Stability and superluminality of cosmological models in generalized Galileon theories

S. Mironov, V. Rubakov, V.Volkova

INR RAS, Moscow

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Cosmological scenarios without initial singularity

- **Motivation**: the unresolved problem of the initial singularity in the Big Bang theory.

- **Examples non-singular solutions**: cosmological bounce and Genesis.

- **Specific properties**: both solutions imply violation of Null Energy condition (NEC)

\[ T_{\mu\nu} k^\mu k^\nu > 0 \quad (g_{\mu\nu} k^\mu k^\nu = 0). \]
Null Energy Condition

For a homogeneous, isotropic Universe:

\[ T_{\mu\nu} k^\mu k^\nu > 0 \quad \iff \quad p + \rho > 0 \]

If NEC holds, then cosmological bounce is impossible:

\[ \dot{H} = -4\pi G (p + \rho) \frac{\kappa}{a^2} \quad \rightarrow \quad \dot{H} = -4\pi G (p + \rho) < 0. \]
Null Energy Condition

For a homogeneous, isotropic Universe:

\[ T_{\mu \nu} k^\mu k^\nu > 0 \quad \iff \quad p + \rho > 0 \]

Another cosmological consequence of NEC: energy density is always decreases in the expanding Universe, which forbids the Genesis scenario

\[ \frac{d\rho}{dt} = -3H(\rho + p) < 0. \]

NEC violation either requires an exotic matter component or results in pathological behaviour of a theory.
Generalized Galileon theories / Horndeski theories


Lagrangian of the theory

\[ S = \int d^4x \sqrt{-g} \left( \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 \right), \]

\[ \mathcal{L}_2 = F(\pi, X), \]

\[ \mathcal{L}_3 = K(\pi, X) \Box \pi, \]

\[ \mathcal{L}_4 = -G_4(\pi, X)R + 2G_4X(\pi, X) \left[ (\Box \pi)^2 - \pi_{;\mu\nu} \pi^{;\mu\nu} \right], \]

\[ \mathcal{L}_5 = G_5(\pi, X)G^{\mu\nu} \pi_{;\mu\nu} + \frac{1}{3}G_5X \left[ (\Box \pi)^3 - 3\Box \pi \pi_{;\mu\nu} \pi^{;\mu\nu} + 2\pi_{;\mu\nu} \pi^{;\mu\rho} \pi^{;\nu}_{;\rho} \right], \]

where \( \pi \) is a scalar field, \( X = g^{\mu\nu} \pi_{,\mu} \pi_{,\nu}, \)

- EOMs are second order (hence, no Ostrogradsky ghosts).
- NEC can be violated with no pathologies arising at the linearized level.
A solution is considered *stable* if there are no ghost and gradient instabilities among the linearized perturbations about the solution.

Background set-up: a homogeneous Galileon field $\pi = \pi_b(t)$ in a spatially-flat FLRW geometry

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2.$$ 

Perturbations

$$h_{00} = 2\alpha, \quad h_{0i} = \partial_i \beta, \quad h_{ij} = -a^2(2\zeta \cdot \delta_{ij} + h^T_{ij}), \quad \delta\pi = \chi.$$ 

- The explicit form of the only scalar DOF depends on the gauge fixing: in unitary gauge ($\chi = 0$) the curvature perturbation $\zeta$ is dynamical.
Stability analysis

**Quadratic action for dynamical DOFs ($\chi = 0$)**

\[
S^{(2)} = \int dt d^3x \ a^3 \left[ \frac{G_T}{8} (\dot{h}_{ij})^2 - \frac{\mathcal{F}_T}{8a^2} (\partial_k h_{ij})^2 + G_S \dot{\zeta}^2 - \mathcal{F}_S \frac{(\nabla \zeta)^2}{a^2} \right]
\]

\[
G_S = \frac{\Sigma G_T^2}{\Theta^2} + 3G_T,
\]

\[
\mathcal{F}_S = \frac{1}{a} \frac{d\xi}{dt} - \mathcal{F}_T,
\]

\[
\xi = \frac{aG_T^2}{\Theta}.
\]

$\Sigma$, $\Theta$, $G_T$ и $\mathcal{F}_T$ are some expressions in terms of Lagrangian functions. Sound speeds squared for tensor and scalar modes:

\[
c_T^2 = \frac{\mathcal{F}_T}{G_T}, \quad c_S^2 = \frac{\mathcal{F}_S}{G_S}
\]
\[ S^{(2)} = \int dt d^3 x \ a^3 \left[ \frac{G_T}{8} \left( \dot{h}_{ij} \right)^2 - \frac{F_T}{8a^2} \left( \partial_k h_{ij}^T \right)^2 + G_S \dot{\zeta}^2 - F_S \frac{(\nabla \zeta)^2}{a^2} \right] \]

\[ c_T^2 = \frac{F_T}{G_T}, \quad c_S^2 = \frac{F_S}{G_S} \]

**Stability conditions**

\[ G_T \geq F_T > 0, \quad G_S \geq F_S > 0 \]

**No-go theorem**

Gradient instabilities inevitably arise about any non-singular solution in Horndeski theory.

*M.Libanov, S.Mironov, V.Rubakov, JCAP 1608 (2016)*

*T.Kobayashi, Phys.Rev. D94 (2016)*
No-go theorem in Horndeski theory

\[ S^{(2)} = \int dt d^3x \ a^3 \left[ \frac{G_T}{8} \left( \dot{h}_{ij}^T \right)^2 - \frac{\mathcal{F}_T}{8a^2} \left( \partial_k h_{ij}^T \right)^2 + G_S \dot{\zeta}^2 - \mathcal{F}_S \frac{(\nabla \zeta)^2}{a^2} \right] \]

\[ \mathcal{F}_S = \frac{1}{a} \frac{d\xi}{dt} - \mathcal{F}_T \quad \rightarrow \quad \frac{d\xi}{dt} = a \ (\mathcal{F}_S + \mathcal{F}_T) > 0 \]

\[ \mathcal{F}_S > \epsilon > 0, \ \mathcal{F}_T > \epsilon > 0 \quad \rightarrow \quad \exists t_0, \ \xi(t_0) = 0 \]

By definition:

\[ \xi = \frac{aG_T^2}{\Theta} \]

In Horndeski theories one cannot make \( \xi(t) \) behave in a way suggested by stability conditions, hence, there are always gradient instabilities arising.
\[ S = \int d^4x \sqrt{-g} (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_{BH}), \]

\[ \mathcal{L}_2 = F(\pi, X), \]
\[ \mathcal{L}_3 = K(\pi, X) \Box \pi, \]
\[ \mathcal{L}_4 = -G_4(\pi, X)R + 2G_4X(\pi, X) \left[ (\Box \pi)^2 - \pi_{;\mu\nu} \pi^{;\mu\nu} \right], \]
\[ \mathcal{L}_5 = G_5(\pi, X)G^{\mu\nu} \pi_{;\mu\nu} + \frac{1}{3}G_5X \left[ (\Box \pi)^3 - 3\Box \pi \pi_{;\mu\nu} \pi^{;\mu\nu} + 2\pi_{;\mu\nu} \pi^{;\mu\rho} \pi^{;\nu}_\rho \right], \]
\[ \mathcal{L}_{BH} = F_4(\pi, X)\epsilon^{\mu\nu\rho}_\sigma \epsilon^{\mu'}\nu'\rho'\sigma' \pi_{;\mu\pi} \mu' \pi^{;\nu\nu'} \pi^{;\rho\rho'} + F_5(\pi, X)\epsilon^{\mu\nu\rho\sigma} \epsilon^{\mu'}\nu'\rho'\sigma' \pi_{;\mu\pi} \mu' \pi^{;\nu\nu'} \pi^{;\rho\rho'} \pi^{;\sigma\sigma'}. \]

Stability analysis

\[ S^{(2)} = \int dt d^3 x a^3 \left[ \frac{G_T}{8} \left( \dot{h}_{ij} \right)^2 - \frac{F_T}{8a^2} \left( \partial_k h_{ij} \right)^2 + G_S \dot{\zeta}^2 - F_S \frac{(\nabla \zeta)^2}{a^2} \right] \]

\[ G_S = \frac{\Sigma G_T^2}{\Theta^2} + 3G_T, \]

\[ F_S = \frac{1}{a} \frac{d\xi}{dt} - F_T, \]

\[ \xi = \frac{aG_T(G_T - D\dot{\pi})}{\Theta}, \]

where

\[ D = 2F_4 X \dot{\pi} + 6HF_5 X^2. \]

Non-trivial \( D \) allows the "correct" behaviour of \( \xi(t) \), which is consistent with stability conditions.
Non-singular cosmological solutions in beyond Horndeski theory

A construction technique for stable non-singular cosmologies:

- Explicit choice of the Hubble parameter $H(t)$:

![Graph showing Hubble parameter H(t) with peaks indicating Matter Contraction and Matter Expansion]

- Reconstruction of the Lagrangian functions $F$, $G_4$, $G_5$, $F_4$ and $F_5$, which comply with the following conditions:
  
  (a) background EOM
  (b) stability conditions ($G_T \geq F_T > 0$, $G_S \geq F_S > 0$)
  (c) * specific form of the asymptotics of the theory
Bouncing solution in beyond Horndeski theory: an example
Superluminality issue

\[ S^{(2)} = \int dt d^3x a^3 \left[ \frac{G_T}{8} (\dot{h}^T_{ij})^2 - \frac{F_T}{8a^2} (\partial_k h^T_{ij})^2 + G_S \dot{\zeta}^2 - F_S \frac{(\nabla \zeta)^2}{a^2} \right] \]

Stability conditions

\[ G_T \geq F_T > 0, \quad G_S \geq F_S > 0 \]

\[ c_T^2 = \frac{F_T}{G_T} \leq 1, \quad c_S^2 = \frac{F_S}{G_S} \leq 1 \]

In the solution above \( G_T = 1 \) and \( F_T = 1 \), hence, \( c_T^2 = 1 \) at all times.

Does the bouncing solution remain safely (sub)luminal even upon minor fluctuations of the solution?
Phase space for $c_S^2(\pi, \dot{\pi})$ (left panel) and $(1 - c_T^2(\pi, \dot{\pi}))$ (right panel). The bouncing solution lies right on the boundary of the superluminal domain.
Unlike the original solution with $\mathcal{G}_T = \mathcal{F}_T = 1$, we reconstruct the Lagrangian functions so that $\mathcal{G}_T < 1$ and $\mathcal{F}_T = 1$ at times around the bouncing epoch.
Phase space \((\pi, \dot{\pi})\) around the \textit{modified} solution.

Phase space for \(c^2_S(\pi, \dot{\pi})\) (left panel) and \((1 - c_T^2(\pi, \dot{\pi}))\) (right panel). The modified bouncing solution is safely away from the superluminal domain.
Conclusions

- One has to go beyond Horndeski to have a *completely* stable non-singular cosmological scenario.

- There is a bouncing solution in beyond Horndeski theory, which is free from ghosts and gradient instabilities during entire evolution (a promising scenario for further phenomenological applications).

- The modified bouncing solution admits strictly subluminal perturbation modes.
Phase space \((\pi, \dot{\pi})\) around the modified solution.

Phase space for \(c_2^g(\pi, \dot{\pi})\) for the modified solution.
Appendix: the reconstructed Lagrangian functions for bouncing solution

\[ F(\pi, X) = f_0(\pi) + f_1(\pi) \cdot X + f_2(\pi) \cdot X^2, \]
\[ G_4(\pi, X) = \frac{1}{2} + g_{40}(\pi) + g_{41}(\pi) \cdot X, \]
\[ F_4(\pi, X) = f_{40}(\pi) + f_{41}(\pi) \cdot X. \]
Appendix: coefficients in the quadratic action

\[ G_T = 2G_4 - 4G_4X X + G_{5\pi} X - 2HG_{5X} X \dot{\pi}, \]
\[ F_T = 2G_4 - 2G_{5X} \dot{X} \dot{\pi} - G_{5\pi} X, \]
\[ D = 2F_4 \dot{X} \dot{\pi} + 6HF_5 X^2, \]
\[ \Theta = -K_X X \dot{\pi} + 2G_4 H - 8HG_4X X - 8HG_{4XX} X^2 + G_{4\pi} \dot{\pi} + 2G_{4\pi X} X \dot{\pi} - 5H^2 G_{5X} X \dot{\pi} - 2H^2 G_{5XX} X^2 \dot{\pi} + 3HG_{5\pi} X + 2HG_{5\pi X} X^2 + 10HF_4 X^2 + 4HF_{4X} X^3 + 21H^2 F_5 X^2 \dot{\pi} + 6H^2 F_{5X} X^3 \dot{\pi}, \]
\[ \Sigma = F_X X + 2F_{XX} X^2 + 12HK_X \dot{X} \dot{\pi} + 6HK_{XX} X^2 \dot{\pi} - K_X X - K_{XX} X^2 \dot{\pi} - 6H^2 G_4 + 42H^2 G_{4X} X + 96H^2 G_{4XX} X^2 + 24H^2 G_{4XXX} X^3 - 6HG_{4\pi} \dot{\pi} - 30HG_{4\pi X} X \dot{\pi} - 12HG_{4\pi XX} X^2 \dot{\pi} + 30H^3 G_{5X} X \dot{\pi} + 26H^3 G_{5XX} X^2 \dot{\pi} + 4H^3 G_{5XXX} X^3 \dot{\pi} - 18H^2 G_{5\pi} X - 27H^2 G_{5\pi X} X^2 - 6H^2 G_{5\pi XX} X^3 - 90H^2 F_4 X^2 - 78H^2 F_{4X} X^3 - 12H^2 F_{4XX} X^4 - 168H^3 F_5 X^2 \dot{\pi} - 102H^3 F_{5X} X^3 \dot{\pi} - 12H^3 F_{5XX} X^4 \dot{\pi}. \]