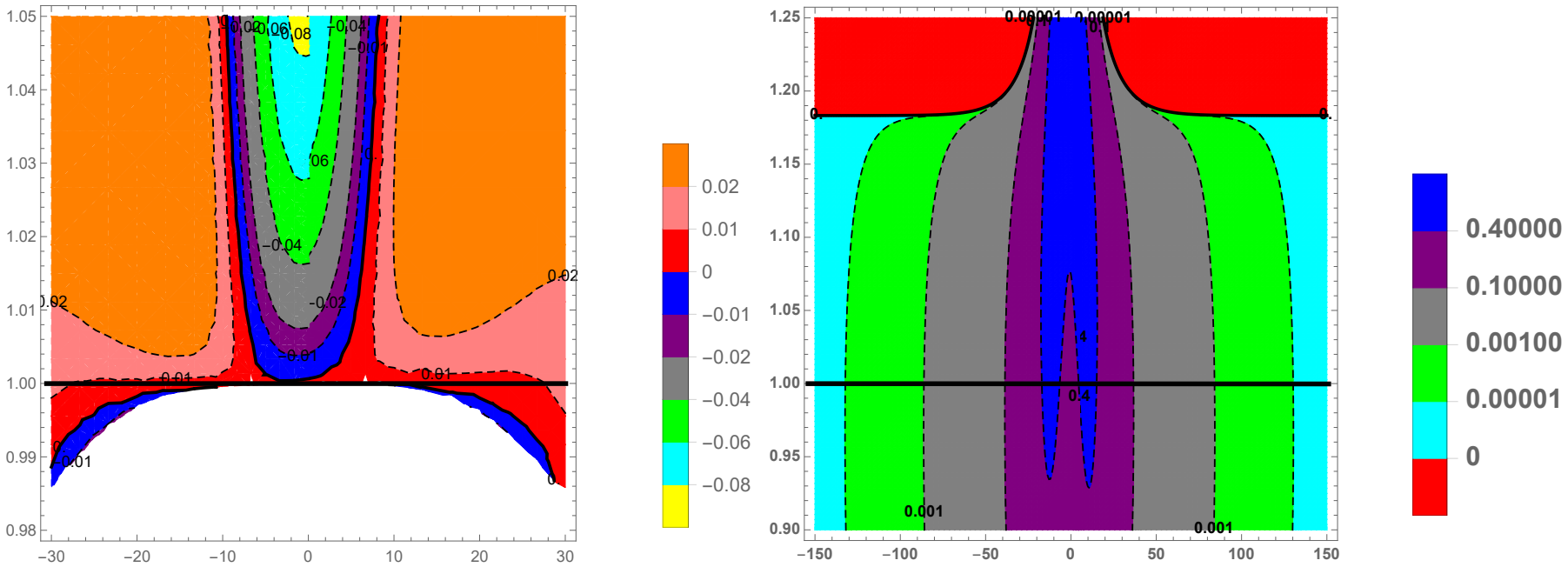
Modified bouncing solution





Unlike the original solution with $\mathcal{G}_{\mathcal{T}} = \mathcal{F}_{\mathcal{T}} = 1$, we reconstruct the Lagrangian functions so that $\mathcal{G}_{\mathcal{T}} < 1$ and $\mathcal{F}_{\mathcal{T}} = 1$ at times around the bouncing epoch.

Phase space $(\pi, \dot{\pi})$ around the *modified* solution

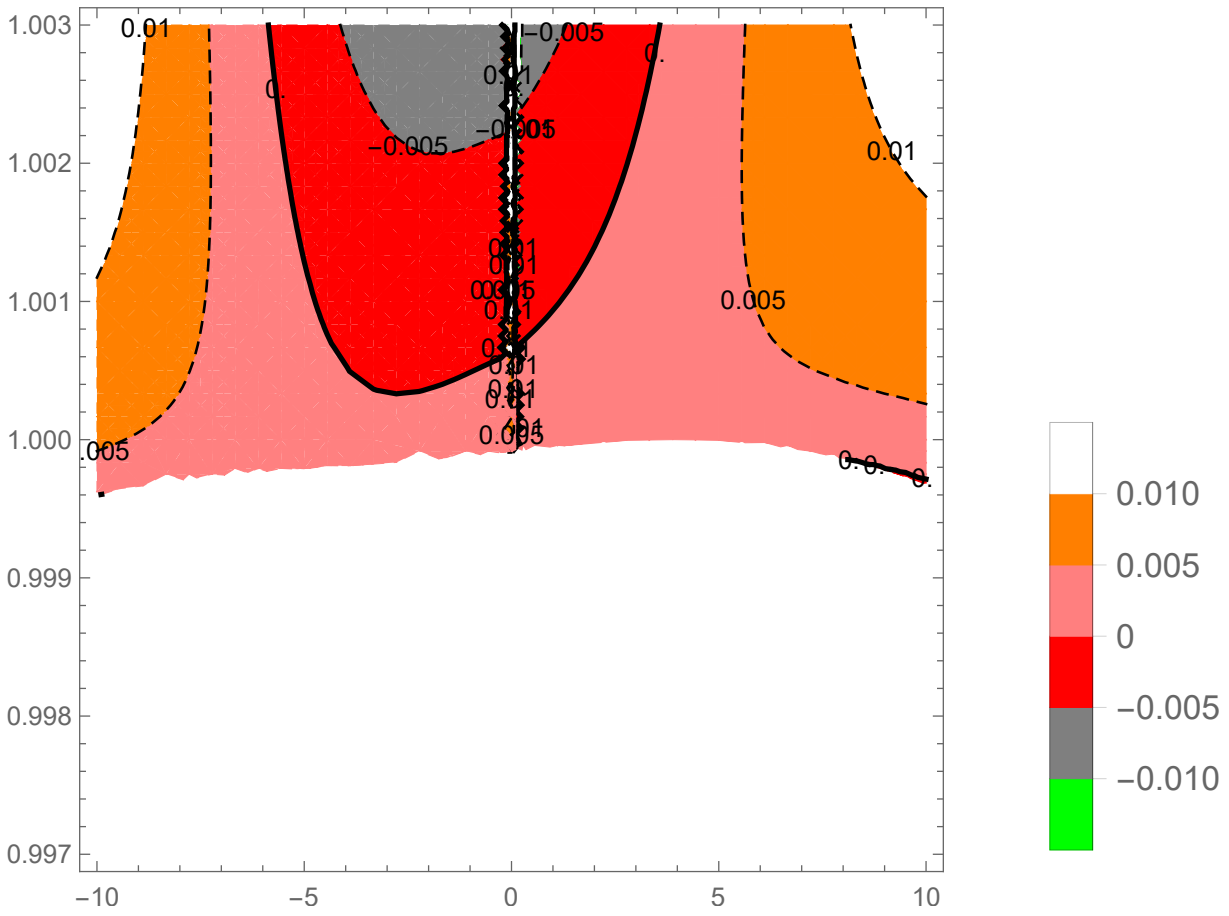


Phase space for $c_{\mathcal{S}}^2(\pi, \dot{\pi})$ (left panel) and $(1 - c_{\mathcal{T}}^2(\pi, \dot{\pi}))$ (right panel). The modified bouncing solution is safely away from the superluminal domain.

Conclusions

- One has to go beyond Horndeski to have a *completely* stable non-singular cosmological scenario.
- There is a bouncing solution in beyond Horndeski theory, which is free from ghosts and gradient instabilities during entire evolution (a promising scenario for further phenomenological applications).
- The modified bouncing solution admits strictly subluminal perturbation modes.

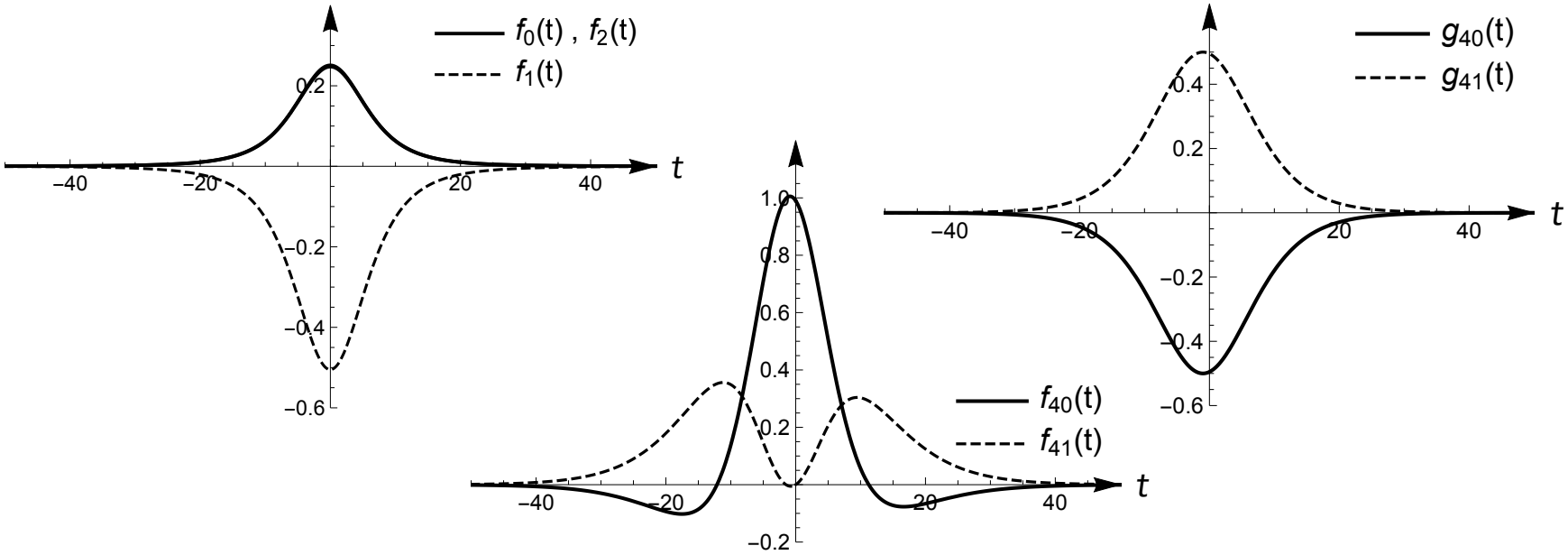
Phase space $(\pi, \dot{\pi})$ around the modified solution



Phase space for $c_{\mathcal{S}}^2(\pi, \dot{\pi})$ for the modified solution.

Appendix: the reconstructed Lagrangian functions for bouncing solution

$$F(\pi, X) = f_0(\pi) + f_1(\pi) \cdot X + f_2(\pi) \cdot X^2,$$
$$G_4(\pi, X) = \frac{1}{2} + g_{40}(\pi) + g_{41}(\pi) \cdot X,$$
$$F_4(\pi, X) = f_{40}(\pi) + f_{41}(\pi) \cdot X.$$



Appendix: coefficients in the quadratic action

$$\mathcal{G}_{\mathcal{T}} = 2G_4 - 4G_{4X}X + G_{5\pi}X - 2HG_{5X}X\dot{\pi},$$
$$\mathcal{F}_{\mathcal{T}} = 2G_4 - 2G_{5X}X\ddot{\pi} - G_{5\pi}X,$$
$$\mathcal{D} = 2F_4X\dot{\pi} + 6HF_5X^2,$$
$$\Theta = -K_XX\dot{\pi} + 2G_4H - 8HG_{4X}X - 8HG_{4XX}X^2 + G_{4\pi}\dot{\pi} + 2G_{4\pi X}X\dot{\pi}$$
$$- 5H^2G_{5X}X\dot{\pi} - 2H^2G_{5XX}X^2\dot{\pi} + 3HG_{5\pi}X + 2HG_{5\pi X}X^2$$
$$+ 10HF_4X^2 + 4HF_{4X}X^3 + 21H^2F_5X^2\dot{\pi} + 6H^2F_{5X}X^3\dot{\pi},$$
$$\Sigma = F_XX + 2F_{XX}X^2 + 12HK_XX\dot{\pi} + 6HK_{XX}X^2\dot{\pi} - K_{\pi}X - K_{\pi X}X^2$$
$$- 6H^2G_4 + 42H^2G_{4X}X + 96H^2G_{4XX}X^2 + 24H^2G_{4XXX}X^3$$
$$- 6HG_{4\pi}\dot{\pi} - 30HG_{4\pi X}X\dot{\pi} - 12HG_{4\pi XX}X^2\dot{\pi} + 30H^3G_{5X}X\dot{\pi}$$
$$+ 26H^3G_{5XX}X^2\dot{\pi} + 4H^3G_{5XXX}X^3\dot{\pi} - 18H^2G_{5\pi}X - 27H^2G_{5\pi X}X^2$$
$$- 6H^2G_{5\pi XX}X^3 - 90H^2F_4X^2 - 78H^2F_{4X}X^3 - 12H^2F_{4XX}X^4$$
$$- 168H^3F_5X^2\dot{\pi} - 102H^3F_{5X}X^3\dot{\pi} - 12H^3F_{5XX}X^4\dot{\pi}.$$